Double posets and the antipode of QSym

Darij Grinberg

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Abstract.
We assign a quasisymmetric function to any double poset (that is, every finite set endowed with two partial orders) and any weight function on its ground set. This generalizes well-known objects such as monomial and fundamental quasisymmetric functions, (skew) Schur functions, dual immaculate functions, and quasisymmetric \((P,\omega)\)-partition enumerators. We then prove a formula for the antipode of this function that holds under certain conditions (which are satisfied when the second order of the double poset is total, but also in some other cases); this restates (in a way that to us seems more natural) a result by Malvenuto and Reutenauer, but our proof is new and self-contained. We generalize it further to an even more comprehensive setting, where a group acts on the double poset by automorphisms.

1. Introduction

Double posets and E-partitions (for E a double poset) have been introduced by Claudia Malvenuto and Christophe Reutenauer \([MalReu09]\) in order to construct a combinatorial Hopf algebra which harbors a noticeable amount of structure, including an analogue of the Littlewood-Richardson rule and a lift of the internal product operation of the Malvenuto-Reutenauer Hopf algebra of permutations. In this note, we shall employ these same notions to restate in a simpler form, and reprove in a more elementary fashion, a formula for the antipode in the Hopf algebra QSym of quasisymmetric functions due to (the same) Malvenuto and Reutenauer (generalizing an earlier result by Gessel), and extend it further to a case in which a group acts on the double poset.

In the present version of the paper, some (classical and/or straightforward) proofs are missing or sketched. A more detailed version exists, in which at least
a few of these proofs are elaborated on more\footnote{It can be downloaded from http://web.mit.edu/~darij/www/algebra/dp-abstr-long.pdf} further details might be added in future versions of this paper or forthcoming work (the author’s PhD thesis).

1.1. Acknowledgments

Katharina Jochemko’s work \cite{Joch13} provoked this research. I learnt a lot about QSym from Victor Reiner.

2. Quasisymmetric functions

Let us first briefly introduce the notations that will be used in the following.

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. A composition means a finite sequence of positive integers. We let Comp be the set of all compositions. For $n \in \mathbb{N}$, a composition of $n$ means a composition whose entries sum to $n$ (that is, a composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$).

Let $\mathbf{k}$ be an arbitrary commutative ring. We shall keep $\mathbf{k}$ fixed throughout this paper. We consider the $\mathbf{k}$-algebra $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ of formal power series in infinitely many (commuting) indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. A monomial shall always mean a monomial (without coefficients) in the variables $x_1, x_2, x_3, \ldots$. \footnote{For the sake of completeness, let us give a detailed definition of monomials and of the topology on $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$. (This definition has been copied from \cite{Grin14} \S 2, essentially unchanged.)

Let $x_1, x_2, x_3, \ldots$ be countably many distinct symbols. We let Mon be the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (written multiplicatively); it consists of elements of the form $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ for finitely supported $(a_1, a_2, a_3, \ldots) \in \mathbb{N}_0^n$ (where “finitely supported” means that all but finitely many positive integers $i$ satisfy $a_i = 0$). A monomial will mean an element of Mon. Thus, a monomial is a combinatorial object, independent of $\mathbf{k}$; it does not carry a coefficient.

We consider the $\mathbf{k}$-algebra $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. By abuse of notation, we shall identify every monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \in$ Mon with the corresponding element $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ of $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ when necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of $\mathbf{k}$). (To be very pedantic, this identification is slightly dangerous, because it can happen that two distinct monomials in Mon get identified with two identical elements of $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$. However, this can only happen when the ring $\mathbf{k}$ is trivial, and even then it is not a real problem unless we infer the equality of monomials from the equality of their counterparts in $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$, which we are not going to do.)

We furthermore endow the ring $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ with the following topology (as in \cite{GriRei14} Section 2.6):

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients (indexed by the set Mon). More precisely, we have a $\mathbf{k}$-module isomorphism

$$\prod_{m \in \text{Mon}} \mathbf{k} \rightarrow \mathbf{k} [[x_1, x_2, x_3, \ldots]], \quad (\lambda_m)_{m \in \text{Mon}} \mapsto \sum_{m \in \text{Mon}} \lambda_m m.$$ We use this isomorphism to transform the product topology on $\prod_{m \in \text{Mon}} \mathbf{k}$ to $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$.

The resulting topology on $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ turns $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ into a polynomial $\mathbf{k}$-algebra; this is the topology that we will be using whenever we make statements about convergence in
Inside the \( k \)-algebra \( k [[x_1, x_2, x_3, \ldots ]] \) is a subalgebra \( k [[x_1, x_2, x_3, \ldots ]]_{\text{bdd}} \) consisting of the bounded-degree formal power series; these are the power series \( f \) for which there exists a \( d \in \mathbb{N} \) such that no monomial of degree \( > d \) appears in \( f \). This \( k \)-subalgebra \( k [[x_1, x_2, x_3, \ldots ]]_{\text{bdd}} \) becomes a topological \( k \)-algebra, by inheriting the topology from \( k [[x_1, x_2, x_3, \ldots ]] \).

Two monomials \( m \) and \( n \) are said to be \textit{pack-equivalent}\(^3\) if they have the forms \( x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \) and \( x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} \) for two strictly increasing sequences \( (i_1 < i_2 < \cdots < i_k) \) and \( (j_1 < j_2 < \cdots < j_k) \) of positive integers and one (common) sequence \( (a_1, a_2, \ldots, a_k) \) of positive integers.\(^4\) A power series \( f \in k [[x_1, x_2, x_3, \ldots ]] \) is said to be \textit{quasisymmetric} if every two pack-equivalent monomials have equal coefficients in front of them in \( f \). It is easy to see that the quasisymmetric power series form a \( k \)-subalgebra of \( k [[x_1, x_2, x_3, \ldots ]] \); but usually, one is interested in the set of quasisymmetric bounded-degree power series in \( k [[x_1, x_2, x_3, \ldots ]] \). This latter set is a \( k \)-subalgebra of \( k [[x_1, x_2, x_3, \ldots ]]_{\text{bdd}} \), and is known as the \( k \)-\textit{algebra of quasisymmetric functions over} \( k \). It is denoted by \( \text{QSym} \). It is clear that symmetric functions (in the usual sense of this word in combinatorics—so, really, symmetric bounded-degree power series in \( k [[x_1, x_2, x_3, \ldots ]] \)) form a \( k \)-subalgebra of \( \text{QSym} \). The quasisymmetric functions have a rich theory which is related to, and often sheds new light on, the classical theory of symmetric functions; expositions can be found in [Stan99, §§7.19, 7.23] and [GriRei14, §§5-6] and other sources.

As a \( k \)-module, \( \text{QSym} \) has a basis \( (M_\alpha)_{\alpha \in \text{Comp}} \) indexed by all compositions, where the quasisymmetric function \( M_\alpha \) for a given composition \( \alpha \) is defined as follows: Writing \( \alpha \) as \( (a_1, a_2, \ldots, a_\ell) \), we set

\[
M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell} = \sum_{m \text{ is a monomial pack-equivalent to } x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}} m
\]

(where the \( i_k \) in the first sum are positive integers). This basis \( (M_\alpha)_{\alpha \in \text{Comp}} \) is known as the \textit{monomial basis} of \( \text{QSym} \), and is the simplest to define among many. (We shall briefly encounter another basis in Example 3.6.)

