The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality

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Abstract

If $H$ is a commutative connected graded Hopf algebra over a commutative ring $\mathbb{k}$, then a certain canonical $\mathbb{k}$-algebra homomorphism $H \to H \otimes \text{QSym}_\mathbb{k}$ is defined, where $\text{QSym}_\mathbb{k}$ denotes the Hopf algebra of quasisymmetric functions. This homomorphism generalizes the "internal comultiplication" on $\text{QSym}_\mathbb{k}$, and extends what Hazewinkel (in §18.24 of his "Witt vectors") calls the Bernstein homomorphism.

We construct this homomorphism with the help of the universal property of $\text{QSym}_\mathbb{k}$ as a combinatorial Hopf algebra (a well-known result by Aguiar, Bergeron and Sottile) and extension of scalars (the commutativity of $H$ allows us to consider, for example, $H \otimes \text{QSym}_\mathbb{k}$ as an $H$-Hopf algebra, and this change of viewpoint significantly extends the reach of the universal property).

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1. Definitions and conventions

For the rest of this note, we fix a commutative ring $k$. All tensor signs ($\otimes$) without a subscript will mean $\otimes_k$. We shall use the notions of $k$-algebras, $k$-coalgebras and $k$-Hopf algebras as defined (e.g.) in [GriRei14, Chapter 1]. We shall also use the notions of graded $k$-algebras, graded $k$-coalgebras and graded $k$-Hopf algebras as defined in [GriRei14, Chapter 1]; in particular, we shall not use the topologists’ sign conventions. The comultiplication and the counit of a $k$-coalgebra $C$ will be denoted by $\Delta_C$ and $\varepsilon_C$, respectively; when the $C$ is unambiguously clear from the context, we will omit it from the notation (so we will just write $\Delta$ and $\varepsilon$).

If $V$ and $W$ are two $k$-modules, then we let $\tau_{V,W}$ be the $k$-linear map $V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$. This $k$-linear map $\tau_{V,W}$ is called the twist map, and is a $k$-module isomorphism.

If $A$ is a $k$-algebra and if $k \in \mathbb{N}$, then we let $m_{(k-1)}^\infty$ denote the “iterated multiplication map” $A \otimes_k A \to A$; see [GriRei14] §1.4 for its definition. (It is the $k$-linear map which sends every $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A \otimes_k$ to $a_1a_2 \cdots a_k$.)

If $A$ is a $k$-coalgebra and if $k \in \mathbb{N}$, then we let $\Delta_{(k-1)}$ denote the “iterated comultiplication map” $A \to A \otimes_k$; again, see [GriRei14] §1.4 for its definition.

A composition shall mean a finite sequence of positive integers. The size of a composition $\alpha = (a_1, a_2, \ldots, a_k)$ is defined to be the nonnegative integer $a_1 + a_2 + \cdots + a_k$, and is denoted by $|\alpha|$. Let $\text{Comp}$ denote the set of all compositions.

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

**Definition 1.1.** Let $H$ be a graded $k$-module. For every $n \in \mathbb{N}$, we let $\pi_n : H \to H_n$ be the canonical projection of $H$ onto the $n$-th graded component $H_n$ of $H$. We shall always regard $\pi_n$ as a map from $H$ to $H_n$, not as a map from $H$ to $H_n$ even though its image is $H_n$.

For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, we let $\pi_{\alpha} : H^{\otimes k} \to H^{\otimes k}$ be the tensor product $\pi_{a_1} \otimes \pi_{a_2} \otimes \cdots \otimes \pi_{a_k}$ of the canonical projections $\pi_{a_i} : H \to H$.

Thus, the image of $\pi_\alpha$ can be identified with $H_{a_1} \otimes H_{a_2} \otimes \cdots \otimes H_{a_k}$.

Let $\text{QSym}_k$ denote the $k$-Hopf algebra of quasisymmetric functions defined over $k$. (This is defined and denoted by $\text{QSym}$ in [ABS03, §3]; it is also defined and denoted by $\text{QSym}$ in [GriRei14, Chapter 5].) We shall follow the notations and conventions of [GriRei14, §5.1] as far as $\text{QSym}_k$ is concerned; in particular,

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1The word “ring” always means “associative ring with 1” in this note. Furthermore, a $k$-algebra (when $k$ is a commutative ring) means a $k$-module $A$ equipped with a ring structure such that the multiplication map $A \times A \to A$ is $k$-bilinear.

2Thus, the twist map $V \otimes V \to V \otimes V$ for a graded $k$-module $V$ sends $v \otimes w \mapsto w \otimes v$, even if $v$ and $w$ are homogeneous of odd degree.
we regard $\mathrm{QSym}_k$ as a subring of the ring $k[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$.

Let $\varepsilon_P$ denote the $k$-linear map $\mathrm{QSym}_k \to k$ sending every $f \in \mathrm{QSym}_k$ to $f(1, 0, 0, 0, \ldots) \in k$. (This map $\varepsilon_P$ is denoted by $\zeta_Q$ in [ABS03 §4] and by $\zeta_Q$ in [GriRei14 Example 7.2].) Notice that $\varepsilon_P$ is a $k$-algebra homomorphism.

**Definition 1.2.** For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we define a power series $M_\alpha \in k[[x_1, x_2, x_3, \ldots]]$ by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

(where the sum is over all strictly increasing $\ell$-tuples $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers). It is well-known (and easy to check) that this $M_\alpha$ belongs to $\mathrm{QSym}_k$. The power series $M_\alpha$ is called the *monomial quasisymmetric function* corresponding to $\alpha$. The family $(M_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the $k$-module $\mathrm{QSym}_k$; this is the so-called *monomial basis* of $\mathrm{QSym}_k$. (See [ABS03 §3] and [GriRei14 §5.1] for more about this basis.)

It is well-known that every $(b_1, b_2, \ldots, b_\ell) \in \text{Comp}$ satisfies

$$\Delta \left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \sum_{i=0}^\ell M_{(b_1, b_2, \ldots, b_i)} \otimes M_{(b_{i+1}, b_{i+2}, \ldots, b_\ell)}$$

(1)

and

$$\varepsilon \left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \begin{cases} 1, & \text{if } \ell = 0; \\ 0, & \text{if } \ell \neq 0. \end{cases}$$

These two equalities can be used as a definition of the $k$-coalgebra structure on $\mathrm{QSym}_k$ (because $(M_\alpha)_{\alpha \in \text{Comp}}$ is a basis of the $k$-module $\mathrm{QSym}_k$, and thus the $k$-linear maps $\Delta$ and $\varepsilon$ are uniquely determined by their values on the $M_\alpha$).

2. **The Aguiar-Bergeron-Sottile theorem**

The cornerstone of the Aguiar-Bergeron-Sottile paper [ABS03] is the following result:

**Theorem 2.1.** Let $k$ be a commutative ring. Let $H$ be a connected graded $k$-Hopf algebra. Let $\zeta : H \to k$ be a $k$-algebra homomorphism.

(a) Then, there exists a unique graded $k$-coalgebra homomorphism $\Psi : H \to \mathrm{QSym}_k$ for which the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\Psi} & \mathrm{QSym}_k \\
\downarrow{\zeta} & & \downarrow{\varepsilon_P} \\
k & & k
\end{array}
$$

holds.
is commutative.

(b) This unique $k$-coalgebra homomorphism $\Psi : H \to \text{QSym}_k$ is a $k$-Hopf algebra homomorphism.

(c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $k$-linear map $\zeta_\alpha : H \to k$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^\otimes k \xrightarrow{\pi_\alpha} H^\otimes k \xrightarrow{\zeta^\otimes k} k \otimes k \xrightarrow{\cong} k.$$  

(Here, the map $k^\otimes k \xrightarrow{\cong} k$ is the canonical $k$-algebra isomorphism from $k^\otimes k$ to $k$. Recall also that $\Delta^{(k-1)} : H \to H^\otimes k$ is the “iterated comultiplication map”; see [GriRei14, §1.4] for its definition. The map $\pi_\alpha : H^\otimes k \to H^\otimes k$ is the one defined in Definition 1.1.)

Then, the unique $k$-coalgebra homomorphism $\Psi$ of Theorem 2.1 (a) is given by the formula

$$\Psi (h) = \sum_{\alpha \in \text{Comp}; \ |\alpha| = n} \zeta_\alpha (h) \cdot M_\alpha \quad \text{whenever } n \in \mathbb{N} \text{ and } h \in H_n.$$  

(Recall that $H_n$ denotes the $n$-th graded component of $H$.)

(d) The unique $k$-coalgebra homomorphism $\Psi$ of Theorem 2.1 (a) is also given by

$$\Psi (h) = \sum_{\alpha \in \text{Comp}} \zeta_\alpha (h) \cdot M_\alpha \quad \text{for every } h \in H$$  

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

(e) Assume that the $k$-coalgebra $H$ is cocommutative. Then, the unique $k$-coalgebra homomorphism $\Psi$ of Theorem 2.1 (a) satisfies $\Psi (H) \subseteq \Lambda_k$, where $\Lambda_k$ is the $k$-algebra of symmetric functions over $k$. (See [GriRei14, §2] for the definition of $\Lambda_k$. We regard $\Lambda_k$ as a $k$-subalgebra of $\text{QSym}_k$ in the usual way.)

Parts (a), (b) and (c) of Theorem 2.1 are proven in [ABS03] proof of Theorem 4.1 and [GriRei14] proof of Theorem 7.3] (although we are using different nota-
tions here and avoiding the standing assumptions of [ABS03] which needlessly require \( k \) to be a field and \( H \) to be of finite type). Theorem 2.1 (d) easily follows from Theorem 2.1 (c). Theorem 2.1 (e) appears in [GriRei14, Remark 7.4] (and something very close is proven in [ABS03, Theorem 4.3]). For the sake of completeness, let me give some details on the proof of Theorem 2.1 (e):

**Proof of Theorem 2.1 (e).** Let \( \varepsilon : \Lambda_k \to k \) be the restriction of the \( k \)-algebra homomorphism \( \varepsilon : \text{QSym}_k \to k \) to \( \Lambda \). From [ABS03, Theorem 4.3], we know that there exists a unique graded \( k \)-coalgebras for which the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Psi'} & \Lambda_k \\
\varepsilon & \circlearrowright & \varepsilon_p \\
\zeta & \downarrow & \kappa
\end{array}
\]

is commutative. Consider this \( \Psi' \). Let \( i : \Lambda_k \to \text{QSym}_k \) be the canonical inclusion map; this is a \( k \)-Hopf algebra homomorphism. Also, \( \varepsilon_p = \varepsilon \circ \varepsilon \circ i \). The commutative diagram (2) yields \( \zeta = \varepsilon_p \circ \Psi' = \varepsilon_p \circ i \circ \Psi' \).

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3The paper [ABS03] defines a combinatorial coalgebra to be a pair \((H, \zeta)\) consisting of a connected graded \( k \)-coalgebra \( H \) (where “connected” means that \( \varepsilon |_{H_0} : H_0 \to k \) is a \( k \)-module isomorphism) and a \( k \)-linear map \( \zeta : H \to k \) satisfying \( \zeta |_{H_0} = \varepsilon |_{H_0} \). Furthermore, it defines a morphism from a combinatorial coalgebra \((H', \zeta')\) to a combinatorial coalgebra \((H, \zeta)\) to be a homomorphism \( \alpha : H' \to H \) of graded \( k \)-coalgebras for which the diagram

\[
\begin{array}{ccc}
H' & \xrightarrow{\alpha} & H \\
\zeta' & \downarrow & \zeta \\
\kappa & \downarrow & \kappa
\end{array}
\]

is commutative. Theorem 2.1 (a) translates into this language as follows: There exists a unique morphism from the combinatorial coalgebra \((H, \zeta)\) to the combinatorial coalgebra \((\text{QSym}_k, \varepsilon_p)\). (Apart from this, [ABS03] is also using the notations \( k, H, \text{QSym} \) and \( \zeta_Q \) for what we call \( k, H, \text{QSym}_k \) and \( \varepsilon_p \).)

4Proof. Let \( \Psi \) be the unique \( k \)-coalgebra homomorphism \( \Psi \) of Theorem 2.1 (a). It is easy to see that every \( n \in \mathbb{N} \), every composition \( \alpha \) with \( |\alpha| \neq n \) and every \( h \in H_n \) satisfy \( \zeta_a(h) = 0 \) (because \( \pi_a \left( \Delta^{(k-1)} \left( \sum_{h \in H_n} h \right) \right) = 0 \) for reasons of gradedness). Hence, for every \( n \in \mathbb{N} \) and every \( h \in H_n \), we have

\[
\sum_{a \in \text{Comp}} \zeta_a(h) \cdot M_a = \sum_{a \in \text{Comp}: |\alpha| = n} \zeta_a(h) \cdot M_a + \sum_{a \in \text{Comp}: |\alpha| \neq n} \zeta_a(h) \cdot M_a
\]

\[= \sum_{a \in \text{Comp}: |\alpha| = n} \zeta_a(h) \cdot M_a = \Psi(h) \] (by Theorem 2.1 (e)).

Both sides of this equality are \( k \)-linear in \( h \); thus, it also holds for every \( h \in H \) (even if \( h \) is not homogeneous). This proves Theorem 2.1 (d).
Now, consider the unique \( k \)-coalgebra homomorphism \( \Psi \) of Theorem 2.1 (a). Due to its uniqueness, it has the following property: If \( \bar{\Psi} \) is any \( k \)-coalgebra homomorphism \( H \to \text{QSym}_k \) for which the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Psi} & \text{QSym}_k \\
\downarrow{\zeta} & & \downarrow{\varepsilon_p} \\
k & & \varepsilon_P
\end{array}
\]

is commutative, then \( \bar{\Psi} = \Psi \). Applying this to \( \bar{\Psi} = \iota \circ \Psi' \), we obtain \( \iota \circ \Psi' = \Psi \) (since the diagram (3) is commutative for \( \bar{\Psi} = \iota \circ \Psi' \) (because \( \zeta = \varepsilon_P \circ \iota \circ \Psi' \))). Hence, \( \underbrace{\Psi}_{=\iota \circ \Psi'} (H) = (\iota \circ \Psi') (H) = \iota \left( \underbrace{\Psi'(H)}_{\subseteq \Lambda_k} \right) \subseteq \iota(\Lambda_k) = \Lambda_k \). This proves Theorem 2.1 (e).

\[\square\]

3. Extension of scalars and \((k, A)\)-coalgebra homomorphism

Various applications of Theorem 2.1 can be found in [ABS03] and [GriRei14, Chapter 7]. We are going to present another application, which we will obtain by “leveraging” Theorem 2.1 through an extension-of-scalars argument.\(^5\) Let us first introduce some more notations.

**Definition 3.1.** Let \( H \) be a \( k \)-algebra (possibly with additional structure, such as a grading or a Hopf algebra structure). Then, \( \underline{H} \) will mean the \( k \)-algebra \( H \) without any additional structure (for instance, the \( k \)-coalgebra structure on \( H \) is forgotten if \( H \) was a \( k \)-bialgebra, and the grading is forgotten if \( H \) was graded). Sometimes we will use the notation \( \underline{H} \) even when \( H \) has no additional structure beyond being a \( H \)-algebra; in this case, it means the same as \( H \), just stressing the fact that it is a plain \( k \)-algebra with nothing up its sleeves.

In other words, \( \underline{H} \) will denote the image of \( H \) under the forgetful functor from whatever category \( H \) belongs to to the category of \( k \)-algebras. We shall often use \( \underline{H} \) and \( H \) interchangeably, whenever \( H \) is merely a \( k \)-algebra or the other structures on \( H \) cannot cause confusion.

**Definition 3.2.** Let \( A \) be a commutative \( k \)-algebra.

(a) If \( H \) is a \( k \)-module, then \( \underline{A} \otimes H \) will be understood to mean the \( A \)-module \( A \otimes \underline{H} \), on which \( A \) acts by the rule

\[ a (b \otimes h) = ab \otimes h \quad \text{for all } a \in A, b \in A \text{ and } h \in H. \]

\(^5\)I have learned this extension-of-scalars trick from Petacci’s [Petra02, proof of Lemma 2.1.1]; similar ideas appear in various other algebraic arguments.
This $A$-module $A \otimes H$ is called the \textit{k-module H with scalars extended to A}.

We can define a functor $\text{Mod}_k \to \text{Mod}_A$ (where $\text{Mod}_B$ denotes the category of $B$-modules) which sends every object $H \in \text{Mod}_k$ to $A \otimes H$ and every morphism $f \in \text{Mod}_k (H_1, H_2)$ to $\text{id} \otimes f \in \text{Mod}_A (A \otimes H_1, A \otimes H_2)$; this functor is called \textit{extension of scalars} (from $k$ to $A$).

(b) If $H$ is a graded $k$-module, then the $A$-module $A \otimes H$ canonically becomes a graded $A$-module (namely, its $n$-th graded component is $A \otimes H_n$, where $H_n$ is the $n$-th graded component of $H$). Notice that even if $A$ is graded, we disregard its grading when defining the grading on $A \otimes H$; this is why we are calling it $A \otimes H$ and not $A \otimes H$.

As before, we can define a functor from the category of graded $k$-modules to the category of graded $A$-modules (which functor sends every object $H$ to $A \otimes H$), which is called \textit{extension of scalars}.

(c) If $H$ is a $k$-algebra, then the $A$-module $A \otimes H$ becomes an $A$-algebra according to the rule

$$(a \otimes h)(b \otimes g) = ab \otimes hg \quad \text{for all } a \in A, b \in A, h \in H \text{ and } g \in H.$$ (This is, of course, the same rule as used in the standard definition of the tensor product $A \otimes H$; but notice that we are regarding $A \otimes H$ as an $A$-algebra, not just as a $k$-algebra.) This $A$-algebra $A \otimes H$ is called the \textit{k-algebra H with scalars extended to A}.

As before, we can define a functor from the category of $k$-algebras to the category of $A$-algebras (which functor sends every object $H$ to $A \otimes H$), which is called \textit{extension of scalars}.

(d) If $H$ is a $k$-coalgebra, then the $A$-module $A \otimes H$ becomes an $A$-coalgebra. Namely, its comultiplication is defined to be

$$\text{id}_A \otimes \Delta_H : A \otimes H \to A \otimes (H \otimes H) \cong (A \otimes H) \otimes_A (A \otimes H),$$

and its counit is defined to be

$$\text{id}_A \otimes \varepsilon_H : A \otimes H \to A \otimes k \cong A$$

(recalling that $\Delta_H$ and $\varepsilon_H$ are the comultiplication and the counit of $H$, respectively). Note that both the comultiplication and the counit are $A$-linear, so this $A$-coalgebra $A \otimes H$ is well-defined. This $A$-coalgebra $A \otimes H$ is called the \textit{k-coalgebra H with scalars extended to A}.

As before, we can define a functor from the category of $k$-coalgebras to the category of $A$-coalgebras (which functor sends every object $H$ to $A \otimes H$), which is called \textit{extension of scalars}.

Notice that $A \otimes H$ is an $A$-coalgebra, not a $k$-coalgebra. If $A$ has a pre-existing $k$-coalgebra structure, then the $A$-coalgebra structure on $A \otimes H$ usually has nothing to do with the $k$-coalgebra structure on $A \otimes H$ obtained by tensoring the $k$-coalgebras $A$ and $H$. 

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If $H$ is a $k$-bialgebra, then the $A$-module $A \otimes H$ becomes an $A$-bialgebra. (Namely, the $A$-algebra structure and the $A$-coalgebra structure previously defined on $A \otimes H$, combined, form an $A$-bialgebra structure.) This $A$-bialgebra $A \otimes H$ is called the $k$-bialgebra $H$ with scalars extended to $A$.

As before, we can define a functor from the category of $k$-bialgebras to the category of $A$-bialgebras (which functor sends every object $H$ to $A \otimes H$), which is called extension of scalars.

Similarly, extension of scalars is defined for $k$-Hopf algebras, graded $k$-bialgebras, etc. Again, all structures on $A$ that go beyond the $k$-algebra structure are irrelevant and can be forgotten.

**Definition 3.3.** Let $A$ be a commutative $k$-algebra.

(a) Let $H$ be a $k$-module, and let $G$ be an $A$-module. For any $k$-linear map $f : H \to G$, we let $f^\sharp$ denote the $A$-linear map

$$A \otimes H \to G, \quad a \otimes h \mapsto af (h).$$

(It is easy to see that this latter map is indeed well-defined and $A$-linear.) For any $A$-linear map $g : A \otimes H \to G$, we let $g^\flat$ denote the $k$-linear map

$$H \to G, \quad h \mapsto g (1 \otimes h).$$

Sometimes we will use the notations $f^{\sharp (A,k)}$ and $g^{\flat (A,k)}$ instead of $f^\sharp$ and $g^\flat$ when the $A$ and the $k$ are not clear from the context.

It is easy to see that $(f^\sharp)^\flat = f$ for any $k$-linear map $f : H \to G$, and that $(g^\flat)^\sharp = g$ for any $A$-linear map $g : A \otimes H \to G$. Thus, the maps

$$\{ \text{k-linear maps } H \to G \} \to \{ \text{A-linear maps } A \otimes H \to G \}, \quad f \mapsto f^\sharp \quad (4)$$

and

$$\{ \text{A-linear maps } A \otimes H \to G \} \to \{ \text{k-linear maps } H \to G \}, \quad g \mapsto g^\flat \quad (5)$$

are mutually inverse.

This is a particular case of an adjunction between functors (namely, the Hom-tensor adjunction, with a slight simplification, also known as the induction-restriction adjunction); this is also the reason why we are using the $\sharp$ and $\flat$ notations. The maps (4) and (5) are natural in $H$ and $G$.

(b) Let $H$ be a $k$-coalgebra, and let $G$ be an $A$-coalgebra. A $k$-linear map $f : H \to G$ is said to be a $(k,A)$-coalgebra homomorphism if the $A$-linear map $f^\sharp : A \otimes H \to G$ is an $A$-coalgebra homomorphism.
**Proposition 3.4.** Let $A$ be a commutative $k$-algebra. Let $H$ be a $k$-algebra. Let $G$ be an $A$-algebra. Let $f : H \to G$ be a $k$-linear map. Then, $f$ is a $k$-algebra homomorphism if and only if $f^\sharp$ is an $A$-algebra homomorphism.

**Proof of Proposition 3.4.** Straightforward and left to the reader. (The main step is to observe that $f^\sharp$ is an $A$-algebra homomorphism if and only if every $a, b \in A$ and $h, g \in H$ satisfy $f^\sharp((a \otimes h)(b \otimes g)) = f^\sharp(a \otimes h)f^\sharp(b \otimes g)$. This is because the tensor product $A \otimes H$ is spanned by pure tensors.)

**Proposition 3.5.** Let $A$ be a commutative $k$-algebra. Let $H$ be a graded $k$-module. Let $G$ be an $A$-module. Let $f : H \to G$ be a $k$-linear map. Then, the $k$-linear map $f$ is graded if and only if the $k$-linear map $f^\sharp$ is graded.

**Proof of Proposition 3.5.** Again, straightforward and therefore omitted.

Let us first prove some easily-checked properties of $(k, A)$-coalgebra homomorphisms.

**Proposition 3.6.** Let $k$ be a commutative ring. Let $A$ be a commutative $k$-algebra. Let $H$ be a $k$-coalgebra. Let $G$ and $I$ be two $A$-coalgebras. Let $f : H \to G$ be a $(k, A)$-coalgebra homomorphism. Let $g : G \to I$ be an $A$-coalgebra homomorphism. Then, $g \circ f$ is a $(k, A)$-coalgebra homomorphism.

**Proof of Proposition 3.6.** Since $f$ is a $(k, A)$-coalgebra homomorphism, the map $f^\sharp : A \otimes H \to G$ is an $A$-coalgebra homomorphism. Now, straightforward elementwise computation (using the fact that the map $f$ is $k$-linear, and the map $g$ is $A$-linear) shows that

$$(g \circ f)^\sharp = g \circ f^\sharp. \quad (6)$$

Thus, $(g \circ f)^\sharp$ is an $A$-coalgebra homomorphism (since $g$ and $f^\sharp$ are $A$-coalgebra homomorphisms). In other words, $g \circ f$ is a $(k, A)$-coalgebra homomorphism. This proves Proposition 3.6.

**Proposition 3.7.** Let $k$ be a commutative ring. Let $A$ be a commutative $k$-algebra. Let $F$ and $H$ be two $k$-coalgebras. Let $G$ be an $A$-coalgebra. Let $f : H \to G$ be a $(k, A)$-coalgebra homomorphism. Let $e : F \to H$ be a $k$-coalgebra homomorphism. Then, $f \circ e$ is a $(k, A)$-coalgebra homomorphism.

**Proof of Proposition 3.7.** Since $f$ is a $(k, A)$-coalgebra homomorphism, the map $f^\sharp : A \otimes H \to G$ is an $A$-coalgebra homomorphism. The map $\text{id}_A \otimes e : A \otimes F \to A \otimes H$ is an $A$-coalgebra homomorphism (since $e : F \to H$ is a $k$-coalgebra homomorphism). Now, straightforward computation shows that $(f \circ e)^\sharp = f^\sharp \circ (\text{id}_A \otimes e)$. Hence, $(f \circ e)^\sharp$ is an $A$-coalgebra homomorphism (since $f^\sharp$ and $\text{id}_A \otimes e$ are $A$-coalgebra homomorphisms). In other words, $f \circ e$ is a $(k, A)$-coalgebra homomorphism. This proves Proposition 3.7.
Proposition 3.8. Let $k$ be a commutative ring. Let $A$ be a commutative $k$-algebra. Let $H$ be a $k$-coalgebra. Let $G$ be an $A$-coalgebra. Let $B$ be a commutative $A$-algebra. Let $p : A \to B$ be an $A$-algebra homomorphism. (Actually, $p$ is uniquely determined by the $A$-algebra structure on $B$.) Let $p_G : G \to B \otimes_A G$ be the canonical $A$-module homomorphism defined as the composition

$$G \xrightarrow{\cong} A \otimes_A G \xrightarrow{p \otimes_A \text{id}} B \otimes_A G.$$ 

Let $f : H \to G$ be a $(k, A)$-coalgebra homomorphism. Then, $p_G \circ f : H \to B \otimes_A G$ is a $(k, B)$-coalgebra homomorphism.

Proof of Proposition 3.8 Since $f$ is a $(k, A)$-coalgebra homomorphism, the map $f^* = f^*_{(A,k)} : A \otimes H \to G$ is an $A$-coalgebra homomorphism. Thus, the map $\text{id}_B \otimes_A f^* : B \otimes_A (A \otimes H) \to B \otimes_A G$ is a $B$-coalgebra homomorphism.

Let $\kappa : B \otimes H \to B \otimes_A (A \otimes H)$ be the canonical $B$-module isomorphism (sending each $b \otimes h \in B \otimes H$ to $b \otimes_A (1 \otimes h)$). It is well-known that $\kappa$ is a $B$-coalgebra isomorphism. Thus, $(\text{id}_B \otimes_A f^*) \circ \kappa$ is a $B$-coalgebra homomorphism (since both $\text{id}_B \otimes_A f^*$ and $\kappa$ are $B$-coalgebra homomorphisms).

The definition of $p_G$ yields that

$$p_G(u) = 1 \otimes_A u \quad (7)$$

for every $u \in G$.

The map $p_G \circ f : H \to B \otimes_A G$ gives rise to a map $(p_G \circ f)^* = (p_G \circ f)^*_{(B,k)} : B \otimes H \to B \otimes_A G$. But easy computations show that $(p_G \circ f)^* \circ (\text{id}_B \otimes_A f^*) = (\text{id}_B \otimes_A f^*) \circ \kappa$ (because the map $(p_G \circ f)^*_{(B,k)}$ sends a pure tensor $b \otimes h \in B \otimes H$ to $b \otimes (p_G \circ f)(h) = p_G(f(h)) = 1 \otimes_A f(h)$ (by (7)))

$$b \otimes (1 \otimes_A f(h)) = b \otimes_A f(h),$$

whereas the map $(\text{id}_B \otimes_A f^*) \circ \kappa$ sends a pure tensor $b \otimes h \in B \otimes H$ to

$$\left( (\text{id}_B \otimes_A f^*) \circ \kappa \right)(b \otimes h) = \left( \text{id}_B \otimes_A f^* \right)(\kappa(b \otimes h)) = \left( \text{id}_B \otimes_A f^* \right)(b \otimes_A (1 \otimes h))$$

as well. Thus, $(p_G \circ f)^*_{(B,k)}$ is a $B$-coalgebra homomorphism (since $(\text{id}_B \otimes_A f^*) \circ \kappa$ is a $B$-coalgebra homomorphism). In other words, $p_G \circ f$ is a $(k, B)$-coalgebra homomorphism. This proves Proposition 3.8 \hfill \Box

\footnote{In fact, it is part of the natural isomorphism $\text{Ind}_P^Q \circ \text{Ind}_k^A \cong \text{Ind}_P^B$, where $\text{Ind}_P^Q$ means extension of scalars from $P$ to $Q$ (as a functor from the category of $P$-coalgebras to the category of $Q$-coalgebras).}
**Proposition 3.9.** Let $k$ be a commutative ring. Let $A$ and $B$ be two commutative $k$-algebras. Let $H$ and $G$ be two $k$-coalgebras. Let $f : H \to A \otimes G$ be a $(k, A)$-coalgebra homomorphism. Let $p : A \to B$ be a $k$-algebra homomorphism. Then, $(p \otimes \text{id}) \circ f : H \to B \otimes G$ is a $(k, B)$-coalgebra homomorphism.