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\( k [[x_1, x_2, x_3, \ldots ]] \) or write down infinite sums of power series. A sequence \( (a_n)_{n \in \mathbb{N}} \) of power series converges to a power series \( a \) with respect to this topology if and only if for every monomial \( m \), all sufficiently high \( n \in \mathbb{N} \) satisfy

\[
(\text{the coefficient of } m \text{ in } a_n) = (\text{the coefficient of } m \text{ in } a).
\]

Note that this topological \( k \)-algebra \( k [[x_1, x_2, x_3, \ldots ]] \) is not the completion of \( k [x_1, x_2, x_3, \ldots] \) with respect to the standard grading (in which all \( x_i \) have degree 1). (They are distinct even as sets.)

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\(^3\)The degree of a monomial \( x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \) is defined to be the nonnegative integer \( a_1 + a_2 + a_3 + \cdots \).

A monomial \( m \) is said to \textit{appear} in a power series \( f \in k [[x_1, x_2, x_3, \ldots ]] \) if and only if the coefficient of \( m \) in \( f \) is nonzero.

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\(^4\)Pack-equivalence and the related notions of packed combinatorial objects that we will encounter below originate in work of Hivert, Novelli and Thibon [NovThi05]. Simple as they are, they are of great help in dealing with quasisymmetric functions.

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\(^5\)For instance, \( x_2^2 x_3 x_4 \) is pack-equivalent to \( x_1^2 x_4 x_8 \) but not to \( x_2 x_3 x_4 \).
The \(k\)-algebra \(QSym\) can be endowed with a structure of a \(k\)-coalgebra which, combined with its \(k\)-algebra structure, turns it into a Hopf algebra. We refer to the literature both for the theory of coalgebras and Hopf algebras (see [Montg93], [GriRei14 §1], [Manchon04 §1-§2], [Abe77], [Sweed69], [DNR01] or [Fresse14, Chapter 7]) and for a deeper study of the Hopf algebra \(QSym\) (see [Malve93], [HaGuKi10 Chp. 6] or [GriRei14, §5]); in this note we shall need but the very basics of this structure, and so it is only them that we introduce.

We define a \(k\)-linear map \(\Delta : QSym \to QSym \otimes QSym\) (here and in the following, all tensor products are over \(k\) by default) by requiring that

\[
\Delta \left( M(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \right) = \sum_{k=0}^{\ell} M(\alpha_1, \alpha_2, \ldots, \alpha_k) \otimes M(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_\ell) \quad (1)
\]

for every \((\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \text{Comp}\).

We further define a \(k\)-linear map \(\varepsilon : QSym \to k\) by requiring that

\[
\varepsilon \left( M(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \right) = \delta_{\ell,0} \quad \text{for every } (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \text{Comp}
\]

(Equivalently, \(\varepsilon\) sends every power series \(f \in QSym\) to the result \(f(0,0,0,\ldots)\) of substituting zeroes for the variables \(x_1, x_2, x_3, \ldots\) in \(f\). The map \(\Delta\) can also be described in such terms, but with greater difficulty [GriRei14 (5.3)].) It is well-known that these maps \(\Delta\) and \(\varepsilon\) make the three diagrams

\[
\begin{array}{ccc}
QSym & \xrightarrow{\Delta} & QSym \otimes QSym \\
\Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\
QSym \otimes QSym & \xrightarrow{\text{id} \otimes \Delta} & QSym \otimes QSym \otimes QSym
\end{array}
\]

\[
\begin{array}{ccc}
QSym & \xrightarrow{\Delta} & QSym \otimes QSym \\
\cong \downarrow \quad \varepsilon \otimes \text{id} & & \cong \downarrow \text{id} \otimes \varepsilon \\
k \otimes QSym & \xrightarrow{\varepsilon} & QSym \otimes k
\end{array}
\]

(where the \(\cong\) arrows are the canonical isomorphisms) commutative, and so \((QSym, \Delta, \varepsilon)\) is what is commonly called a \(k\)-coalgebra. Furthermore, \(\Delta\) and \(\varepsilon\) are \(k\)-algebra homomorphisms, which is what makes this \(k\)-coalgebra \(QSym\) into a \(k\)-bialgebra. Finally, let \(m : QSym \otimes QSym \to QSym\) be the \(k\)-linear map sending every pure tensor \(a \otimes b\) to \(ab\), and let \(u : k \to QSym\) be the \(k\)-linear map sending \(1 \in k\) to \(1 \in QSym\).

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\(^6\)This definition relies on the fact that \((M_\alpha)_{\alpha \in \text{Comp}}\) is a basis of the \(k\)-module \(QSym\).

\(^7\)Here, \(\delta_{u,v}\) is defined to be \(\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}\) whenever \(u\) and \(v\) are two objects.
Then, there exists a unique $k$-linear map $S : QSym \to QSym$ making the diagram

$$
\begin{array}{ccc}
QSym \otimes QSym & \xrightarrow{\Delta} & QSym \otimes QSym \\
\downarrow & & \downarrow \\
QSym & \xrightarrow{\epsilon} & k \\
\downarrow & & \downarrow \\
QSym \otimes QSym & \xrightarrow{id \otimes S} & QSym \otimes QSym
\end{array}
$$

(2)

commutative. This map $S$ is known as the antipode of $QSym$. It is known to be an involution and an algebra automorphism of $QSym$, and its action on the various quasisymmetric functions defined combinatorially is the main topic of this note. The existence of the antipode $S$ makes $QSym$ into a Hopf algebra.

3. Double posets

Next, we shall introduce the notion of a double poset, following Malvenuto and Reutenauer [MalReu09].

**Definition 3.1.** (a) We shall encode posets as pairs $(P, <)$, where $P$ is a set and $<$ is a strict partial order relation (i.e., an irreflexive, transitive and antisymmetric binary relation) on the set $P$; this relation $<$ will be regarded as the smaller relation of the poset. (All binary relations will be written in infix notation: i.e., we write "$a < b$" for "$a$ is related to $b$ by the relation $<$".)

(b) If $<$ is a strict partial order relation on a set $P$, and if $a$ and $b$ are two elements of $P$, then we say that $a$ and $b$ are $<$-comparable if we have either $a < b$ or $a = b$ or $b < a$. A strict partial order relation $<$ on a set $P$ is said to be a total order if and only if every two elements of $P$ are $<$-comparable.

(c) If $<$ is a strict partial order relation on a set $P$, and if $a$ and $b$ are two elements of $P$, then we say that $a$ is $<$-covered by $b$ if we have $a < b$ and there exists no $c \in P$ satisfying $a < c < b$. (For instance, if $<$ is the standard smaller relation on $\mathbb{Z}$, then each $i \in \mathbb{Z}$ is $<$-covered by $i + 1$.)

(d) A double poset is defined as a triple $(E, <_1, <_2)$ where $E$ is a finite set and $<_1$ and $<_2$ are two strict partial order relations on $E$.

(e) A double poset $(E, <_1, <_2)$ is said to be special if the relation $<_2$ is a total order.

(f) A double poset $(E, <_1, <_2)$ is said to be semispecial if every two $<_1$-comparable elements of $E$ are $<_2$-comparable.
(g) A double poset \((E, <_1, <_2)\) is said to be tertispecial if it satisfies the following condition: If \(a\) and \(b\) are two elements of \(E\) such that \(a\) is \(<_1\)-covered by \(b\), then \(a\) and \(b\) are \(<_2\)-comparable.

(h) If \(<\) is a binary relation on a set \(P\), then the opposite relation of \(<\) is defined to be the binary relation \(>\) on the set \(P\) which is defined as follows: For any \(e \in P\) and \(f \in P\), we have \(e > f\) if and only if \(f < e\). Notice that if \(<\) is a strict partial order relation, then so is the opposite relation \(>\) of \(<\).