**Proof of Proposition 3.9.** Consider $B$ as an $A$-algebra by means of the $k$-algebra homomorphism $p : A \to B$. Thus, $p$ becomes an $A$-algebra homomorphism $A \to B$. Now, $A \otimes G$ is an $A$-coalgebra. Let $p_{A \otimes G} : A \otimes G \to B \otimes_A (A \otimes G)$ be the canonical $A$-module homomorphism defined as the composition

$$A \otimes G \xrightarrow{\sim} A \otimes_A (A \otimes G) \xrightarrow{p \otimes \text{id}} B \otimes_A (A \otimes G).$$

Proposition 3.8 (applied to $A \otimes G$ and $p_{A \otimes G}$ instead of $G$ and $p_G$) shows that $p_{A \otimes G} \circ f : H \to B \otimes_A (A \otimes G)$ is a $(k, B)$-coalgebra homomorphism.

But let $\phi$ be the canonical $B$-module isomorphism $B \otimes_A (A \otimes G) \to (B \otimes_A A) \otimes G \cong B \otimes G$. Then, $\phi$ is a $B$-coalgebra homomorphism, and has the property that $p \otimes \text{id} = \phi \circ p_{A \otimes G}$ as maps $A \otimes G \to B \otimes G$ (this can be checked by direct computation). Now,

$$(p \otimes \text{id}) \circ f = \phi \circ p_{A \otimes G} \circ f = \phi \circ (p_{A \otimes G} \circ f) = \phi \circ (\phi \circ p_{A \otimes G} \circ f).$$

must be a $(k, B)$-coalgebra homomorphism (by Proposition 3.6, since $p_{A \otimes G} \circ f$ is a $(k, B)$-coalgebra homomorphism and since $\phi$ is a $B$-coalgebra homomorphism). This proves Proposition 3.9. \hfill \square

**Proposition 3.10.** Let $k$ be a commutative ring. Let $A$ and $B$ be two commutative $k$-algebras. Let $H$ be an $A$-coalgebra. Let $G$ be an $A$-coalgebra. Let $f : H \to G$ be a $(k, A)$-coalgebra homomorphism. Then, $\text{id} \otimes f : B \otimes H \to B \otimes G$ is a $(B, B \otimes A)$-coalgebra homomorphism.

**Proof of Proposition 3.10.** Since $f$ is a $(k, A)$-coalgebra homomorphism, the map $f^* = f^*(A, k) : A \otimes H \to G$ is an $A$-coalgebra homomorphism. Thus, the map $\text{id}_B \otimes f^* : B \otimes (A \otimes H) \to B \otimes G$ is a $B$-coalgebra homomorphism.

But the $B$-linear map $\text{id} \otimes f : B \otimes H \to B \otimes G$ gives rise to a $B \otimes A$-linear map $(\text{id} \otimes f)^*_{(B \otimes A, B)} : (B \otimes A) \otimes_B (B \otimes H) \to B \otimes G$.

Now, let $\gamma$ be the canonical $B$-module isomorphism $(B \otimes A) \otimes_B (B \otimes H) \to B \otimes (A \otimes H)$ (sending each $(b \otimes a) \otimes_B (b' \otimes h) \in (B \otimes A) \otimes_B (B \otimes H)$ to $bb' \otimes (a \otimes h)$). Then, $\gamma$ is a $B$-coalgebra isomorphism (this is easy to check). Hence, $(\text{id}_B \otimes f^*) \circ \gamma$ is a $B$-coalgebra isomorphism (since $\text{id}_B \otimes f^*$ and $\gamma$ are $B$-coalgebra isomorphisms).
Now, it is straightforward to see that \((\text{id} \otimes f)^{\sharp(B \oplus A, B)} = (\text{id}_B \otimes f^\sharp) \circ \gamma\) \(^7\) Hence, the map \((\text{id} \otimes f)^{\sharp(B \oplus A, B)}\) is a \(B\)-coalgebra homomorphism (since \((\text{id}_B \otimes f^\sharp) \circ \gamma\) is a \(B\)-coalgebra homomorphism). In other words, \(\text{id} \otimes f : B \otimes H \to B \otimes G\) is a \((B, B \otimes A)\)-coalgebra homomorphism. This proves Proposition 3.10 \(\square\)

**Proposition 3.11.** Let \(k\) be a commutative ring. Let \(A\) be a commutative \(k\)-algebra. Let \(B\) be a commutative \(A\)-algebra. Let \(H\) be a \(k\)-coalgebra. Let \(G\) be an \(A\)-coalgebra. Let \(I\) be a \(B\)-coalgebra. Let \(f : H \to G\) be a \((A, B)\)-coalgebra homomorphism. Let \(g : G \to I\) be an \((A, B)\)-coalgebra homomorphism. Then, \(g \circ f : H \to I\) is a \((k, B)\)-coalgebra homomorphism.

**Proof of Proposition 3.11** Since \(f\) is a \((k, A)\)-coalgebra homomorphism, the map \(f^{\sharp(A,k)} : A \otimes H \to G\) is an \(A\)-coalgebra homomorphism. Thus, the map \(\text{id}_B \otimes A f^{\sharp(A,k)} : B \otimes_A (A \otimes H) \to B \otimes_A G\) is a \(B\)-coalgebra homomorphism.

Since \(g : G \to I\) is an \((A, B)\)-coalgebra homomorphism, the map \(g^{\sharp(B,A)} : B \otimes_A G \to I\) is a \(B\)-coalgebra homomorphism.

Let \(\delta : B \otimes H \to B \otimes_A (A \otimes H)\) be the canonical \(B\)-module isomorphism (sending each \(b \otimes h\) to \(b \otimes_A (1 \otimes h)\)). Then, \(\delta\) is a \(B\)-coalgebra isomorphism. Straightforward elementwise computation shows that \((g \circ f)^{\sharp(B,k)} = g^{\sharp(B,A)} \circ (\text{id}_B \otimes A f^{\sharp(A,k)}) \circ \delta\). Hence, \((g \circ f)^{\sharp(B,k)}\) is a \(B\)-coalgebra homomorphism (since \(g^{\sharp(B,A)}\) and \(f^{\sharp(A,k)}\) and \(\delta\) are \(B\)-coalgebra homomorphisms). In other words, \(g \circ f : H \to I\) is a \((k, B)\)-coalgebra homomorphism. This proves Proposition 3.11 \(\square\)

With these basics in place, we can now “escalate” Theorem 2.1 to the following setting:

**Corollary 3.12.** Let \(k\) be a commutative ring. Let \(H\) be a connected graded \(k\)-Hopf algebra. Let \(A\) be a commutative \(k\)-algebra. Let \(\zeta : H \to A\) be a \(k\)-coalgebra homomorphism.

(a) Then, there exists a unique graded \((k, A)\)-coalgebra homomorphism \(\Xi : H \to A \otimes \text{QSym}_k\) for which the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Xi} & A \otimes \text{QSym}_k \\
\downarrow{\zeta} & & \downarrow{\text{id}_A \otimes \varepsilon_P} \\
A & &
\end{array}
\]

is commutative (where we regard \(\text{id}_A \otimes \varepsilon_P : A \otimes \text{QSym}_k \to A \otimes k\) as a map from \(A \otimes \text{QSym}_k\) to \(A\), by canonically identifying \(A \otimes k\) with \(A\)).

\(^7\)Indeed, it suffices to check it on pure tensors, i.e., to prove that \((\text{id} \otimes f)^{\sharp(B \oplus A, B)}((b \otimes a) \otimes_B (b' \otimes h)) = (\text{id}_B \otimes f^\sharp)((b \otimes a) \otimes_B (b' \otimes h))\) for each \(b \in B\), \(a \in A\), \(b' \in B\) and \(h \in H\). But this is easy (both sides turn out to be \(bb' \otimes_B af(h)\)).
This unique \((k, A)\)-coalgebra homomorphism \(\Xi : H \to A \otimes \mathrm{QSym}_k\) is a \(k\)-algebra homomorphism.

For every composition \(\alpha = (a_1, a_2, \ldots, a_k)\), define a \(k\)-linear map \(\xi_\alpha : H \to A\) (not to \(k\) !) as the composition

\[
\begin{align*}
H &\xrightarrow{\Delta^{(k-1)}} H \otimes H \\
&\xrightarrow{\pi_\alpha} H \otimes k \\
&\xrightarrow{\xi \otimes k} (k - 1) \\
&\xrightarrow{m^{(k-1)}} A.
\end{align*}
\]

(Recall that \(\Delta^{(k-1)} : H \to H \otimes k\) and \(m^{(k-1)} : A \otimes k \to A\) are the “iterated comultiplication and multiplication maps”; see [GriRei14, §1.4] for their definitions. The map \(\pi_\alpha : H \otimes k \to H \otimes k\) is the one defined in Definition 1.1.)

Then, the unique \((k, A)\)-coalgebra homomorphism \(\Xi\) of Corollary 3.12 (a) is given by

\[
\Xi(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \quad \text{for every } h \in H
\]

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

If the \(k\)-coalgebra \(H\) is cocommutative, then \(\Xi(H)\) is a subset of the subring \(A \otimes \Lambda_k\) of \(A \otimes \mathrm{QSym}_k\), where \(\Lambda_k\) is the \(k\)-algebra of symmetric functions over \(k\).

**Proof of Corollary 3.12** We have \(A \otimes \mathrm{QSym}_k \cong \mathrm{QSym}_A\) as \(A\)-bialgebras canonically (since \(\mathrm{QSym}_k\) is defined functorially in \(k\), with a basis that is independent of \(k\)).

Recall that we have defined a \(k\)-algebra homomorphism \(\varepsilon_P : \mathrm{QSym}_k \to k\). We shall now denote this \(\varepsilon_P\) by \(\varepsilon_P, k\) in order to stress that it depends on \(k\). Similarly, an \(m\)-algebra homomorphism \(\varepsilon_{P, m} : \mathrm{QSym}_m \to m\) is defined for any commutative ring \(m\). In particular, an \(A\)-algebra homomorphism \(\varepsilon_{P, A} : \mathrm{QSym}_A \to A\) is defined. The definitions of \(\varepsilon_{P, m}\) for all \(m\) are essentially identical; thus, the map \(\varepsilon_{P, A} : \mathrm{QSym}_A \to A\) can be identified with the map \(\mathrm{id}_{A} \otimes \varepsilon_{P, k} : A \otimes \mathrm{QSym}_k \to A \otimes k\) (if we identify \(A \otimes \mathrm{QSym}_k\) with \(\mathrm{QSym}_A\) and identify \(A \otimes k\) with \(A\)). We shall use this identification below.

The \(k\)-linear map \(\zeta : H \to A\) induces an \(A\)-linear map \(\zeta^\#: A \otimes H \to A\) (defined by \(\zeta^\#(a \otimes h) = a \zeta(h)\) for all \(a \in A\) and \(h \in H\)). Proposition 3.4 (applied to \(G = A\) and \(f = \zeta\)) shows that \(\zeta^\#\) is an \(A\)-algebra homomorphism (since \(\zeta\) is a \(k\)-algebra homomorphism).

Theorem 2.1 (a) (applied to \(A\), \(A \otimes H\) and \(\zeta^\#\) instead of \(k\), \(H\) and \(\zeta\)) shows that there exists a unique graded \(A\)-coalgebra homomorphism \(\Psi : A \otimes H \to \mathrm{QSym}_A\) for which the diagram

\[
\begin{array}{ccc}
A \otimes H & \xrightarrow{\Psi} & \mathrm{QSym}_A \\
\downarrow{\zeta^\#} & & \downarrow{\varepsilon_{P, A}} \\
A & \xrightarrow{\varepsilon_P, A} & \mathrm{QSym}_A
\end{array}
\]

(9)
is commutative. Since we are identifying the map $\varepsilon_{P,A} : QSym_A \to A$ with the map $\text{id}_A \otimes \varepsilon_{P,k} : A \otimes QSym_k \to A \otimes k = A$, we can rewrite this as follows: There exists a unique graded $A$-coalgebra homomorphism $\Psi : A \otimes H \to A \otimes QSym_k$ for which the diagram

\[
\begin{array}{ccc}
A \otimes H & \xrightarrow{\Psi} & A \otimes QSym_k \\
\downarrow{\xi^2} & & \downarrow{\text{id}_A \otimes \varepsilon_{P,k}} \\
A
\end{array}
\]

is commutative. In other words, there exists a unique graded $A$-coalgebra homomorphism $\Psi : A \otimes H \to A \otimes QSym_k$ such that $(\text{id}_A \otimes \varepsilon_{P,k}) \circ \Psi = \xi^2$. Let us refer to this observation as the \textit{intermediate universal property}.