Clearly, every special double poset is semispecial, and every semispecial double poset is tertispecial.

**Definition 3.2.** If \(E = (E, <_1, <_2)\) is a double poset, then an **\(E\)-partition** shall mean a map \(\phi : E \to \{1, 2, 3, \ldots\}\) such that:

- every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) satisfy \(\phi(e) \leq \phi(f)\);
- every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) and \(f <_2 e\) satisfy \(\phi(e) < \phi(f)\).

**Example 3.3.** The notion of an \(E\)-partition (which was inspired by the earlier notions of \(P\)-partitions and \((P, \omega)\)-partitions as studied by Gessel and Stanley) generalizes various well-known combinatorial concepts. For example:

- If \(<_2\) is the same order as \(<_1\) (or any extension of this order), then \(E\)-partitions are weakly increasing maps from the poset \((E, <_1)\) to the totally ordered set \(\{1, 2, 3, \ldots\}\).
- If \(<_2\) is the opposite relation of \(<_1\) (or any extension of this opposite relation), then \(E\)-partitions are strictly increasing maps from the poset \((E, <_1)\) to the totally ordered set \(\{1, 2, 3, \ldots\}\).

For a more interesting example, let \(\mu = (\mu_1, \mu_2, \mu_3, \ldots)\) and \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)\) be two partitions such that \(\mu \subseteq \lambda\). (See [GriRei14 §2] for the notations we are using here.) The skew Young diagram \(Y(\lambda/\mu)\) is then defined as the set of all \((i, j) \in \{1, 2, 3, \ldots\}^2\) satisfying \(\mu_i < j \leq \lambda_i\). On this set \(Y(\lambda/\mu)\), we define two partial order relations \(<_1\) and \(<_2\) by

\[
(i, j) <_1 (i', j') \iff (i \leq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j'))
\]

and

\[
(i, j) <_2 (i', j') \iff (i \geq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j')).
\]

The notions of a double poset and of a special double poset come from [MalReu09]. The notion of a “tertispecial double poset” (Dog Latin for “slightly less special than semispecial”) appears to be new and arguably sounds artificial, but is the most suitable setting for some of the results below (and appears in nature, beyond the particular case of special double posets – see Example 3.3). We shall not use semispecial double posets in the following; they were only introduced as a middle ground between special and tertispecial double posets with a less daunting definition.
The resulting double poset $Y(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$ has the property that the $Y(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape $\lambda/\mu$. (Again, see [GriRei14 §2] for the meaning of these words.)

This double poset $Y(\lambda/\mu)$ is not special (in general), but it is tertispecial. (Indeed, if $a$ and $b$ are two elements of $Y(\lambda/\mu)$ such that $a$ is $<_1$-covered by $b$, then $a$ is either the left neighbor of $b$ or the top neighbor of $b$, and thus we have either $a <_2 b$ (in the former case) or $b <_2 a$ (in the latter case).) Some authors prefer to use a special double poset instead, which is defined as follows: We define a total order $<_h$ on $Y(\lambda/\mu)$ by

$$(i, j) <_h (i', j') \iff (i > i' \text{ or } (i = i' \text{ and } j < j')).$$

Then, $Y_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$ is a special double poset, and the $Y_h(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape $\lambda/\mu$.

We now assign a certain formal power series to every double poset:

**Definition 3.4.** If $E = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is a map, then we define a power series $\Gamma(E, w) \in k[[x_1, x_2, x_3, \ldots]]$ by

$$\Gamma(E, w) = \sum_{\pi \text{ is an } E\text{-partition}} x_{\pi, w}, \text{ where } x_{\pi, w} = \prod_{e \in E} x^{w(e)}_{\pi(e)}.$$

The following fact is easy to see (but will be reproven below):

**Proposition 3.5.** Let $E = (E, <_1, <_2)$ be a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then, $\Gamma(E, w) \in QSym$.

**Example 3.6.** The power series $\Gamma(E, w)$ generalize various well-known quasisymmetric functions.

(a) If $E = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is the constant function sending everything to 1, then $\Gamma(E, w) = \sum_{\pi \text{ is an } E\text{-partition}} x_{\pi}$, where $x_{\pi} = \prod_{e \in E} x^{\pi(e)}_{\pi(e)}$. We shall denote this power series $\Gamma(E, w)$ by $\Gamma(E)$; it is exactly what has been called $\Gamma(E)$ in [MalReu09 §2.2]. All results proven below for $\Gamma(E, w)$ can be applied to $\Gamma(E)$, yielding simpler (but less general) statements.

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9See [Gessel15] for the history of these notions, and see [Gessel84], [Stan71], [Stan11 §3.15] and [Stan99 §7.19] for some of their theory. Mind that these sources use different and sometimes incompatible notations – e.g., the $P$-partitions of [Stan11 §3.15] and [Gessel15] differ from those of [Gessel84] by a sign reversal.
(b) If \( E = \{1, 2, \ldots, \ell\} \) for some \( \ell \in \mathbb{N} \), if \(<_1\) is the usual total order inherited from \( \mathbb{Z} \), and if \(<_2\) is the opposite relation of \(<_1\), then the special double poset \( E = (E, <_1, <_2) \) satisfies \( \Gamma (E, w) = M_\alpha \), where \( \alpha \) is the composition \((w(1), w(2), \ldots, w(\ell))\). Thus, the elements of the monomial basis \( (M_\alpha)_{\alpha \in \text{Comp}} \) are special cases of the functions \( \Gamma (E, w) \).

(c) Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) be a composition of a nonnegative integer \( n \). Let \( D(\alpha) \) be the set \( \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \). Let \( E \) be the set \( \{1, 2, \ldots, n\} \), and let \(<_1\) be the total order inherited on \( E \) from \( \mathbb{Z} \). Let \(<_2\) be some partial order on \( E \) with the property that
\[
i + 1 <_2 i \quad \text{for every } i \in D(\alpha)
\]
and
\[
i <_2 i + 1 \quad \text{for every } i \in \{1, 2, \ldots, n - 1\} \setminus D(\alpha).
\]
(There are several choices for such an order; in particular, we can find one which is a total order.) Then,
\[
\Gamma ((E, <_1, <_2)) = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\
i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}
= \sum_{\beta \text{ is a composition of } n; \; D(\beta) \supseteq D(\alpha)} M_\beta.
\]
This power series is known as the \( \alpha \)-th fundamental quasisymmetric function, usually called \( F_\alpha \) (in [BBSSZ13, §2.4] and [Grin14, §2]) or \( L_\alpha \) (in [Stan99, §7.19] or [GriRei14, Def. 5.15]).

(d) Let \( E \) be one of the two double posets \( Y(\lambda/\mu) \) and \( Y_h(\lambda/\mu) \) defined as in Example 3.3 for two partitions \( \mu \) and \( \lambda \). Then, \( \Gamma (E) \) is the skew Schur function \( s_{\lambda/\mu} \).

(e) Similarly, dual immaculate functions as defined in [BBSSZ13, §3.7] can be realized as \( \Gamma (E) \) for conveniently chosen \( E \) (see [Grin14, Proposition 4.4]), which helped the author to prove one of their properties [Grin14]. (The \( E \)-partitions here are the so-called immaculate tableaux.)

(f) When the relation \(<_2\) of a double poset \( E = (E, <_1, <_2) \) is a total order (i.e., when the double poset \( E \) is special), the \( E \)-partitions are precisely the reverse \((P, \omega)\)-partitions (for \( P = (E, <_1) \) and \( \omega \) being a labelling of \( P \) dictated by \(<_2\)) in the terminology of [Stan99, §7.19], and the power series \( \Gamma (E) \) is the \( K_{P, \omega} \) of [Stan99, §7.19]. This can also be rephrased using the notations of [GriRei14, §5.2]: When the relation \(<_2\) of a double poset \( E = (E, <_1, <_2) \) is a total order, we can relabel the elements of \( E \) by the integers \( 1, 2, \ldots, n \) in such a way that \( 1 <_2 2 <_2 \cdots <_2 n \); then, the \( E \)-partitions are the \( P \)-partitions in the terminology of [GriRei14, Def. 5.12].
where $P$ is the labelled poset $(E, <_1)$; and furthermore, our $\Gamma(E)$ is the $F_P(x)$ of [GriRei14, Def. 5.12]. Conversely, if $P$ is a labelled poset, then the $F_P(x)$ of [GriRei14, Def. 5.12] is our $\Gamma((P, <_P, <_Z))$.

4. The antipode theorem

We now come to the main results of this note. We first state a theorem and a corollary which are not new, but will be reproven in a more self-contained way which allows them to take their (well-deserved) place as fundamental results rather than afterthoughts in the theory of QSym.

**Definition 4.1.** We let $S$ denote the antipode of QSym.

**Theorem 4.2.** Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots \}$. Then, $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$, where $>_1$ denotes the opposite relation of $<_1$.

**Corollary 4.3.** Let $(E, <_1, <_2)$ be a tertispecial double poset. Then, $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$, where $>_1$ denotes the opposite relation of $<_1$.

We shall give examples for consequences of these facts shortly (Example 4.7), but let us first explain where they have already appeared. Corollary 4.3 is equivalent to [GriRei14, Corollary 5.27] (a result apparently due to Gessel). Theorem 4.2 is equivalent to Malvenuto’s and Reutenauer’s [MalReu98] Theorem 3.1.

It is easiest to derive [GriRei14, Corollary 5.27] from our Corollary 4.3, as this only requires setting $E = (P, <_P, <_Z)$ (this is a special double poset, thus in particular a tertispecial one) and noticing that $\Gamma((P, <_P, <_Z)) = F_P(x)$ and $\Gamma((P, >_P, <_Z)) = F_{opp}(x)$, where all unexplained notations are defined in [GriRei14, Chp. 5]. But one can also proceed in the opposite direction.

This equivalence requires a bit of work to set up. To derive [MalReu98] Theorem 3.1 from our Theorem 4.2, it is enough to contract all undirected edges in $G$, denoting the vertex set of the new graph by $E$, and then define two order relations $<_1$ and $<_2$ on $E$ by

$$(a <_1 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G)$$

and

$$(a <_2 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G').$$

The map $w$ sends every $e \in E$ to the number of vertices of $G$ that became $e$ when the edges were contracted. To show that the resulting double poset $(E, <_1, <_2)$ is tertispecial, we must notice that if $a$ is $<_1$-covered by $b$, then $G$ had an edge from one of the vertices that became $a$ to one of the vertices that became $b$. The “$x$’s in $X$ satisfying a set of conditions” (in the language of [MalReu98, Section 3]) are then in 1-to-1 correspondence with $(E, <_1, <_2)$-partitions (at least when $X = \{1, 2, 3, \ldots \}$); this is not immediately obvious but not hard to check either (the acyclicity of $G$ and $G'$ is used in the proof). As a result, [MalReu98] Theorem 3.1 follows from Theorem 4.2 above. With some harder work, one can conversely derive our Theorem 4.2 from [MalReu98] Theorem 3.1.
nevertheless believe that our versions of these facts are more natural and simpler than the ones appearing in existing literature\footnote{That said, we would not be surprised if Malvenuto and Reutenauer are aware of them and just have not published them; after all, they have discovered both the original version of Theorem 4.2 in \cite{MalReu98} and the notion of double posets in \cite{MalReu09}.} and if not, then at least their proofs below are more in the nature of things.

To these known results, we add another, which seems to be unknown so far (probably because it is far harder to state in the terminologies of \((P, \omega)\)-partitions or equality-and-inequality conditions appearing in literature). First, we need to introduce some notation:

**Definition 4.4.** Let \(G\) be a group, and let \(E\) be a \(G\)-set.

(a) Let \(<\) be a strict partial order relation on \(E\). We say that \(G\) preserves the relation \(<\) if the following holds: For every \(g \in G\), \(a \in E\) and \(b \in E\) satisfying \(a < b\), we have \(ga < gb\).

(b) Let \(w : E \to \{1, 2, 3, \ldots\}\). We say that \(G\) preserves \(w\) if every \(g \in G\) and \(e \in E\) satisfy \(w(ge) = w(e)\).

(c) Let \(g \in G\). Assume that the set \(E\) is finite. We say that \(g\) is \(E\)-even if the action of \(g\) on \(E\) (that is, the permutation of \(E\) that sends every \(e \in E\) to \(ge\)) is an even permutation of \(E\).

(d) If \(X\) is any set, then the set \(X^E\) of all maps \(E \to X\) becomes a \(G\)-set in the following way: For any \(\pi \in X^E\) and \(g \in G\), we define the element \(g\pi \in X^E\) to be the map sending each \(e \in E\) to \(\pi(g^{-1}e)\).

(e) Let \(F\) be a further \(G\)-set. Assume that the set \(E\) is finite. An element \(\pi \in F\) is said to be \(E\)-coeven if every \(g \in G\) satisfying \(g\pi = \pi\) is \(E\)-even. A \(G\)-orbit \(O\) on \(F\) is said to be \(E\)-coeven if all elements of \(O\) are \(E\)-coeven.

Before we come to the promised result, let us state a simple fact:

**Lemma 4.5.** Let \(G\) be a group. Let \(F\) and \(E\) be \(G\)-sets such that \(E\) is finite. Let \(O\) be a \(G\)-orbit on \(F\). Then, \(O\) is \(E\)-coeven if and only if at least one element of \(O\) is \(E\)-coeven.

**Theorem 4.6.** Let \(E = (E, <_1, <_2)\) be a tertispecial double poset. Let \(\text{Par}_E\) denote the set of all \(E\)-partitions. Let \(w : E \to \{1, 2, 3, \ldots\}\). Let \(G\) be a finite group which acts on \(E\). Assume that \(G\) preserves both relations \(<_1\) and \(<_2\), and also preserves \(w\). Then, \(G\) acts also on the set \(\text{Par}_E\) of all \(E\)-partitions; namely, \(\text{Par}_E\) is a \(G\)-subset of the \(G\)-set \(\{1, 2, 3, \ldots\}^E\) (see Definition 4.4 (d) for the definition of the latter). For any \(G\)-orbit \(O\) on \(\text{Par}_E\), we define a monomial \(x_{O,w}\) by

\[
x_{O,w} = x_{\pi,w} \quad \text{for some element } \pi \text{ of } O
\]
Double posets and the antipode of QSym

Let \( \Gamma(E, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \Par E} x_{O, w} \) and \( \Gamma^+(E, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \Par E} x_{O, w} \).

Then, \( \Gamma(E, w, G) \) and \( \Gamma^+(E, w, G) \) belong to QSym and satisfy

\[
S(\Gamma(E, w, G)) = (-1)^{|E|} \Gamma^+(((E, >_1, <_2), w, G)).
\]

This theorem, which combines Theorem 4.2 with the ideas of Pólya enumeration, is inspired by Jochemko’s reciprocity result for order polynomials [Joch13, Theorem 2.8], which can be obtained from it by specializations (see Section 8 for the details of how Jochemko’s result follows from ours).

We shall now review a number of particular cases of Theorem 4.2. Details on most of them will be provided in forthcoming work.