The $(k,A)$-coalgebra homomorphisms $H \to A \otimes QSym_k$ are in a 1-to-1 correspondence with the $A$-coalgebra homomorphisms $A \otimes H \to A \otimes QSym_k$, which is the same as the $A$-coalgebra homomorphisms $A \otimes H \to QSym_A$ (since $A \otimes QSym_k \cong QSym_A$). The correspondence is given by sending a $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ to the $A$-coalgebra homomorphism $\Xi^2 : A \otimes H \to A \otimes QSym_k$. Moreover, this correspondence has the property that $\Xi$ is graded if and only if $\Xi^2$ is (according to Proposition 3.5). Thus, this correspondence restricts to a correspondence between the graded $(k,A)$-coalgebra homomorphisms $H \to A \otimes QSym_k$ and the graded $A$-coalgebra homomorphisms $A \otimes H \to A \otimes QSym_k$. Using this correspondence, we can rewrite the intermediate universal property as follows: There exists a unique graded $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ such that $(\text{id}_A \otimes \varepsilon_{P,k}) \circ \Xi^2 = \xi^2$. In other words, there exists a unique graded $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ such that $((\text{id}_A \otimes \varepsilon_{P,k}) \circ \Xi)^2 = (\text{id}_A \otimes \varepsilon_{P,k}) \circ \Xi^2$. In other words, there exists a unique graded $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ such that $(\text{id}_A \otimes \varepsilon_{P,k}) \circ \Xi = \xi$ (since the map (b) is a bijection). In other words, there exists a unique graded $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ for which the diagram (b) is commutative. This proves Corollary 3.12 (a).

By tracing back the above argument, we see that it yields an explicit construction of the unique graded $(k,A)$-coalgebra homomorphism $\Xi : H \to A \otimes QSym_k$ for which the diagram (b) is commutative: Namely, it is defined by $\Xi^2 = \Psi$, where $\Psi$ is the unique graded $A$-coalgebra homomorphism $\Psi : A \otimes H \to QSym_A$ for which the diagram (f) is commutative. Consider these $\Xi$ and $\Psi$.

Theorem 2.1 (b) (applied to $A, A \otimes H$ and $\xi^2$ instead of $k$, $H$ and $\xi$) shows that $\Psi : A \otimes H \to QSym_A$ is an $A$-Hopf algebra homomorphism, thus an $A$-algebra homomorphism. In other words, $\Xi^2 : A \otimes H \to A \otimes QSym_k$ is an $A$-algebra homomorphism (since $\Xi^2 : A \otimes H \to A \otimes QSym_k$ is the same as $\Psi : A \otimes H \to QSym_A$, up to our identifications). Hence, $\Xi : H \to A \otimes QSym_k$ is a $k$-algebra homomorphism as well (by Proposition 3.4 applied to $A, A \otimes QSym_k$ and $\Xi$ instead of $A, G$ and $f$). This proves Corollary 3.12 (b).
Theorem 2.1 (d) (applied to $A$, $A \otimes H$ and $\xi^\sharp$ instead of $k$, $H$ and $\zeta$) shows that $\Psi$ is given by

$$\Psi(h) = \sum_{\alpha \in \text{Comp}} (\xi^\sharp)_\alpha(h) \cdot M_\alpha$$

for every $h \in A \otimes H$, \hfill (10)

where the map $(\xi^\sharp)_\alpha : A \otimes H \rightarrow A$ is defined in the same way as the map $\zeta_\alpha : H \rightarrow k$ was defined in Theorem 2.1 (d) (but for $A$, $A \otimes H$ and $\xi^\sharp$ instead of $k$, $H$ and $\zeta$). Notice that (10) is an equality inside $\text{QSym}_A$. Recalling that we are identifying $\text{QSym}_A$ with $A \otimes \text{QSym}_k$, we can rewrite it as an equality in $A \otimes \text{QSym}_k$; it then takes the form

$$\Psi(h) = \sum_{\alpha \in \text{Comp}} (\xi^\sharp)_\alpha(h) \otimes M_\alpha$$

for every $h \in A \otimes H$. \hfill (11)

Let $\iota_H$ be the $k$-module homomorphism

$$H \rightarrow A \otimes H, \quad h \mapsto 1 \otimes h.$$

Also, for every $k \in \mathbb{N}$, we let $\iota_k$ be the $k$-module homomorphism

$$H^\otimes_k \rightarrow (A \otimes H)^\otimes k,$$

$g \mapsto 1 \otimes g \in A \otimes H^\otimes k \cong (A \otimes H)^\otimes k$

(where $U^\otimes k$ denotes the $k$-th tensor power of an $A$-module $U$); this homomorphism sends every $h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H^\otimes k$ to $(1 \otimes h_1) \otimes_A (1 \otimes h_2) \otimes_A \cdots \otimes_A (1 \otimes h_k)$.

On the other hand, fix some $\alpha \in \text{Comp}$. Write the composition $\alpha$ in the form $\alpha = (a_1, a_2, \ldots, a_k)$. The diagram

$$
\begin{array}{cccccccc}
H & \xrightarrow{\Delta^{(k-1)}} & H^\otimes_k & \xrightarrow{\pi_\alpha} & H^\otimes_k & \xrightarrow{\xi^\otimes k} & A^\otimes k & \xrightarrow{\pi} & A \\
\downarrow{\iota_H} & & \downarrow{\iota_k} & & \downarrow{\iota_k} & & \downarrow{id} \\
A \otimes H & \xrightarrow{\Delta^{(k-1)}} & (A \otimes H)^\otimes k & \xrightarrow{\pi_\alpha} & (A \otimes H)^\otimes k & \xrightarrow{(\xi^\sharp)^\otimes k} & A \otimes k & \cong & A \\
\end{array}
$$

is commutative.\(^8\) Therefore, $(\xi^\sharp)_\alpha \circ \iota_H = \text{id} \circ (\xi^\sharp)_\alpha = \xi_\alpha$.

\(^8\)Proof. In fact:

- Its upper pentagon is commutative (by the definition of $\xi_\alpha$).
Now, forget that we fixed $\alpha$. We thus have shown that
\[
\left(\xi^\sharp\right)_\alpha \circ \iota_H = \xi_\alpha \quad \text{for every } \alpha \in \text{Comp}.
\]  

- Its lower pentagon is commutative (by the definition of $\left(\xi^\sharp\right)_\alpha$).
- Its left square is commutative (since the operation $\Delta^{(k-1)}$ on a $k$-coalgebra is functorial with respect to the base ring, i.e., commutes with extension of scalars).
- Its middle square is commutative (since the operation $\pi_\alpha$ on a graded $k$-module is functorial with respect to the base ring, i.e., commutes with extension of scalars).
- Its right rectangle is commutative. (Indeed, every $h_1, h_2, \ldots, h_k \in H$ satisfy

\[
\left(\text{id} \circ m^{(k-1)} \circ \xi^{\otimes k}\right) (h_1 \otimes h_2 \otimes \cdots \otimes h_k) = m^{(k-1)} \left(\xi (h_1) \otimes \xi (h_2) \otimes \cdots \otimes \xi (h_k)\right)
\]

and thus

\[
\left(\xi^\sharp\right)^{\otimes k} \circ \iota_k \left(\xi (h_1) \otimes \xi (h_2) \otimes \cdots \otimes \xi (h_k)\right) = \xi (h_1) \otimes \xi (h_2) \otimes \cdots \otimes \xi (h_k).
\]

Hence, $\left(\xi^\sharp\right)^{\otimes k} \circ \iota_k = \text{id} \circ m^{(k-1)} \circ \xi^{\otimes k}$. In other words, the right rectangle is commutative.)
Now, every $h \in H$ satisfies
\[
\Xi(h) = \sum_{\alpha \in \text{Comp}} \left( \xi^\sharp \right)_\alpha (1 \otimes h) \otimes M_\alpha = \sum_{\alpha \in \text{Comp}} \left( \xi^\sharp \right)_\alpha (t_H(h)) \otimes M_\alpha.
\]
(by (11), applied to $1 \otimes h$ instead of $h$)

This proves Corollary 3.12 (c).

(d) Assume that the $k$-coalgebra $H$ is cocommutative. Then, the $A$-coalgebra $A \otimes H$ is cocommutative as well.

Let us first see why $A \otimes \Lambda_k$ is a subring of $A \otimes \text{QSym}_k$. Indeed, recall that we are using the standard $k$-Hopf algebra isomorphism $A \otimes \text{QSym}_k \rightarrow \text{QSym}_A$ to identify $\text{QSym}_A$ with $A \otimes \text{QSym}_k$. Similarly, let us use the standard $k$-Hopf algebra isomorphism $A \otimes \Lambda_k \rightarrow \Lambda_A$ to identify $\Lambda_A$ with $A \otimes \Lambda_k$. Now, $A \otimes \Lambda_k = \Lambda_A \subseteq \text{QSym}_A = A \otimes \text{QSym}_k$.

Theorem 2.1 (e) (applied to $A$, $A \otimes H$ and $\xi^\sharp$ instead of $k$, $H$ and $\xi$) shows that $\Psi(A \otimes H) \subseteq \Lambda_A = A \otimes \Lambda_k$. Since $\Psi = \Xi^\sharp$, this rewrites as $\Xi^\sharp(A \otimes H) \subseteq A \otimes \Lambda_k$. But $\Xi(H) \subseteq \Xi^\sharp(A \otimes H)$ (since every $h \in H$ satisfies $\Xi(h) = \Xi^\sharp(1 \otimes h) \in \Xi^\sharp(A \otimes H)$). Hence, $\Xi(H) \subseteq \Xi^\sharp(A \otimes H) \subseteq A \otimes \Lambda_k$. This proves Corollary 3.12 (d).

4. The second comultiplication on $\text{QSym}_k$

**Convention 4.1.** In the following, we do not identify compositions with infinite sequences, as several authors do. As a consequence, the composition $(1,3)$ does not equal the vector $(1,3,0)$ or the infinite sequence $(1,3,0,0,0,\ldots)$.

We now recall the definition of the second comultiplication (a.k.a. internal comultiplication) of $\text{QSym}_k$. Several definitions of this operation appear in the literature; we shall use the one in [Haz08 §11.39]:

**Definition 4.2.** (a) Given a $u \times v$-matrix $A = (a_{ij})_{1 \leq i \leq u, 1 \leq j \leq v}$ (where $u, v \in \mathbb{N}$) with nonnegative entries, we define three tuples of nonnegative integers:

- The $v$-tuple column $A \in \mathbb{N}^v$ is the $v$-tuple whose $j$-th entry is $\sum_{i=1}^u a_{ij}$ (that
is, the sum of all entries in the $j$-th column of $A$) for each $j$. (In other words, column $A$ is the sum of all rows of $A$, regarded as vectors.)

- The $u$-tuple row $A \in \mathbb{N}^u$ is the $u$-tuple whose $i$-th entry is $\sum_{j=1}^v a_{i,j}$ (that is, the sum of all entries in the $i$-th row of $A$) for each $i$. (In other words, row $A$ is the sum of all columns of $A$, regarded as vectors.)

- The $uv$-tuple read $A \in \mathbb{N}^{uv}$ is the $uv$-tuple whose $(v(i-1)+j)$-th entry is $a_{i,j}$ for all $i \in \{1,2,\ldots,u\}$ and $j \in \{1,2,\ldots,v\}$. In other words,
  \[
  \text{read } A = (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots, a_{u,1}, a_{u,2}, \ldots, a_{u,v}).
  \]

We say that the matrix $A$ is column-reduced if column $A$ is a composition (i.e., contains no zero entries). Equivalently, $A$ is column-reduced if and only if no column of $A$ is the 0 vector.

We say that the matrix $A$ is row-reduced if row $A$ is a composition (i.e., contains no zero entries). Equivalently, $A$ is row-reduced if and only if no row of $A$ is the 0 vector.

We say that the matrix $A$ is reduced if $A$ is both column-reduced and row-reduced.

(b) If $w \in \mathbb{N}^k$ is a $k$-tuple of nonnegative integers (for some $k \in \mathbb{N}$), then $w^{\text{red}}$ shall mean the composition obtained from $w$ by removing each entry that equals 0. For instance, $(3,1,0,1,0,0,2)^{\text{red}} = (3,1,1,2)$.

(c) Let $\mathbb{N}_{\text{red}}^{\bullet \bullet}$ denote the set of all reduced matrices in $\mathbb{N}^{u \times v}$, where $u$ and $v$ both range over $\mathbb{N}$. In other words, we set
  \[
  \mathbb{N}_{\text{red}}^{\bullet \bullet} = \bigcup_{(u,v) \in \mathbb{N}^2} \{A \in \mathbb{N}^{u \times v} \mid A \text{ is reduced}\}.
  \]

(d) Let $\Delta_P : \text{QSym}_k \to \text{QSym}_k \otimes \text{QSym}_k$ be the $k$-linear map defined by setting
  \[
  \Delta_P(M_\alpha) = \sum_{\substack{A \in \mathbb{N}_{\text{red}}^{\bullet \bullet} \colon (\text{read } A)^{\text{red}} = \alpha}} M_{\text{row } A} \otimes M_{\text{column } A}
  \]
  for each $\alpha \in \text{Comp}$.

This map $\Delta_P$ is called the second comultiplication (or internal comultiplication) of $\text{QSym}_k$.

(e) Let $\tau$ denote the twist map $\tau_{\text{QSym}_k,\text{QSym}_k} : \text{QSym}_k \otimes \text{QSym}_k \to \text{QSym}_k \otimes \text{QSym}_k$. Let $\Delta'_P = \tau \circ \Delta_P : \text{QSym}_k \to \text{QSym}_k \otimes \text{QSym}_k$.

\[
\text{Figure 100}
\]

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Proposition 4.3. The \( k \)-algebra \( \text{QSym}_k \), equipped with comultiplication \( \Delta_p \) and counit \( \varepsilon_p \), is a \( k \)-bialgebra (albeit not a connected graded one, and not a Hopf algebra).

Proposition 4.3 is a well-known fact, but we shall actually derive it further below using our results.

5. The (generalized) Bernstein homomorphism

Let us now define the Bernstein homomorphism of a commutative connected graded \( k \)-Hopf algebra, generalizing [Haz08 \( \S \)18.24]:

\[
\text{Definition 5.1.} \text{ Let } k \text{ be a commutative ring. Let } H \text{ be a commutative connected graded } k \text{-Hopf algebra. For every composition } \alpha = (a_1, a_2, \ldots, a_k), \text{ define a } k \text{-linear map } \xi_{\alpha} : H \to H \text{ (not to } k!) \text{ as the composition}
\]

\[
H \xrightarrow{\Delta^{(k-1)}} H \otimes^k \xrightarrow{\pi_{\alpha}} H \otimes^k m^{(k-1)} \xrightarrow{\pi_{\alpha}} H.
\]

(Recall that \( \Delta^{(k-1)} : H \to H \otimes^k \) and \( m^{(k-1)} : H \otimes^k \to H \) are the "iterated comultiplication and multiplication maps"; see [GriRei14 \( \S \)1.4] for their definitions. The map \( \pi_{\alpha} : H \otimes^k \to H \otimes^k \) is the one defined in Definition 1.1) Define a map \( \beta_H : H \to H \otimes \text{QSym}_k \) by

\[
\beta_H (h) = \sum_{\alpha \in \text{Comp}} \xi_{\alpha} (h) \otimes M_{\alpha} \quad \text{for every } h \in H. \tag{13}
\]

It is easy to see that this map \( \beta_H \) is well-defined (i.e., the sum on the right hand side of (13) has only finitely many nonzero addends\(^9\)) and \( k \)-linear.