**Example 4.7.** (a) Corollary 4.3 follows from Theorem 4.2 by letting \( w \) be the function which is constantly 1.

(b) Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) be a composition of a nonnegative integer \( n \), and let \( E = (E, <_1, <_2) \) be the double poset defined in Example 3.6 (b). Let \( w : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\} \) be the map sending every \( i \) to \( \alpha_i \). As Example 3.6 (b) shows, we have \( \Gamma(E, w) = M_\alpha \). Thus, applying Theorem 4.2 to these \( E \) and \( w \) yields

\[
S(M_\alpha) = (-1)^\ell \Gamma(((E, >_1, <_2), w) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = (-1)^\ell \sum_{\gamma \text{ is a composition of } n; D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))} M_\gamma.
\]

This is the formula for \( S(M_\alpha) \) given in [Malve93, (4.26)], in [GriRei14, Theorem 5.11], and in [BenSag14, Theorem 4.1] (originally due to Ehrenborg and to Malvenuto and Reutenauer). It also shows that the \( \Gamma(E, w) \) for varying \( E \) and \( w \) span the \( k \)-module QSym.

(c) Applying Corollary 4.3 to the double poset of Example 3.6 (c) (where the relation \(<_2 \) is chosen to be a total order) yields the formula for the antipode of a fundamental quasisymmetric function ([Malve93, (4.27)], [GriRei14, (5.9)], [BenSag14, Theorem 5.1]).

(d) Let us use the notations of Example 3.3. For any partition \( \lambda \), let \( \lambda^t \) denote the conjugate partition of \( \lambda \). Let \( \mu \) and \( \lambda \) be two partitions satisfying \( \mu \subset \lambda \).
Then, there is a bijection $\tau : Y(\lambda/\mu) \to Y(\lambda^t/\mu^t)$ sending each $(i,j) \in Y(\lambda/\mu)$ to $(j,i)$. This bijection is an isomorphism of double posets from $(Y(\lambda/\mu), >_1, <_2)$ to $(Y(\lambda^t/\mu^t), >_1, >_2)$. Thus, applying Corollary 4.3 to the tertispecial double poset $Y(\lambda/\mu)$, we obtain

$$S(\Gamma(Y(\lambda/\mu))) = (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda/\mu), >_1, <_2))$$

$$= (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda^t/\mu^t), >_1, >_2)). \tag{3}$$

But from Example 3.6 (d), we know that $\Gamma(Y(\lambda/\mu)) = s_{\lambda/\mu}$. Moreover, a similar argument using [GriRei14, Remark 2.12] shows that $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$. Hence, (3) rewrites as

$$S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}. \tag{4}$$

This is a well-known formula, and is usually stated for $S$ being the antipode of the Hopf algebra of symmetric (rather than quasisymmetric) functions, but the latter antipode is a restriction of the antipode of QSym.

It is also possible (but more difficult) to derive (4) by using the double poset $Y_h(\lambda/\mu)$ instead of $Y(\lambda/\mu)$. (This boils down to what was done in [GriRei14, proof of Corollary 5.29].)

(e) Two results of Benedetti and Sagan [BenSag14, Theorems 8.1–8.2] on the antipodes of immaculate functions can be obtained from Corollary 4.3 using dualization.

5. Lemmas: packed $E$-partitions and comultiplications

We shall now prepare for the proofs of our results. To this end, we introduce the notion of a packed map.

**Definition 5.1.** (a) An initial interval will mean a set of the form $\{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$.

(b) If $T$ is a set and $\pi : T \to \{1, 2, 3, \ldots\}$ is a map, then $\pi$ is said to be packed if $\pi(T)$ is an initial interval. Clearly, this initial interval must be $\{1, 2, \ldots, |\pi(T)|\}$.

**Proposition 5.2.** Let $E = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. For every packed map $\pi : E \to \{1, 2, 3, \ldots\}$, we define $ev_w \pi$ to be the composition $(a_1, a_2, \ldots, a_\ell)$, where $\ell = |\pi(E)|$ (so that $\pi(E) = \{1, 2, \ldots, \ell\}$,
since \( \pi \) is packed), and where each \( \alpha_i \) is defined as \( \sum_{e \in \pi^{-1}(i)} w(e) \). Then,

\[
\Gamma(E, w) = \sum_{\varphi \text{ is a packed } E\text{-partition}} M_{\text{ev}_w \varphi}.
\] (5)

**Proof of Proposition 5.2.** For every finite subset \( T \) of \( \{1, 2, 3, \ldots\} \), there exists a unique strictly increasing bijection \( \{1, 2, \ldots, |T|\} \to T \). We shall denote this bijection by \( r_T \). For every map \( \pi : E \to \{1, 2, 3, \ldots\} \), we define the packing of \( \pi \) as the map \( r_{\pi(E)} \circ \pi : E \to \{1, 2, 3, \ldots\} \); this is a packed map (indeed, its image is \( \{1, 2, \ldots, |\pi(E)|\} \)), and will be denoted by \( \text{pack} \pi \). This map \( \text{pack} \pi \) is an \( E \)-partition if and only if \( \pi \) is an \( E \)-partition.

We shall show that for every packed \( E \)-partition \( \varphi \), we have

\[
\sum_{\pi \text{ is an } E\text{-partition}; \text{pack} \pi = \varphi} x_{\pi, w} = M_{\text{ev}_w \varphi}.
\] (6)

Once this is proven, it will follow that

\[
\Gamma(E, w) = \sum_{\pi \text{ is an } E\text{-partition}} x_{\pi, w} = \sum_{\varphi \text{ is a packed } E\text{-partition}} \sum_{\pi \text{ is an } E\text{-partition}; \text{pack} \pi = \varphi} x_{\pi, w} = M_{\text{ev}_w \varphi} \quad \text{(by (6))}
\]

(since \( \text{pack} \pi \) is a packed \( E \)-partition for every \( E \)-partition \( \pi \))

\[
= \sum_{\varphi \text{ is a packed } E\text{-partition}} M_{\text{ev}_w \varphi},
\]

and Proposition 5.2 will be proven.

So it remains to prove (6). Let \( \varphi \) be a packed \( E \)-partition. Let \( \ell = |\varphi(E)| \); thus \( \varphi(E) = \{1, 2, \ldots, \ell\} \) (since \( \varphi \) is packed). Let \( \alpha_i = \sum_{e \in \varphi^{-1}(i)} w(e) \) for every \( i \in \{1, 2, \ldots, \ell\} \); thus, \( \text{ev}_w \varphi = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) (by the definition of \( \text{ev}_w \varphi \)). Then, the

---

\[\footnote{Indeed, \( \text{pack} \pi = r_{\pi(E)}^{-1} \circ \pi \). Since \( r_{\pi(E)} \) is strictly increasing, we thus see that, for any given \( e \in E \) and \( f \in E \), the equivalences}

\[(\text{pack} \pi)(e) \leq (\text{pack} \pi)(f) \iff (\pi(e) \leq \pi(f))
\]

and

\[(\text{pack} \pi)(e) < (\text{pack} \pi)(f) \iff (\pi(e) < \pi(f))
\]

hold. Hence, \( \text{pack} \pi \) is an \( E \)-partition if and only if \( \pi \) is an \( E \)-partition.
right hand side of (6) rewrites as follows:

\[
\begin{align*}
M_{\text{ev}} \varphi &= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{a_k} = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(r) \\
&= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) \\
&= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) \\
&= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} x_{i_k}^{\varphi^{-1}(k)} w(e) \\
&= \sum_{T \subseteq \{1,2,3,\ldots\}; |T|=\ell} \prod_{e \in T} x_{r_T(\varphi(e))}^{w(e)} = \sum_{T \subseteq \{1,2,3,\ldots\}; |T|=\ell} \prod_{e \in T} x_{r_T(\varphi(e))}^{w(e)} \\
&= \prod_{e \in E} x_{r_T(\varphi(e))}^{w(e)}
\end{align*}
\]