The graded \( k \)-Hopf algebra \( \text{QSym}_k \) is commutative and connected; thus, Definition 5.1 (applied to \( H = \text{QSym}_k \)) constructs a \( k \)-linear map \( \beta_{\text{QSym}_k} : \text{QSym}_k \to \text{QSym}_k \otimes \text{QSym}_k \). We shall now prove that this map is identical with the \( \Delta_p' \) from Definition 4.2 (e):

\[
\text{Proposition 5.2.} \text{ We have } \beta_{\text{QSym}_k} = \Delta_p'.
\]

Before we prove this, let us recall a basic formula for multiplication of monomial quasisymmetric functions:

\( ^{9} \text{Proof.} \text{ Let } h \in H. \text{ Then, there exists some } N \in \mathbb{N} \text{ such that } h \in \bigoplus_{i \in \mathbb{N}} H_i. \text{ Consider this } N. \text{ Now, it is easy to see that every composition } \alpha = (a_1, a_2, \ldots, a_k) \text{ of size } \geq N \text{ satisfies } \left( \pi_{\alpha} \circ \Delta^{(k-1)} \right) (h) = 0 \text{ (because } \Delta^{(k-1)} (h) \text{ is concentrated in the first } N \text{ homogeneous components of the graded } k \text{-module } H \otimes^k, \text{ and all of these components are annihilated by } \pi_{\alpha} \text{ and therefore } \xi_{\alpha} (h) = 0. \text{ Thus, the sum on the right hand side of (13) has only finitely many nonzero addends (namely, all its addends with } |\alpha| \geq N \text{ are } 0. \)
Proposition 5.3. Let \( k \in \mathbb{N} \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be \( k \) compositions. Let \( \mathbb{N}^{k,\bullet}_{\text{Cred}} \) denote the set of all column-reduced matrices in \( \mathbb{N}^{k \times v} \) with \( v \) ranging over \( \mathbb{N} \). In other words, let

\[
\mathbb{N}^{k,\bullet}_{\text{Cred}} = \bigcup_{v \in \mathbb{N}} \left\{ A \in \mathbb{N}^{k \times v} \mid A \text{ is column-reduced} \right\}.
\]

Then,

\[
M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} = \sum_{A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}} M_{\text{column } A}.
\]

Here, \( A_{i,\bullet} \) denotes the \( i \)-th row of \( A \) (regarded as a list of nonnegative integers).

Notice that the \( k = 2 \) case of Proposition 5.3 is a restatement of the standard formula for the multiplication of monomial quasisymmetric functions (e.g., [GriRei14, Proposition 5.3] or [Haz08, §11.26]). The general case is still classical, but since an explicit proof is hard to locate in the literature, let me sketch it here.

Proof of Proposition 5.3. We begin by introducing notations:

- Let \( \mathbb{N}^{k,\infty} \) denote the set of all matrices with \( k \) rows (labelled 1, 2, \ldots, \( k \)) and countably many columns (labelled 1, 2, 3, \ldots) whose entries all belong to \( \mathbb{N} \).
- Let \( \mathbb{N}^{k,\infty}_{\text{fin}} \) denote the set of all matrices in \( \mathbb{N}^{k,\infty} \) which have only finitely many nonzero entries.
- Let \( \mathbb{N}^{\infty} \) denote the set of all infinite sequences \( (a_1, a_2, a_3, \ldots) \) of elements of \( \mathbb{N} \).
- Let \( \mathbb{N}^{\infty}_{\text{fin}} \) denote the set of all sequences in \( \mathbb{N}^{\infty} \) which have only finitely many nonzero entries.
- For every \( B \in \mathbb{N}^{k,\infty}_{\text{fin}} \) and \( i \in \{1, 2, \ldots, k\} \), we let \( B_{i,\bullet} \in \mathbb{N}^{\infty}_{\text{fin}} \) be the \( i \)-th row of \( B \).
- For every \( B = (b_{i,j})_{1 \leq i \leq k, 1 \leq j \leq} \in \mathbb{N}^{k,\infty}_{\text{fin}} \), we let column \( B \in \mathbb{N}^{\infty}_{\text{fin}} \) be the sequence whose \( j \)-th entry is \( \sum_{i=1}^{k} a_{i,j} \) (that is, the sum of all entries in the \( j \)-th column of \( B \)) for each \( j \). (In other words, column \( B \) is the sum of all rows of \( B \), regarded as vectors.)

\(^{10}\) Actually, [GriRei14, Proposition 5.3] is slightly more general (the \( k = 2 \) case of Proposition 5.3 is obtained from [GriRei14, Proposition 5.3] by setting \( I = \{1, 2, 3, \ldots\} \)). That said, our proof can easily be extended to work in this greater generality.
• We extend Definition 4.2 (b) to the case when \( w \in \mathbb{N}_\infty^\infty \): If \( w \in \mathbb{N}_\infty^\infty \), then \( w^{\text{red}} \) shall mean the composition obtained from \( w \) by removing each entry that equals 0.\(^{11}\)

• For every \( \beta = (b_1, b_2, b_3, \ldots) \in \mathbb{N}_\infty^\infty \), we define a monomial \( x^\beta \) in the indeterminates \( x_1, x_2, x_3, \ldots \) by

\[
x^\beta = x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots .
\]

Then, it is easy to see that

\[
M_\alpha = \sum_{\substack{\beta \in \mathbb{N}_\infty^\infty; \\
\beta^{\text{red}} = \alpha}} x^\beta \quad \text{for every composition } \alpha . \quad (14)
\]

Now,

\[
M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k}
\]

\[
= \prod_{g=1}^{k} M_{\alpha_g}
= \prod_{g=1}^{k} \sum_{\substack{\beta \in \mathbb{N}_\infty^\infty; \\
\beta^{\text{red}} = \alpha_g}} x^\beta
= \sum_{\substack{\beta_1, \beta_2, \ldots, \beta_k \in (\mathbb{N}_\infty^\infty)^k; \\
(\beta_1)_{\text{red}} = \alpha_1 \text{ for each } g}} x^{\beta_1} x^{\beta_2} \cdots x^{\beta_k}
= \sum_{B \in \mathbb{N}_\infty^{k,\infty}} \left( \begin{array}{c}
B_1, B_2, \ldots, B_k
\end{array} \right)
= \sum_{B \in \mathbb{N}_\infty^{k,\infty}} \left( \begin{array}{c}
(x^{\text{column } B})_1, x^{\text{column } B}_2, \ldots, x^{\text{column } B}_k
\end{array} \right)
= \sum_{\substack{\beta \in \mathbb{N}_\infty^\infty; \\
(\beta_1, \beta_2, \ldots, \beta_k) \text{ in the sum, since the map} \\
\mathbb{N}_\infty^\infty \to (\mathbb{N}_\infty^\infty)^k, \ B \mapsto (B_1, B_2, \ldots, B_k) \text{ is a bijection} \}
\]

\[
= \sum_{B \in \mathbb{N}_\infty^{k,\infty}} x^{\text{column } B} \quad (15)
\]

Now, let us introduce one more notation: For every matrix \( B \in \mathbb{N}_\infty^{k,\infty} \), let \( B^{\text{Cred}} \) be the matrix obtained from \( B \) by removing all zero columns (i.e., all columns

\[^{11}\]Here is a more rigorous definition of \( w^{\text{red}} \): Let \( w = (w_1, w_2, w_3, \ldots) \). Let \( J \) be the set of all positive integers \( j \) such that \( w_j \neq 0 \). Let \( (j_1 < j_2 < \cdots < j_h) \) be the list of all elements of \( J \), in increasing order. Then, \( w^{\text{red}} \) is defined to be the composition \( (w_{j_1}, w_{j_2}, \ldots, w_{j_h}) \).

This rigorous definition of \( w^{\text{red}} \) has the additional advantage of making sense in greater generality than “remove each entry that equals 0”; namely, it still works when \( w \in \mathbb{N}_\infty^I \) for some totally ordered set \( I \).
containing only zeroes. It is easy to see that $B^{\text{Cred}} \in \mathbb{N}^{k,\ast}_{\text{Cred}}$ for every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$. Moreover, every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$ satisfies the following fact: If $A = B^{\text{Cred}}$, then

$$(B_{g,\bullet})^{\text{red}} = (A_{g,\bullet})^{\text{red}} \text{ for each } g$$

(16)

(Indeed, $A_{g,\bullet}$ is obtained from $B_{g,\bullet}$ by removing some zero entries). Now, (15) becomes

$$M_{A_1}M_{A_2} \cdots M_{A_k} = \sum_{B \in \mathbb{N}^{k,\infty}_{\text{fin}}} \sum_{B \in \mathbb{N}^{k,\infty}_{\text{fin}}} x^{\text{column } B}$$

$$= \sum_{A \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

$$= \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

$$= \sum_{A \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

$$= \sum_{A_1 \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

$$= \sum_{A_1 \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

$$= \sum_{A_1 \in \mathbb{N}^{k,\ast}_{\text{Cred}}} \sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B}$$

(17)

But for every matrix $A \in \mathbb{N}^{k,\ast}_{\text{Cred}}$, we have

$$\sum_{B \in \mathbb{N}^{k,\ast}_{\text{Cred}}} x^{\text{column } B} = M_{\text{column } A}.$$ 

(18)

---

12 Again, we can define $B^{\text{Cred}}$ more rigorously as follows: Let $J$ be the set of all positive integers $j$ such that the $j$-th column of $B$ is nonzero. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements of $J$, in increasing order. Then, $B^{\text{Cred}}$ is defined to be the $k \times h$-matrix whose columns (from left to right) are the $j_1$-th column of $B$, the $j_2$-nd column of $B$, \ldots, the $j_h$-th column of $B$. 

---

22
Proof of (18): Let $A \in \mathbb{N}^k_{\text{cred}}$. We need to prove (18).

For every $B \in \mathbb{N}^k_{\text{fin}}$, we have $(\text{column } B)_{\text{red}} = \text{column } (B_{\text{cred}})$ (because first taking the sum of each column of $B$ and then removing the zeroes among these sums results in the same list as first removing the zero columns of $B$ and then taking the sum of each remaining column). Thus, for every $B \in \mathbb{N}^k_{\text{fin}}$ satisfying $B_{\text{cred}} = A$, we have column $B \in \mathbb{N}_{\text{fin}}$ and $(\text{column } B)_{\text{red}} = \text{column } (B_{\text{cred}})_{\text{red}} = A$

column $A$. Hence, the map

$$
\{ B \in \mathbb{N}^k_{\text{fin}} \mid B_{\text{cred}} = A \} \rightarrow \{ \beta \in \mathbb{N}^\infty_{\text{fin}} \mid \beta_{\text{red}} = \text{column } A \},
$$

is well-defined.

On the other hand, if $\beta \in \mathbb{N}^\infty_{\text{fin}}$ satisfies $\beta_{\text{red}} = \text{column } A$, then there exists a unique $B \in \mathbb{N}^k_{\text{fin}}$ satisfying $B_{\text{cred}} = A$ and column $B = \beta$ \[13\] In other words, the map (19) is bijective. Thus, we can substitute $\beta$ for column $B$ in the sum

$$
\sum_{B \in \mathbb{N}^k_{\text{fin}}, \quad B_{\text{cred}} = A} x^{\text{column } B} = \sum_{\beta \in \mathbb{N}^\infty_{\text{fin}}, \quad \beta_{\text{red}} = \text{column } A} x^\beta = M_{\text{column } A}
$$

(by (14), applied to $\alpha = \text{column } A$). This proves (18).

\[13\] Namely, this $B$ can be computed as follows: Write the sequence $\beta$ in the form $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$. Let $(i_1 < i_2 < \cdots < i_h)$ be the list of all $c$ satisfying $\beta_c \neq 0$, written in increasing order. Then, $\beta$ shall be the matrix whose $i_1$-st, $i_2$-nd, $\ldots$, $i_h$-th columns are the columns of $A$ (from left to right), whereas all its other columns are 0.

Let us briefly sketch a proof of the fact that this $B$ is indeed an element of $\mathbb{N}^k_{\text{fin}}$ satisfying $B_{\text{cred}} = A$ and column $B = \beta$:

Indeed, it is clear that $B \in \mathbb{N}^k_{\text{fin}}$.

We shall now show that

$$(\text{the } j\text{-th entry of column } B) = \beta_j \quad (20)$$

for every $j \in \{1, 2, 3, \ldots\}$.

Proof of (20): Let $j \in \{1, 2, 3, \ldots\}$. We must prove (20). We are in one of the following two cases:

Case 1: We have $j \in \{i_1, i_2, \ldots, i_h\}$.

Case 2: We have $j \notin \{i_1, i_2, \ldots, i_h\}$.

Let us first consider Case 1. In this case, we have $j \in \{i_1, i_2, \ldots, i_h\}$. Hence, there exists a
\( g \in \{1, 2, \ldots, h\} \) such that \( j = i_g \). Consider this \( g \). Now,

\[
\begin{align*}
\text{(the } j\text{-th entry of column } B) &= \\
&= \left( \text{the sum of the entries of the } j\text{-th column of } B \right)_{=i_g} \\
&= \left( \text{the sum of the entries of the } i_g\text{-th column of } B \right)_{=(\text{the } g\text{-th column of } A)} \\
&= (\text{the sum of the entries of the } g\text{-th column of } A) \\
&= \left( \text{the } g\text{-th entry of column } A \right)_{=\beta_{\text{red}}=(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_h})} \\
&= (\text{the } g\text{-th entry of } (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_h})) = \beta_{i_g} = \beta_j \quad (\text{since } i_g = j) .
\end{align*}
\]

Thus, (20) is proven in Case 1.