On the other hand, recall that \( \varphi \) is an \( E \)-partition. Hence, every map \( \pi \) satisfying \( \text{pack} \pi = \varphi \) is an \( E \)-partition (because, as we know, pack \( \pi \) is an \( E \)-partition if and only if \( \pi \) is an \( E \)-partition). Thus, the \( E \)-partitions \( \pi \) satisfying \( \text{pack} \pi = \varphi \) are precisely the maps \( \pi : E \to \{1,2,3,\ldots\} \) satisfying \( \text{pack} \pi = \varphi \). Hence,

\[
\sum_{\pi \text{ is an } E\text{-partition}; \text{pack } \pi = \varphi} x_{\pi,\varphi} = \sum_{\pi : E \to \{1,2,3,\ldots\}; \text{pack } \pi = \varphi} x_{\pi,\varphi}
\]

(because if \( \pi : E \to \{1,2,3,\ldots\} \) is a map satisfying \( \text{pack} \pi = \varphi \), then \( |\pi(E)| = \ell \)). But for every \( \ell \)-element subset \( T \) of \( \{1,2,3,\ldots\} \), there exists exactly one \( \pi : T \to \{1,2,3,\ldots\} \).

In the second-to-last equality, we have used the fact that the strictly increasing sequences \((i_1 < i_2 < \cdots < i_\ell)\) of positive integers are in bijection with the subsets \( T \subseteq \{1,2,3,\ldots\} \) such that \(|T| = \ell\). The bijection sends a sequence \((i_1 < i_2 < \cdots < i_\ell)\) to the set of its entries; its inverse map sends every \( T \) to the sequence \((r_T(1), r_T(2), \ldots, r_T(|T|))\).

**Proof.** Let \( \pi : E \to \{1,2,3,\ldots\} \) be a map satisfying \( \text{pack} \pi = \varphi \). The definition of pack \( \pi \) yields \( \text{pack} \pi = r^{-1}_{\pi(E)} \circ \pi \). Hence, \(|(\text{pack} \pi) (E)| = |(r^{-1}_{\pi(E)} \circ \pi) (E)| = |r^{-1}_{\pi(E)}(\pi(E))| = |\pi(E)| \) (since \( r^{-1}_{\pi(E)} \) is a bijection). Since pack \( \pi = \varphi \), this rewrites as \(|\varphi(E)| = |\pi(E)| \). Hence, \(|\pi(E)| = |\varphi(E)| = \ell\), qed.
\[ E \to \{1,2,3,\ldots\} \] satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \): namely, \( \pi = r_T \circ \varphi \).

Therefore, for every \( \ell \)-element subset \( T \) of \( \{1,2,3,\ldots\} \), we have
\[
\sum_{\pi: E \to \{1,2,3,\ldots\}; \text{pack } \pi = \varphi; \pi(E) = T} x_{\pi,w} = x_{r_T \circ \varphi,w}.
\]

Hence,
\[
\sum_{\pi \text{ is an } E\text{-partition; pack } \pi = \varphi} x_{\pi,w} = \sum_{T \subseteq \{1,2,3,\ldots\}; |T| = \ell} \sum_{\pi: E \to \{1,2,3,\ldots\}; \text{pack } \pi = \varphi; \pi(E) = T} x_{\pi,w} = \sum_{T \subseteq \{1,2,3,\ldots\}; |T| = \ell} x_{r_T \circ \varphi,w} = M_{\text{ev}_w \varphi}
\]
(by (7)). Thus, (6) is proven, and with it Proposition 3.5.

**Proof of Proposition 3.5** Proposition 3.5 follows immediately from Proposition 5.2.

We shall now describe the coproduct of \( \Gamma (E, w) \), essentially giving the proof that is left to the reader in [MalReu09, Theorem 2.2].

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16Proof. Let \( T \) be an \( \ell \)-element subset of \( \{1,2,3,\ldots\} \). We need to show that there exists exactly one \( \pi : E \to \{1,2,3,\ldots\} \) satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \): namely, \( \pi = r_T \circ \varphi \). In other words, we need to prove the following two claims:

**Claim 1:** The map \( r_T \circ \varphi \) is a map \( \pi : E \to \{1,2,3,\ldots\} \) satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \).

**Claim 2:** If \( \pi : E \to \{1,2,3,\ldots\} \) is a map satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \), then \( \pi = r_T \circ \varphi \).

**Proof of Claim 1.** We have \( (r_T \circ \varphi) (E) = r_T \left( \varphi \left( E \right) \right) \left( = \{1,2,\ldots,\ell\} \right) \)
\begin{align*}
&= r_T \left( \left\{ 1,2,\ldots,\ell \right\} \right) \left( \text{since } T \text{ is } \ell\text{-element} \right) \\
&= r_T \left( \{1,2,\ldots,|T|\} \right) = T \text{ (by the definition of } r_T). \\
&= \left\{ 1,2,\ldots,\ell \right\} \left( = \text{the definition of pack } (r_T \circ \varphi) \right)
\end{align*}

Thus, the map \( r_T \circ \varphi : E \to \{1,2,3,\ldots\} \) satisfies \( \text{pack } (r_T \circ \varphi) = \varphi \) and \( (r_T \circ \varphi) (E) = T \). In other words, \( r_T \circ \varphi \) is a map \( \pi : E \to \{1,2,3,\ldots\} \) satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \). This proves Claim 1.

**Proof of Claim 2.** Let \( \pi : E \to \{1,2,3,\ldots\} \) be a map satisfying \( \text{pack } \pi = \varphi \) and \( \pi (E) = T \).

The definition of pack \( \pi \) shows that \( \text{pack } \pi = r_{\pi(E)}^{-1} \circ \pi = r_T^{-1} \circ \pi \) (since \( \pi (E) = T \)). Hence, \( r_T^{-1} \circ \pi = \text{pack } \pi = \varphi \), so that \( \pi = r_T \circ \varphi \). This proves Claim 2.

Now, both Claims 1 and 2 are proven; hence, our proof is complete.
Definition 5.3. Let $E = (E, \prec_1, \prec_2)$ be a double poset.

(a) Then, $\text{Adm} E$ will mean the set of all pairs $(P, Q)$, where $P$ and $Q$ are subsets of $E$ satisfying $P \cap Q = \emptyset$ and $P \cup Q = E$ and having the property that no $p \in P$ and $q \in Q$ satisfy $q \prec_1 p$. These pairs $(P, Q)$ are called the admissible partitions of $E$. (In the terminology of [MalReu09], they are the decompositions of $(E, \prec_1)$.)

(b) For any subset $T$ of $E$, we let $E \mid_T$ denote the double poset $(T, \prec_1, \prec_2)$, where $\prec_1$ and $\prec_2$ (by abuse of notation) denote the restrictions of the relations $\prec_1$ and $\prec_2$ to $T$.

Proposition 5.4. Let $E = (E, \prec_1, \prec_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta (\Gamma (E, w)) = \sum_{(P, Q) \in \text{Adm} E} \Gamma (E \mid_P, w \mid_P) \otimes \Gamma (E \mid_Q, w \mid_Q).$$

A particular case of Proposition 5.4 (namely, the case when $w(e) = 1$ for each $e \in E$) appears in [Malve93, Théorème 4.16].

We shall now outline a proof of this fact. The proof relies on a simple bijection that an experienced combinatorialist will have no trouble finding (and proving even less); let us just give a brief outline of the argument\(^\text{17}\).

Proof of Proposition 5.4. Whenever $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition and $k \in \{0, 1, \ldots, \ell\}$, we introduce the notation $\alpha [\cdot : k]$ for the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, and the notation $\alpha [k :]$ for the composition $(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_\ell)$. Now, the formula (9) can be rewritten as follows:

$$\Delta (M_\alpha) = \sum_{k=0}^{\ell} M_\alpha[k] \otimes M_\alpha[k]$$

for every $\ell \in \mathbb{N}$ and every composition $\alpha$ with $\ell$ entries.

Now, applying $\Delta$ to the equality (5) yields

$$\Delta (\Gamma (E, w)) = \sum_{\varphi \text{ is a packed } E\text{-partition}} \Delta (M_{\text{ev}_w \varphi})$$

$$= \sum_{k=0}^{\varphi(E)} M_{(\text{ev}_w \varphi)[k]} \otimes M_{(\text{ev}_w \varphi)[k]} \quad \text{(by (9))}$$

$$= \sum_{\varphi \text{ is a packed } E\text{-partition}} \sum_{k=0}^{\varphi(E)} M_{(\text{ev}_w \varphi)[k]} \otimes M_{(\text{ev}_w \varphi)[k]}.$$
On the other hand, rewriting each of the tensorands on the right hand side of \((8)\) using \((5)\), we obtain

\[
\sum_{(P,Q)\in \text{Adm } E} \Gamma (E \vert_p, w \vert_p) \otimes \Gamma (E \vert_Q, w \vert_Q)
\]

\[
= \sum_{(P,Q)\in \text{Adm } E} \left( \sum_{\varphi \text{ is a packed } E\vert_p\text{-partition}} M_{\text{ev}_w\varphi} \right) \otimes \left( \sum_{\varphi \text{ is a packed } E\vert_Q\text{-partition}} M_{\text{ev}_w\varphi} \right)
\]

\[
= \sum_{(P,Q)\in \text{Adm } E} \left( \sum_{\sigma \text{ is a packed } E\vert_p\text{-partition}} M_{\text{ev}_w\sigma} \right) \otimes \left( \sum_{\tau \text{ is a packed } E\vert_Q\text{-partition}} M_{\text{ev}_w\tau} \right)
\]

\[
= \sum_{(P,Q)\in \text{Adm } E} \sum_{\sigma \text{ is a packed } E\vert_p\text{-partition}} \sum_{\tau \text{ is a packed } E\vert_Q\text{-partition}} M_{\text{ev}_w\sigma} \otimes M_{\text{ev}_w\tau}.
\]

We need to prove that the right hand sides of this equality and of \((10)\) are equal (because then, it will follow that so are the left hand sides, and thus Proposition 5.4 will be proven). For this, it is clearly enough to exhibit a bijection between

- the pairs \((\varphi, k)\) consisting of a packed \(E\)-partition \(\varphi\) and a \(k \in \{0, 1, \ldots, |\varphi(E)|\}\)

and

- the triples \(((P,Q), \sigma, \tau)\) consisting of a \((P,Q)\) \(\in\) \text{Adm } \(E\), a packed \(E \vert_p\text{-partition } \sigma\) and a packed \(E \vert_Q\text{-partition } \tau\)

which bijection has the property that whenever it maps \((\varphi, k)\) to \(((P,Q), \sigma, \tau)\), we have the equalities \(\text{ev}_w \varphi [k] = \text{ev}_w \sigma[\tau]\) and \(\text{ev}_w \varphi [k] = \text{ev}_w \tau[k]\). Such a bijection is easy to construct: Given \((\varphi, k)\), it sets \(P = \varphi^{-1}([1, 2, \ldots, k])\), \(Q = \varphi^{-1}([k+1, k+2, \ldots, |\varphi(E)|])\), \(\sigma = \varphi \vert_p\) and \(\tau = \text{pack } \varphi \vert_Q\)\(^{18}\) Conversely, given \(((P,Q), \sigma, \tau)\), the inverse bijection sets \(k = |\sigma(P)|\) and constructs \(\varphi\) as the map \(E \to \{1, 2, 3, \ldots\}\) which sends every \(e \in E\) to \(\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q. \end{cases}\) Proving that this alleged bijection and its alleged inverse bijection are well-defined and actually mutually inverse is straightforward and left to the reader\(^{19}\).

\(^{18}\)We notice that these \(P, Q, \sigma\) and \(\tau\) satisfy \(\sigma(e) = \varphi(e)\) for every \(e \in P\), and \(\tau(e) = \varphi(e) - k\) for every \(e \in Q\).

\(^{19}\)The only part of the argument that is a bit trickier is proving the well-definedness of the inverse bijection. We need to show that if \(((P,Q), \sigma, \tau)\) is a triple consisting of a \((P,Q) \in \text{Adm } E\), a packed \(E \vert_p\text{-partition } \sigma\) and a packed \(E \vert_Q\text{-partition } \tau\), and if we set \(k = |\sigma(P)|\), then the map \(\varphi : E \to \{1, 2, 3, \ldots\}\) which sends every \(e \in E\) to \(\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}\) is actually a packed \(E\)-partition.

Indeed, it is clear that this map \(\varphi\) is packed. It remains to show that it is an \(E\)-partition. To do so, we must prove the following two claims:

**Claim 1:** Every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) satisfy \(\varphi(e) \leq \varphi(f)\).
We note in passing that there is also a rule for multiplying quasisymmetric functions of the form \( \Gamma(E,w) \). Namely, if \( E \) and \( F \) are two double posets and \( u \) and \( v \) are corresponding maps, then \( \Gamma(E,u) \Gamma(F,v) = \Gamma(\text{EF},w) \) for a map \( w \) which is defined to be \( u \) on the subset \( E \) of \( \text{EF} \), and \( v \) on the subset \( F \) of \( \text{EF} \). Here, \( \text{EF} \) is a double poset defined as in \cite[\S2.1]{MalReu09}. Combined with Proposition 3.5, this fact gives a combinatorial proof for the fact that \( \text{QSym} \) is a \( \mathbb{k} \)-algebra, as well as for some standard formulas for multiplications of quasisymmetric functions; similarly, Proposition 5.4 can be used to derive the well-known formulas for \( \Delta M_{\alpha}, \Delta L_{\alpha}, \Delta s_{\lambda/\mu} \) etc. (although, of course, we have already used the formula for \( \Delta M_{\alpha} \) in our proof of Proposition 5.4).

6. Proof of Theorem 4.2

Before we come to the proof of Theorem 4.2, let us state three simple lemmas:

**Lemma 6.1.** Let \( E = (E, <_1, <_2) \) be a double poset. Let \( P \) and \( Q \) be subsets of \( E \) such that \( P \cap Q = \emptyset \) and \( P \cup Q = E \). Assume that there exist no \( p \in P \) and \( q \in Q \) such that \( q \) is \( <_1 \)-covered by \( p \). Then, \( (P, Q) \in \text{Adm} E \).