Let us now consider Case 2. In this case, we have \( j \notin \{i_1, i_2, \ldots, i_h\} \). Hence, \( j \) does not belong to the list \((i_1 < i_2 < \cdots < i_h)\). In other words, \( j \) does not belong to the list of all \( c \in \{1, 2, 3, \ldots\} \) satisfying \( \beta_c \neq 0 \) (since this list is \((i_1 < i_2 < \cdots < i_h)\)). Hence, \( \beta_j = 0 \).

Recall that \( j \notin \{i_1, i_2, \ldots, i_h\} \). Hence, the \( j\)-th column of \( B \) is the 0 vector (by the definition of \( B \)). Now,

\[
\begin{align*}
\text{(the } j\text{-th entry of column } B) &= \\
&= \left( \text{the sum of the entries of the } j\text{-th column of } B \right)_{=(\text{the } 0\text{ vector})} \\
&= (\text{the sum of the entries of the } 0\text{ vector}) \\
&= 0 = \beta_j.
\end{align*}
\]

Thus, (20) is proven in Case 2.

Hence, (20) is proven in both Cases 1 and 2. Thus, the proof of (20) is complete.

Now, from (20), we immediately obtain column \( B = (\beta_{i_1}, \beta_{i_2}, \ldots) = \beta \).

It remains to prove that \( B_{\text{cred}} = A \). This can be done as follows: We have \( A \in \mathbb{N}^{k \times h}_{\text{cred}} \), thus, the matrix \( A \) is column-reduced. Hence, no column of \( A \) is the zero vector. Therefore, none of the \( i_1\)-st, \( i_2\)-nd, \ldots, \( i_h\)-th columns of \( B \) is the zero vector (since these columns are the columns of \( A \)). On the other hand, each of the remaining columns of \( B \) is the zero vector (due to the definition of \( B \)). Thus, the set of all positive integers \( j \) such that the \( j\)-th column of \( B \) is nonzero is precisely \( \{i_1, i_2, \ldots, i_h\} \). The list of all elements of this set, in increasing order, is \((i_1 < i_2 < \cdots < i_h)\). Hence, the definition of \( B_{\text{cred}} \) shows that \( B_{\text{cred}} \) is the \( k \times h \)-matrix whose columns (from left to right) are the \( i_1\)-th column of \( B \), the \( i_2\)-nd column of \( B \), \ldots, the \( i_h\)-th
Now, (17) becomes

\[ M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} = \sum_{A \in \mathbb{N}^{k}_{\text{red},\text{fin}}} \sum_{B \in \mathbb{N}^{k}_{\text{fin}}}^{\text{column } B} \sum_{C \in \mathbb{N}^{k}_{\text{fin}}}^{\text{row } C} M_{\text{column } A} \cdot \]

This proves Proposition 5.3.

We need one more piece of notation:

**Definition 5.4.** We define a (multiplicative) monoid structure on the set \( \text{Comp} \) as follows: If \( \alpha = (a_1, a_2, \ldots, a_n) \) and \( \beta = (b_1, b_2, \ldots, b_m) \) are two compositions, then we set \( \alpha \beta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m) \). Thus, \( \text{Comp} \) becomes a monoid with neutral element \( \emptyset \) (the empty composition).

**Proposition 5.5.** Let \( \gamma \in \text{Comp} \) and \( k \in \mathbb{N} \). Then,

\[ \Delta^{(k-1)} M_\gamma = \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k} \]

**Proof of Proposition 5.5 (sketched).** We can rewrite (1) as follows:

\[ \Delta M_\beta = \sum_{(\sigma, \tau) \in \text{Comp} \times \text{Comp}; \sigma \tau = \beta} M_\sigma \otimes M_\tau \quad \text{for every } \beta \in \text{Comp}. \quad (21) \]

Proposition 5.5 can easily be proven by induction using (21).

**Proof of Proposition 5.2** Fix \( \alpha \in \text{Comp} \) and \( \gamma \in \text{Comp} \). Write \( \alpha \) in the form \( \alpha = (a_1, a_2, \ldots, a_k) \); thus, a \( k \)-linear map \( \tilde{\xi}_\alpha : \text{QSym}_k \rightarrow \text{QSym}_k \) is defined (as in Definition 5.1, applied to \( H = \text{QSym}_k \)).

We shall prove that

\[ \tilde{\xi}_\alpha (M_\gamma) = \sum_{A \in \mathbb{N}^{k}_{\text{red},\text{fin}}}^{\text{column } B} M_{\text{column } A} \cdot \]

column of \( B \). Since these columns are precisely the columns of \( A \), this entails that \( B^{\text{cred}} \) is the matrix \( A \). In other words, \( B^{\text{cred}} = A \).

Thus, we have proven that \( B \) is an element of \( \mathbb{N}^{k}_{\text{fin},\text{red}} \) satisfying \( B^{\text{cred}} = A \) and column \( B = \beta \).

It is fairly easy to see that it is the only such element (because the condition column \( B = \beta \) determines which columns of \( B \) are nonzero, whereas the condition \( B^{\text{cred}} = A \) determines the precise values of these columns).
**Proof of (22):** The definition of $\xi_{\alpha}$ yields

$$\xi_{\alpha} (M_{\gamma}) = (m^{(k-1)} \circ \pi_{\alpha} \circ \Delta^{(k-1)}) (M_{\gamma})$$

$$= (m^{(k-1)} \circ \pi_{\alpha}) \left( \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k} \right)$$

(by Proposition 5.5)

$$= \left( m^{(k-1)} \circ \pi_{\alpha} \right) \left( \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k} \right)$$

$$= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} m^{(k-1)} \left( \pi_{\alpha} \left( M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k} \right) \right)$$

$$= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} m^{(k-1)} \left( \pi_{a_1}(M_{\gamma_1}) \otimes \pi_{a_2}(M_{\gamma_2}) \otimes \cdots \otimes \pi_{a_k}(M_{\gamma_k}) \right)$$

(since $\pi_{\alpha} = \pi_{a_1} \otimes \pi_{a_2} \otimes \cdots \otimes \pi_{a_k}$)

$$= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} \prod_{g=1}^{k} \pi_{a_g}(M_{\gamma_g})$$

$$= \begin{cases} M_{\gamma_g}, & \text{if } |\gamma_g| = a_g; \\ 0, & \text{if } |\gamma_g| \neq a_g \end{cases}$$

(since the power series $M_{\gamma_g}$ is homogeneous of degree $|\gamma_g|$)
\[= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} \prod_{g=1}^{k} \begin{cases} M_{\gamma_g}, & \text{if } |\gamma_g| = a_g; \\ 0, & \text{if } |\gamma_g| \neq a_g \end{cases} = \begin{cases} \prod_{g=1}^{k} M_{\gamma_g}, & \text{if } |\gamma_g| = a_g \text{ for all } g; \\ 0, & \text{otherwise} \end{cases} \]

\[= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} \left\{ \prod_{g=1}^{k} M_{\gamma_g}, \text{ if } |\gamma_g| = a_g \text{ for all } g; 0, \text{ otherwise} \right\} \]

\[= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} \left\{ \prod_{g=1}^{k} M_{\gamma_g}, \text{ if } |\gamma_g| = a_g \text{ for all } g; 0, \text{ otherwise} \right\} \]

\[= \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma; |\gamma_g| = a_g \text{ for all } g} \prod_{g=1}^{k} M_{\gamma_g} = \sum_{A \in N^k_{\text{Cred}}} \prod_{g=1}^{k} M_{\gamma_g} \text{column } A \]

\[= (A_{g, a})_{\text{red}} = \gamma_g \text{ for each } g \]

(by Proposition 5.3 applied to \(a_i = \gamma_i\))

(here, we have filtered out the zero addends)
\[
\sum_{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \text{Comp}^k; \gamma_1 \gamma_2 \cdots \gamma_k = \gamma} \sum_{A \in \mathbb{N}_\text{cred}^k; (A_g)_{\text{red}} = \gamma_g \text{ for each } g} \frac{\sum_{\chi_1 \chi_2 \cdots \chi_k = \gamma} (A_{\chi_1})_{\text{red}} (A_{\chi_2})_{\text{red}} \cdots (A_{\chi_k})_{\text{red}} = \gamma; |\gamma_g| = a_g \text{ for all } g}{\sum_{\chi_1 \chi_2 \cdots \chi_k = \gamma} (A_{\chi_1})_{\text{red}} (A_{\chi_2})_{\text{red}} \cdots (A_{\chi_k})_{\text{red}} = \gamma; |\gamma_g| = a_g \text{ for all } g}
\]

Now, we observe that every \( A \in \mathbb{N}_\text{cred}^k \) satisfies

\[
(A_{\chi_1})_{\text{red}} (A_{\chi_2})_{\text{red}} \cdots (A_{\chi_k})_{\text{red}} = (\text{read } A)_{\text{red}}
\]
Also, for every every \( A \in \mathbb{N}_\text{Cred}^k \), we have the logical equivalence
\[
\left( \left| (A_g)_\text{red} \right| = a_g \text{ for all } g \right) \iff (\text{row } A = \alpha) \quad (25)
\]

Also, every \( A \in \mathbb{N}_\text{Cred}^k \) satisfying row \( A = \alpha \) belongs to \( \mathbb{N}_\text{red}^\cdot \). Conversely, every \( A \in \mathbb{N}_\text{red}^\cdot \) satisfying row \( A = \alpha \) belongs to \( \mathbb{N}_\text{Cred}^k \). Combining these

**Proof of (24):** Let \( A \in \mathbb{N}_\text{Cred}^k \).
Let \( \mathbb{N}^\ast \) be the set of all finite lists of nonnegative integers. Then, \( \text{Comp} \subseteq \mathbb{N}^\ast \). In Definition 5.4 we have defined a monoid structure on the set \( \text{Comp} \). We can extend this monoid structure to the set \( \mathbb{N}^\ast \) (by the same rule: namely, if \( \alpha = (a_1,a_2,\ldots,a_n) \) and \( \beta = (b_1,b_2,\ldots,b_m) \), then \( \alpha\beta = (a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_m) \)). (Of course, this monoid \( \mathbb{N}^\ast \) is just the free monoid on the set \( \mathbb{N} \).
Using the latter structure, we can rewrite the definition of \( \text{read} \) as follows:
\[
\text{read } A = A_1 \cdot A_2 \cdot \cdots \cdot A_k.
\]

Clearly, the map \( \mathbb{N}^\ast \to \text{Comp} \), \( \beta \mapsto \beta_{\text{red}} \) is a monoid homomorphism. Thus,
\[
(A_1)_\text{red} \cdot (A_2)_\text{red} \cdots (A_k)_\text{red} = \underbrace{(A_1 \cdot A_2 \cdot \cdots \cdot A_k)}_{= \text{read } A} = (\text{read } A)_{\text{red}}.
\]

This proves (24).

**Proof of (25):** Let \( A \in \mathbb{N}_\text{Cred}^k \). Then, every \( g \in \{1,2,\ldots,k\} \) satisfies
\[
\left| (A_g)_\text{red} \right| = \left( \text{sum of all entries of } (A_g)_\text{red} \right) = \left( \text{sum of all nonzero entries in } A_g \right) \\
\quad \text{by the definition of } (A_g)_\text{red} \\
\quad = \left( \text{sum of all nonzero entries in the } g\text{-th row of } A \right) \\
\quad = \left( \text{sum of all entries in the } g\text{-th row of } A \right) = (\text{the } g\text{-th entry of row } A).
\]

Hence, we have the following chain of equivalences:
\[
\left( \left| (A_g)_\text{red} \right| = a_g \text{ for all } g \right) = (\text{the } g\text{-th entry of row } A) \\
\iff \left( (\text{the } g\text{-th entry of row } A) = a_g \text{ for all } g \right) \\
\iff \left( \text{row } A = (a_1,a_2,\ldots,a_k) \right) = (\text{row } A = \alpha).
\]

This proves (25).

**Proof.** Let \( A \in \mathbb{N}_\text{Cred}^k \) be such that row \( A = \alpha \). We must show that \( A \in \mathbb{N}_\text{red}^\cdot \). The sequence row \( A = \alpha \) is a composition; hence, \( A \) is row-reduced. Since \( A \) is also column-reduced (because \( A \in \mathbb{N}_\text{Cred}^k \)), this shows that \( A \) is reduced. Hence, \( A \in \mathbb{N}_\text{red}^\cdot \). qed.

**Proof.** Let \( A \in \mathbb{N}_\text{red}^\cdot \) be such that row \( A = \alpha \). We must show that \( A \in \mathbb{N}_\text{Cred}^k \). The number of
two observations, we see that
\[
\left( \begin{array}{c}
\text{the matrices } A \in \mathbb{N}_{\text{Cred}}^k \text{ satisfying row } A = \alpha \\
\text{are precisely the matrices } A \in \mathbb{N}_{\text{red}}^k \text{ satisfying row } A = \alpha
\end{array} \right).
\]  

(26)

Now, (23) becomes
\[
\xi_\alpha (M_\gamma) = \sum_{A \in \mathbb{N}_{\text{Cred}}^k; \begin{array}{c}
(A_1, \ldots, A_k) \text{ \_red } = \gamma; \\
\text{row } A = \alpha 
\end{array}}
M_{\text{column } A}
\]
\[
= \sum_{A \in \mathbb{N}_{\text{Cred}}^k; \begin{array}{c}
\text{row } A = \alpha 
\end{array}}
M_{\text{column } A}
= \sum_{A \in \mathbb{N}_{\text{red}}^k; \begin{array}{c}
\text{read } A = \gamma; \\
\text{row } A = \alpha
\end{array}}
M_{\text{column } A}.
\]

This proves (22).

rows of \( A \) is clearly the length of the vector row \( A \) (where the “length” of a vector just means its number of entries). But this length is \( k \) (since row \( A = \alpha = (a_1, a_2, \ldots, a_k) \)). Therefore, the number of rows of \( A \) is \( k \). Also, \( A \) is reduced (since \( A \in \mathbb{N}_{\text{red}}^k \)) and therefore column-reduced. Hence, \( A \in \mathbb{N}_{\text{Cred}}^k \) (since \( A \) is column-reduced and the number of rows of \( A \) is \( k \)), qed.
Now, forget that we fixed $\alpha$ and $\gamma$. For every $\gamma \in \text{Comp}$, we have

$$
\beta_{\text{QSym}_k} (M_\gamma) = \sum_{a \in \text{Comp}} \sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A = a} \xi_a (M_\gamma) \otimes M^\text{column}_A
$$

(by the definition of $\beta_{\text{QSym}_k}$)

$$
= \sum_{a \in \text{Comp}} \sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A = a} M^\text{column}_A \otimes M^\text{row}_A
$$

(by (22))

$$
= \sum_{a \in \text{Comp}} \sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A = a} M^\text{column}_A \otimes M^\text{row}_A
$$

$$
= \sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A \in \text{Comp}} M^\text{column}_A \otimes M^\text{row}_A.
$$

(27)

But every $A \in N^\bullet_{\text{red}'}$ satisfies row $A \in \text{Comp}$. Hence, the summation sign on the right hand side of (27) can be replaced by $\sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A \in \text{Comp}}$. Thus, (27) becomes

$$
\beta_{\text{QSym}_k} (M_\gamma) = \sum_{A \in N^\bullet_{\text{red}'} \atop \text{(read } A\text{)}_{\text{red}} = \gamma; \text{row } A \in \text{Comp}} M^\text{column}_A \otimes M^\text{row}_A
$$

(28)

18Proof. Let $A \in N^\bullet_{\text{red}'}$. Then, the matrix $A$ is reduced, and therefore row-reduced. In other words, row $A$ is a composition. In other words, row $A \in \text{Comp}$, qed.
On the other hand, every $\gamma \in \text{Comp}$ satisfies

$$\Delta_p' (M_\gamma) = (\tau \circ \Delta_p) (M_\gamma) = \tau$$

(by the definition of $\Delta_p$)

$$= \sum_{A \in N_{\text{red}}} \left( M_{\text{row} A} \otimes M_{\text{column} A} \right)$$

(by the definition of $\tau$)

$$= \beta_{\text{QSym}_k} (M_\gamma)$$

(by (28)).

Since both maps $\Delta_p'$ and $\beta_{\text{QSym}_k}$ are $k$-linear, this yields $\Delta_p' = \beta_{\text{QSym}_k}$ (since $(M_\gamma)_{\gamma \in \text{Comp}}$ is a basis of the $k$-module $\text{QSym}_k$). This proves Proposition 5.2.

The next theorem is an analogue for $\text{QSym}$ of the Bernstein homomorphism ([Haz08, §18.24]) for the symmetric functions:

**Theorem 5.6.** Let $k$ be a commutative ring. Let $H$ be a commutative connected graded $k$-Hopf algebra. For every composition $\alpha$, define a $k$-linear map $\xi_\alpha : H \to H$ as in Definition 5.1. Define a map $\beta_H : H \to H \otimes \text{QSym}_k$ as in Definition 5.1.

(a) The map $\beta_H$ is a $k$-algebra homomorphism $H \to H \otimes \text{QSym}_k$ and a graded $(k, H)$-coalgebra homomorphism.

(b) We have $(\text{id} \otimes \varepsilon_p) \circ \beta_H = \text{id}$, where we regard $\text{id} \otimes \varepsilon_p : H \otimes \text{QSym}_k \to H \otimes k$ as a map from $H \otimes \text{QSym}_k$ to $H$ (by identifying $H \otimes k$ with $H$).

(c) Define a map $\Delta_p : \text{QSym}_k \to \text{QSym}_k \otimes \text{QSym}_k$ as in Definition 4.2 (e). The diagram

$$\begin{array}{ccc}
H & \xrightarrow{\beta_H} & H \otimes \text{QSym}_k \\
\downarrow{\beta_H} & & \downarrow{\beta_H \circ \text{id}} \\
H \otimes \text{QSym}_k & \xrightarrow{\text{id} \otimes \Delta_p} & H \otimes \text{QSym}_k \otimes \text{QSym}_k
\end{array}$$

is commutative.

(d) If the $k$-coalgebra $H$ is cocommutative, then $\beta_H (H)$ is a subset of the subring $H \otimes \Lambda_k$ of $H \otimes \text{QSym}_k$, where $\Lambda_k$ is the $k$-algebra of symmetric functions over $k$.  

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Parts (b) and (c) of Theorem 5.6 can be combined into “$\beta_H$ makes $H$ into a $\text{QSym}_2$-comodule, where $\text{QSym}_2$ is the coalgebra $(\text{QSym}, \Delta'_H, \varepsilon)_p$” (the fact that this $\text{QSym}_2$ is actually a coalgebra follows from Proposition 4.3).

What Hazewinkel actually calls the Bernstein homomorphism in [Haz08, §18.24] is the $k$-algebra homomorphism $H \to H \otimes \Lambda_k$ obtained from our map $\beta_H : H \to H \otimes \text{QSym}_k$ by restricting the codomain when $H$ is both commutative and cocommutative.\(^\dagger\) His observation that the second comultiplication of $\Lambda_k$ is a particular case of the Bernstein homomorphism is what gave the original motivation for the present note; its analogue for $\text{QSym}_k$ is our Proposition 5.2.

**Proof of Theorem 5.6.** Set $A = H$ and $\xi = \text{id}$. Then, the map $\xi_{\alpha}$ defined in Corollary 3.12 (c) is precisely the map $\xi_{\alpha}$ defined in Definition 5.1 (because $\xi \otimes_k = \text{id} \otimes_k = \text{id}$). Thus, we can afford calling both maps $\xi_{\alpha}$ without getting confused.

(a) Corollary 3.12 (a) shows that there exists a unique graded $(k, A)$-coalgebra homomorphism $\Xi : H \to A \otimes \text{QSym}_k$ for which the diagram (8) is commutative. Since $A = H$ and $\xi = \text{id}$, we can rewrite this as follows: There exists a unique graded $(k, H)$-coalgebra homomorphism $\Xi : H \to H \otimes \text{QSym}_k$ for which the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\Xi} & H \otimes \text{QSym}_k \\
\downarrow \text{id} & & \downarrow \text{id} \\
H & \xrightarrow{id \otimes \varepsilon_P} & H \otimes \varepsilon_P
\end{array}
$$

(29)

is commutative. Consider this $\Xi$. Corollary 3.12 (c) shows that this homomorphism $\Xi$ is given by

$$
\Xi(h) = \sum_{\alpha \in \text{Comp}} \xi_{\alpha}(h) \otimes M_{\alpha} \quad \text{for every } h \in H.
$$

Comparing this equality with (13), we obtain $\Xi(h) = \beta_H(h)$ for every $h \in H$. In other words, $\Xi = \beta_H$. Thus, $\beta_H$ is a graded $(k, H)$-coalgebra homomorphism (since $\Xi$ is a graded $(k, H)$-coalgebra homomorphism).

Corollary 3.12 (b) shows that $\Xi$ is a $k$-algebra homomorphism. In other words, $\beta_H$ is a $k$-algebra homomorphism (since $\Xi = \beta_H$). This completes the proof of Theorem 5.6 (a).

(b) Consider the map $\Xi$ defined in our above proof of Theorem 5.6 (a). We have shown that $\Xi = \beta_H$.

The commutative diagram (29) shows that $(\text{id} \otimes \varepsilon_P) \circ \Xi = \text{id}$. In other words, $(\text{id} \otimes \varepsilon_P) \circ \beta_H = \text{id}$ (since $\Xi = \beta_H$). This proves Theorem 5.6 (b).

(c) Theorem 5.6 (a) shows that the map $\beta_H$ is a $k$-algebra homomorphism $H \to H \otimes \text{QSym}_k$ and a graded $(k, H)$-coalgebra homomorphism. Theorem

\(^\dagger\)Hazewinkel neglects to require the cocommutativity of $H$ in [Haz08, §18.24], but he uses it nevertheless.
5.6. (a) (applied to $QSym_k$ instead of $H$) shows that the map $\beta_{QSym_k}$ is a $k$-algebra homomorphism $QSym_k \to QSym_k \otimes QSym_k$ and a graded $\left( k, QSym_k \right)$-coalgebra homomorphism. Since $\Delta'_p = \beta_{QSym_k}$ (by Proposition 5.2), this rewrites as follows: The map $\Delta'_p$ is a $k$-algebra homomorphism $QSym_k \to QSym_k \otimes QSym_k$ and a graded $\left( k, QSym_k \right)$-coalgebra homomorphism.

Applying Corollary 5.12 (a) to $H \otimes QSym_k$ and $\beta_H$ instead of $A$ and $\xi$, we see that there exists a unique graded $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphism $\Xi : H \to H \otimes QSym_k \otimes QSym_k$ for which the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\Xi} & H \otimes QSym_k \otimes QSym_k \\
\downarrow{\beta_H} & & \downarrow{\id_{H \otimes QSym_k} \otimes \varepsilon_p} \\
H \otimes QSym_k & & 
\end{array}
$$

is commutative. Thus, if we have two graded $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphisms $\Xi : H \to H \otimes QSym_k \otimes QSym_k$ for which the diagram (30) is commutative, then these two homomorphisms must be identical. We will now show that the two homomorphisms $(\beta_H \otimes \id) \circ \beta_H$ and $(\id \otimes \Delta'_p) \circ \beta_H$ both fit the bill; this will then yield that $(\beta_H \otimes \id) \circ \beta_H = (\id \otimes \Delta'_p) \circ \beta_H$, and thus Theorem 5.6 (c) will follow.

Recall that $\beta_H$ and $\Delta'_p$ are graded maps. Thus, so are $(\beta_H \otimes \id) \circ \beta_H$ and $(\id \otimes \Delta'_p) \circ \beta_H$. Moreover, $\beta_H$ is a $(k, H)$-coalgebra homomorphism, and $\Delta'_p$ is a $\left( k, QSym_k \right)$-coalgebra homomorphism. From this, it is easy to see that $(\beta_H \otimes \id) \circ \beta_H$ and $(\id \otimes \Delta'_p) \circ \beta_H$ are $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphisms.

---

**Proof.** Proposition 3.9 (applied to $H, H \otimes QSym_k, H, QSym_k, \beta_H$ and $\beta_H$ instead of $A, B, H, G, f$ and $p$) shows that $(\beta_H \otimes \id) \circ \beta_H$ is a $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphism. It remains to show that $(\id \otimes \Delta'_p) \circ \beta_H$ is a $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphism.

Recall that $\Delta'_p$ is a $\left( k, QSym_k \right)$-coalgebra homomorphism $QSym_k \to QSym_k \otimes QSym_k$. Hence, Proposition 3.10 (applied to $QSym_k, H, QSym_k, QSym_k \otimes QSym_k$ and $\Delta'_p$ instead of $A, B, H, G$ and $f$) shows that $\id \otimes \Delta'_p : H \otimes QSym_k \to H \otimes QSym_k \otimes QSym_k$ is an $(H, H \otimes QSym_k)$-coalgebra homomorphism. Therefore, Proposition 3.11 (applied to $H, H \otimes QSym_k, H, H \otimes QSym_k, H \otimes QSym_k \otimes QSym_k, \beta_H$ and $\id \otimes \Delta'_p$ instead of $A, B, H, G, I, f$ and $g$) shows that $(\id \otimes \Delta'_p) \circ \beta_H$ is a $\left( k, H \otimes QSym_k \right)$-coalgebra homomorphism. This completes the proof.

---

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Now, we shall show that the diagrams

\[
\begin{array}{c}
H \\ \downarrow \beta_H \\
H \otimes \text{QSym}_k \\
\end{array}
\xrightarrow{(\beta_H \otimes \text{id}) \circ \beta_H}
\begin{array}{c}
H \otimes \text{QSym}_k \otimes \text{QSym}_k \\
\downarrow \text{id}_{H \otimes \text{QSym}_k} \otimes \varepsilon_p \\
\end{array}
\]

(31)

and

\[
\begin{array}{c}
H \\ \downarrow \beta_H \\
H \otimes \text{QSym}_k \\
\end{array}
\xrightarrow{(\text{id} \otimes \Delta'_P) \circ \beta_H}
\begin{array}{c}
H \otimes \text{QSym}_k \otimes \text{QSym}_k \\
\downarrow \text{id}_{H \otimes \text{QSym}_k} \otimes \varepsilon_p \\
\end{array}
\]

(32)

are commutative. This follows from the computations

\[
\left(\text{id}_{H \otimes \text{QSym}_k} \otimes \varepsilon_P\right) \circ (\beta_H \otimes \text{id}) \circ \beta_H = \beta_H \circ (\text{id} \otimes \varepsilon_P) \circ \beta_H = \beta_H
\]

(by Theorem 5.6(b))

and

\[
\left(\text{id}_{H \otimes \text{QSym}_k} \otimes \varepsilon_P\right) \circ \left(\text{id} \otimes \Delta'_P\right) \circ \beta_H = \left(\text{id} \otimes \beta_{\text{QSym}_k}\right) \circ \beta_H
\]

(33)

Thus, we know that \((\beta_H \otimes \text{id}) \circ \beta_H\) and \((\text{id} \otimes \Delta'_P) \circ \beta_H\) are two graded \((k, H \otimes \text{QSym}_k)\)-coalgebra homomorphisms \(\Xi : H \to H \otimes \text{QSym}_k \otimes \text{QSym}_k\) for which the diagram (30) is commutative (since the diagrams (31) and (32) are commutative). But we have shown before that any two such homomorphisms must be identical. Thus, we conclude that \((\beta_H \otimes \text{id}) \circ \beta_H = (\text{id} \otimes \Delta'_P) \circ \beta_H\). This completes the proof of Theorem 5.6(c).
(d) Consider the map $\Xi$ defined in our above proof of Theorem 5.6 (a). We have shown that $\Xi = \beta_H$.

Assume that $H$ is cocommutative. Corollary 3.12 (d) then shows that $\Xi(H)$ is a subset of the subring $A \otimes \Lambda_k$ of $A \otimes \text{QSym}_k$. In other words, $\beta_H(H)$ is a subset of the subring $H \otimes \Lambda_k$ of $H \otimes \text{QSym}_k$ (since $\Xi = \beta_H$ and $A = H$). This proves Theorem 5.6 (d).

(Alternatively, we could prove (d) by checking that for any element $h$ of a commutative cocommutative Hopf algebra $H$, the element $\xi_{\alpha}(h)$ of $A$ depends only on the result of sorting $\alpha$, rather than on the composition $\alpha$ itself.)

Proof of Proposition 4.3. Let $\tau$ be the twist map $\tau_{\text{QSym}_k,\text{QSym}_k} : \text{QSym}_k \otimes \text{QSym}_k \rightarrow \text{QSym}_k \otimes \text{QSym}_k$. This twist map clearly satisfies $\tau \circ \tau = \text{id}$. Hence, $\tau \circ \Delta'_P = \text{id} \circ \Delta_P = \Delta_P$. Theorem 5.6 (c) (applied to $H = \text{QSym}_k$) shows that the diagram

$$
\begin{array}{ccc}
\text{QSym}_k & \xrightarrow{\beta_{\text{QSym}_k}} & \text{QSym}_k \otimes \text{QSym}_k \\
\beta_{\text{QSym}_k} & & \beta_{\text{QSym}_k} \otimes \text{id} \\
\text{QSym}_k \otimes \text{QSym}_k & \xrightarrow{\text{id} \otimes \Delta'_P} & \text{QSym}_k \otimes \text{QSym}_k \otimes \text{QSym}_k
\end{array}
$$

is commutative. In other words, $(\text{id} \otimes \Delta'_P) \circ \beta_{\text{QSym}_k} = (\beta_{\text{QSym}_k} \otimes \text{id}) \circ \beta_{\text{QSym}_k}$. Since $\beta_{\text{QSym}_k} = \Delta'_P$ (by Proposition 5.2), this rewrites as $(\text{id} \otimes \Delta'_P) \circ \Delta'_P = (\Delta'_P \otimes \text{id}) \circ \Delta'_P$. Thus, the operation $\Delta'_P$ is coassociative. Therefore, the operation $\Delta_P = \tau \circ \Delta'_P$ is also coassociative (because the coassociativity of a map $H \rightarrow H \otimes H$ does not change if we compose this map with the twist map $\tau_{H,H} : H \otimes H \rightarrow H \otimes H$). It is furthermore easy to see that the operation $\epsilon_P$ is counital with respect to the operation $\Delta_P$ (see, for example, [Haz08, §11.45]). Hence, the $k$-module $\text{QSym}_k'$, equipped with the comultiplication $\Delta_P$ and the counit $\epsilon_P$, is a $k$-coalgebra. Our goal is to prove that it is a $k$-bialgebra. Hence, it remains to show that $\Delta_P$ and $\epsilon_P$ are $k$-algebra homomorphisms. For $\epsilon_P$, this is again obvious (indeed, $\epsilon_P$ sends any $f \in \text{QSym}_k$ to $f(1,0,0,0,\ldots)$). It remains to prove that $\Delta_P$ is a $k$-algebra homomorphism.

The map $\beta_{\text{QSym}_k}$ is a $k$-algebra homomorphism $\text{QSym}_k \rightarrow \text{QSym}_k \otimes \text{QSym}_k$ (by Theorem 5.6 (a), applied to $H = \text{QSym}_k$). In other words, the map $\Delta'_P$ is a $k$-algebra homomorphism $\text{QSym}_k \rightarrow \text{QSym}_k \otimes \text{QSym}_k$ (since $\beta_{\text{QSym}_k} = \Delta'_P$, and since $\text{QSym}_k = \text{QSym}_k$ as $k$-algebras). Thus, $\Delta_P = \tau \circ \Delta'_P$ is also a $k$-algebra homomorphism (since both $\tau$ and $\Delta'_P$ are $k$-algebra homomorphisms). This completes the proof of Proposition 4.3. □
6. Remark on antipodes

We have hitherto not really used the antipode of a Hopf algebra; thus, we could just as well have replaced the words “Hopf algebra” by “bialgebra” throughout the entire preceding text. Let us now connect the preceding results with antipodes.

The antipode of any Hopf algebra \( H \) will be denoted by \( S_H \).

**Proposition 6.1.** Let \( k \) be a commutative ring. Let \( A \) be a commutative \( k \)-algebra. Let \( H \) be a \( k \)-Hopf algebra. Let \( G \) be an \( A \)-Hopf algebra. Then, every \( k \)-algebra homomorphism \( f : H \to G \) which is a \((k,A)\)-coalgebra homomorphism must also satisfy \( f \circ S_H = S_G \circ f \).

**Proof of Proposition 6.1.** We know that \( H \) is a \( k \)-Hopf algebra. Thus, \( A \otimes H \) is an \( A \)-Hopf algebra. Its definition by extending scalars yields that its antipode is given by \( S_{A \otimes H} = \text{id}_A \otimes S_H \).

Let \( f : H \to G \) be a \( k \)-algebra homomorphism which is a \((k,A)\)-coalgebra homomorphism. Then, \( f^\sharp : A \otimes H \to G \) is an \( A \)-algebra homomorphism (since \( f \) is a \((k,A)\)-coalgebra homomorphism and \( A \) is a \( k \)-algebra). Hence, \( f^\sharp \) is an \( A \)-bialgebra homomorphism, thus an \( A \)-Hopf algebra homomorphism (since every \( A \)-bialgebra homomorphism between two \( A \)-Hopf algebras is an \( A \)-Hopf algebra homomorphism). Thus, \( f^\sharp \) commutes with the antipodes, i.e., satisfies \( f^\sharp \circ S_{A \otimes H} = S_G \circ f^\sharp \).

Now, let \( \iota \) be the canonical \( k \)-module homomorphism \( H \to A \otimes H, h \mapsto 1 \otimes h \). Then, \( (\text{id}_A \otimes S_H) \circ \iota = \iota \circ S_H \). On the other hand, \( f^\sharp \circ \iota = f \) (this is easy to check). Thus,

\[
\begin{align*}
(f \circ S_H) & = f^\sharp \circ \iota \circ S_H \\
& = (\text{id}_A \otimes S_H) \circ \iota \\
& = S_{A \otimes H} \circ \iota \\
& = S_G \circ f^\sharp \\
& = f \circ S_H \\
& = S_G \circ f.
\end{align*}
\]

This proves Proposition 6.1. \( \square \)

**Corollary 6.2.** Let \( k \) be a commutative ring. Let \( H \) be a commutative connected graded \( k \)-Hopf algebra. Define a map \( \beta_H : H \to H \otimes QSym_k \) as in Definition 5.1. Then,

\[
\beta_H \circ S_H = \left( \text{id}_H \otimes S_{QSym_k} \right) \circ \beta_H.
\]
Proof of Corollary \[6.2\] Theorem \[5.6\] (\(a\)) shows that the map \(\beta_H\) is a \(k\)-algebra homomorphism \(H \rightarrow H \otimes \text{QSym}_k\) and a graded \((k, H)\)-coalgebra homomorphism. Thus, Proposition \[6.1\] (applied to \(A = H, G = H \otimes \text{QSym}_k\) and \(f = \beta_H\)) shows that \(\beta_H \circ S_H = S_{H \otimes \text{QSym}_k} \circ \beta_H\).

But the \(H\)-Hopf algebra \(H \otimes \text{QSym}_k\) is defined by extension of scalars; thus, its antipode is given by \(S_{H \otimes \text{QSym}_k} = \text{id}_H \otimes S_{\text{QSym}_k}\). Hence,

\[
\beta_H \circ S_H = \underbrace{S_{H \otimes \text{QSym}_k}}_{= \text{id}_H \otimes S_{\text{QSym}_k}} \circ \beta_H = \left(\text{id}_H \otimes S_{\text{QSym}_k}\right) \circ \beta_H.
\]

This proves Corollary \[6.2\]. \(\square\)

7. Questions

Let me finish with some questions, which unfortunately are not very well-posed (and likely not all very meaningful).

**Question 7.1.** It is well-known that the graded Hopf-algebraic dual of the graded Hopf algebra \(\text{QSym}\) is a graded Hopf algebra \(\text{NSym}\). The second comultiplication \(\Delta_P\) and the second counit \(\epsilon_P\) on \(\text{QSym}\) dualize to a second multiplication \(m_P\) and a second unit \(u_P\) on \(\text{NSym}\), albeit \(u_P\) does not really live inside \(\text{NSym}\) (in fact, it lives in the completion of \(\text{NSym}\) with respect to its grading). We denote the “almost-\(k\)-bialgebra” \((\text{NSym}, m_P, u_P, \Delta, \epsilon)\) ("almost" because \(u_P \notin \text{NSym}\)) by \(\text{NSym}_2\).

We can more or less dualize Theorem \[5.6\]. As a result, instead of a \(\text{QSym}_2\)-comodule structure on every commutative graded connected Hopf algebra \(H\), we obtain an \(\text{NSym}_2\)-module structure on every cocommutative graded connected Hopf algebra \(H\). This structure is rather well-known: (I believe) it has \(H_\alpha \in \text{NSym}_2\) act as the convolution product \(\pi_{a_1} \ast \pi_{a_2} \ast \cdots \ast \pi_{a_k} \in \text{End} H\) for every composition \(\alpha = (a_1, a_2, \ldots, a_k)\) (where \(\ast\) denotes the convolution product in \(\text{End} H\)). This should be somewhere in the papers by Patras and Reutenauer on descent algebras; it is essentially the way to transfer information from the descent algebra \(\text{NSym}_2\) to a descent algebra \((\text{End}_{\text{graded}} H, \circ)\) of a cocommutative graded connected Hopf algebra \(H\).

I am wondering if it is possible to prove that this works using universal properties like I have done above for Theorem \[5.6\] (Just saying “dualize Theorem \[5.6\]” is not enough because dualization over arbitrary commutative rings is a heuristic, not a proof strategy; there does not seem to be a general theorem stating that “the dual of a correct result is correct”, at least when the result has assumptions about gradedness and similar things.)

If the answer is positive, can we use this to give a slick proof of Solomon’s Mackey formula? (I am not saying that there is need for slick proofs of this...
formula – not after those by Gessel and Bidigare –, but it would be interesting to have a new one. I am thinking of letting both NSym\(_2\) and the symmetric groups act on the tensor algebra \( T(V) \) of an infinite-dimensional free \( k \)-module \( V \); one then only needs to check that the actions match.)

Note that if \( u \) and \( v \) are two elements of NSym\(_2\), then the action of the NSym-product \( uv \) (not the NSym\(_2\)-product!) on \( H \) is the convolution of the actions of \( u \) and \( v \). So the action map NSym\(_2\) \( \rightarrow \) \( \text{End} \) \( H \) takes the multiplication of NSym\(_2\) to composition, and the multiplication of NSym to convolution.

**Question 7.2.** In Question 7.1 we found a map NSym\(_2\) \( \rightarrow \) \( \text{End} \) \( H \) for every cocommutative connected graded Hopf algebra \( H \). This is functorial in \( H \), and so is really a map from the constant functor NSym\(_2\) to the functor

\[
\{\text{cocommutative connected graded Hopf algebras}\} \rightarrow \{k\text{-modules}\},
\]

\( H \mapsto \text{End} \) \( H \).

Does the image of this action span (up to topology) the whole functor? I guess I am badly abusing categorical language here, so let me restate the question in simpler terms: If I have a natural endomorphism of \( H \) for every cocommutative connected graded Hopf algebra \( H \), and this endomorphism is known to annihilate all homogeneous components \( H_m \) for sufficiently high \( m \) (this is what I mean by “up to topology”), then must there be an element \( v \) of NSym\(_2\) such that this endomorphism is the action of \( v \)?

If the answer is “No”, then does it change if we require the endomorphism of \( H \) to be graded? If we require \( k \) to have characteristic 0?

What if we restrict ourselves to commutative cocommutative connected graded Hopf algebras? At least then, if \( k \) is a finite field \( \mathbb{F}_q \), there are more natural endomorphisms of \( H \), such as the Frobenius morphism \( x \mapsto x^q \) and its powers. One can then ask for the graded endomorphisms of \( H \), but actually it is also interesting to ask how the full \( k \)-algebra of natural endomorphisms looks like (how do the endomorphisms coming from NSym\(_2\) interact with the Frobenii?). And what about characteristic 0 here?

**Question 7.3.** What are the natural endomorphisms of connected graded Hopf algebras, without any cocommutativity or commutativity assumption? I assume they will form a connected graded Hopf algebra, with two multiplications (one for composition and the other for convolution), but now with a basis indexed by “mopiscotions” (i.e., pairs \((\alpha, \sigma)\) of a composition \( \alpha \) and a permutation \( \sigma \in \mathfrak{S}_{\ell(\alpha)} \))? 

**Question 7.4.** Can we extend the map \( \beta_H : H \rightarrow H \otimes \text{QSym}_k \) to a map \( H \rightarrow H \otimes U \) for some combinatorial Hopf algebra \( U \) bigger than \( \text{QSym}_k \)? What if we require some additional (say, dendriform?) structure on \( H \)? Can we achieve \( U = \text{NCQSym}_k \) or \( U = \text{DoublePosets}_k \) (the combinatorial Hopf
algebra of double posets, which is defined for $k = \mathbb{Z}$ and denoted by $\mathbb{Z}D$ in [MalReu11], and can be similarly defined over any $k$? (I am singling out these two Hopf algebras because they have fairly nice internal comultiplications. Actually, the internal comultiplication of $\text{NCQSym}_k$ is the key to Bidigare’s proof of Solomon’s Mackey formula [Schock04, §2], and I feel it will tell us more if we let it speak.)

**Question 7.5.** Do we gain anything from applying Corollary 6.2 to $H = \text{QSym}_k$ (thus getting a statement about $\Delta'_p$)? Probably not much for $\Delta'_p$ that the Marne-la-Vallée people haven’t already found using virtual alphabets (the dual version is the statement that $S(a \ast b) = a \ast S(b)$ for all $a, b \in \text{NSym}_k$, where $\ast$ is the internal product).

**Question 7.6.** From Theorem 5.6(a) and Proposition 5.2, we can conclude that $\Delta'_p$ is a $\left(k, \text{QSym}_k\right)$-coalgebra homomorphism. If I am not mistaken, this can be rewritten as the equality

$$(AB) \ast G = \sum (A \ast G_{(1)}) (B \ast G_{(2)})$$

(using Sweedler’s notation) for any three elements $A, B$ and $G$ of $\text{NSym}$. This is the famous splitting formula.

Now, it is known from [DHNT08, §7] that the same splitting formula holds when $A$ and $B$ are elements of $\text{FQSym}$ (into which $\text{NSym}$ is known to inject), as long as $G$ is still an element of $\text{NSym}$ (actually, it can be an element of the bigger Patras-Reutenauer algebra, but let us settle for $\text{NSym}$ so far). Can this be proven in a similar vein? How much of the Marne-la-Vallée theory follows from Theorem 2.1?

**References**


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