**Proof of Lemma 6.1** For any \( a \in E \) and \( b \in E \), we let \([a,b]\) denote the subset \( \{ e \in E \mid a <_1 e <_1 b \} \) of \( E \). It is clear that if \( a, b \) and \( c \) are three elements of \( E \) satisfying \( a <_1 c <_1 b \), then both \([a,c]\) and \([c,b]\) are proper subsets of \([a,b]\), and therefore both numbers \(|[a,c]|\) and \(|[c,b]|\) are smaller than \(|[a,b]|\). (11)

A pair \((p,q) \in P \times Q\) is said to be a *malposition* if it satisfies \( q <_1 p \). Now, let us assume (for the sake of contradiction) that there exists a malposition. Fix a malposition \((u,v)\) for which the value \(|[u,v]|\) is minimum. Thus, \( u \in P, v \in Q \) and \( v <_1 u \), but \( v \) is not \( <_1 \)-covered by \( u \) (since there exist no \( p \in P \) and \( q \in Q \) such that \( q \) is \( <_1 \)-covered by \( p \)). Hence, there exists a \( w \in E \) such that \( v <_1 w <_1 u \)

---

Claim 2: Every \( e \in E \) and \( f \in E \) satisfying \( e <_1 f \) and \( f <_2 e \) satisfy \( \varphi(e) < \varphi(f) \).

We shall only prove Claim 1 (as the proof of Claim 2 is similar). So let \( u \in E \) and \( f \in E \) be such that \( e <_1 f \). We need to show that \( \varphi(e) \leq \varphi(f) \). We are in one of the following four cases:

- **Case 1:** We have \( e \in P \) and \( f \in P \).
- **Case 2:** We have \( e \in P \) and \( f \in Q \).
- **Case 3:** We have \( e \in Q \) and \( f \in P \).
- **Case 4:** We have \( e \in Q \) and \( f \in Q \).

In Case 1, our claim \( \varphi(e) \leq \varphi(f) \) follows from the assumption that \( \sigma \) is an \( E \mid p \)-partition (because in Case 1, we have \( \varphi(e) = \sigma(e) \) and \( \varphi(f) = \sigma(f) \)). In Case 4, it follows from the assumption that \( \tau \) is an \( E \mid q \)-partition (since in Case 4, we have \( \varphi(e) = \tau(e) + k \) and \( \varphi(f) = \tau(f) + k \)). In Case 2, it clearly holds (indeed, if \( e \in P \), then the definition of \( \varphi \) yields \( \varphi(e) = \sigma(e) \leq k \), and if \( f \in Q \), then the definition of \( \varphi \) yields \( \varphi(f) = \tau(f) + k > k \); therefore, in Case 2, we have \( \varphi(e) \leq k < \varphi(f) \)). Finally, Case 3 is impossible (because having \( e \in Q \) and \( f \in P \) and \( e <_1 f \) would contradict \((P, Q) \in \text{Adm} E \)). Thus, we have proven the claim in each of the four cases, and consequently Claim 1 is proven. As we have said above, Claim 2 is proven similarly.
(since \( v <_1 u \)). Consider this \( w \). Applying (11) to \( a = v, c = w \) and \( b = u \), we see that both numbers \( |[u, w]| \) and \( |[w, v]| \) are smaller than \( |[u, v]| \), and therefore neither \((u, w)\) nor \((w, v)\) is a malposition (since we picked \((u, v)\) to be a malposition with minimum \( |[u, v]| \)). But \( w \in E = P \cup Q \), so that either \( w \in P \) or \( w \in Q \). If \( w \in P \), then \((w, v)\) is a malposition; if \( w \in Q \), then \((u, w)\) is a malposition. In either case, we obtain a contradiction to the fact that neither \((u, w)\) nor \((w, v)\) is a malposition. This contradiction shows that our assumption was wrong. Hence, there exists no malposition. Consequently, \((P, Q) \in \text{Adm } E \). \( \square \)

**Lemma 6.2.** Let \( E = (E, <_1, <_2) \) be a tertispecial double poset. Let \((P, Q) \in \text{Adm } E \). Then, \( E |_P \) is a tertispecial double poset.

**Proof of Lemma 6.2** We need to show that the double poset \( E |_P = (P, <_1, <_2) \) is tertispecial. In other words, we need to show that if \( a \) and \( b \) are two elements of \( P \) such that \( a \) is \( <_1 \)-covered by \( b \) as element of the set \( P \), then \( a \) and \( b \) are \( <_2 \)-comparable.

Let \( a \) and \( b \) be two elements of \( P \) such that \( a \) is \( <_1 \)-covered by \( b \) as element of the set \( P \). Thus, \( a <_1 b \), and

there exists no \( c \in P \) satisfying \( a <_1 c <_1 b \). \( (12) \)

Now, if \( c \in E \) is such that \( a <_1 c <_1 b \), then \( c \) must belong to \( P \) \( \square \) which entails a contradiction to \((12) \). Thus, there is no \( c \in E \) satisfying \( a <_1 c <_1 b \). Therefore (and because we have \( a <_1 b \)), we see that \( a \) is \( <_1 \)-covered by \( b \) as element of the set \( E \). Since \( E \) is tertispecial, this yields that \( a \) and \( b \) are \( <_2 \)-comparable.

Thus, we have shown that if \( a \) and \( b \) are two elements of \( P \) such that \( a \) is \( <_1 \)-covered by \( b \) as element of the set \( P \), then \( a \) and \( b \) are \( <_2 \)-comparable. This proves Lemma 6.2.

(We could similarly show that \( E |_Q \) is a tertispecial double poset; but we will not use this.) \( \square \)

**Lemma 6.3.** Let \( E = (E, <_1, <_2) \) be a double poset. Let \( w : E \to \{1, 2, 3, \ldots\} \) be a map.

(a) If \( E = \emptyset \), then \( \Gamma (E, w) = 1 \).

(b) If \( E \neq \emptyset \), then \( \varepsilon (\Gamma (E, w)) = 0 \).

---

\( ^{20} \)Proof. Assume the contrary. Thus, \( c \notin P \). But \((P, Q) \in \text{Adm } E \). Thus, \( P \cap Q = \emptyset \), \( P \cup Q = E \), and no \( p \in P \) and \( q \in Q \) satisfy \( q <_1 p \). \( (13) \)

From \( c \in E \) and \( c \notin P \), we obtain \( c \in E \setminus P \subseteq Q \) (since \( P \cup Q = E \)). Applying \((13) \) to \( p = b \) and \( q = c \), we thus conclude that we cannot have \( c <_1 b \). This contradicts \( c <_1 b \). This contradiction shows that our assumption was false, qed.
Proof of Lemma 6.3 (a) Part (a) is obvious (since there is only one \(E\)-partition when \(E = \varnothing\)).

(b) Observe that \(\Gamma (E, w)\) is a homogeneous power series of degree \(\sum_{e \in E} w(e)\).

When \(E \neq \varnothing\), this degree is \(> 0\), and thus the power series \(\Gamma (E, w)\) is annihilated by \(\varepsilon\) (since \(\varepsilon\) annihilates any homogeneous power series in \(Q\text{Sym}\) whose degree is \(> 0\)).

Proof of Theorem 4.2 We shall prove Theorem 4.2 by induction over \(|E|\). The induction base \(|E| = 0\) is left to the reader; we start with the induction step.

Consider a tertispecial double poset \(E = (E, <_1, <_2)\) with \(|E| > 0\) and a map \(w : E \to \{1, 2, 3, \ldots\}\), and assume that Theorem 4.2 is proven for all tertispecial double posets of smaller size.

We have \(|E| > 0\) and thus \(E \neq \varnothing\). Hence, Lemma 6.3 (b) shows that \(\varepsilon(\Gamma (E, w)) = 0\). Thus, \((u \circ \varepsilon)(\Gamma (E, w)) = u(\varepsilon(\Gamma (E, w))) = u(0) = 0\).

The upper commutative pentagon of (2) shows that \(u \circ \varepsilon = m \circ (S \otimes \text{id}) \circ \Delta\). Applying both sides of this equality to \(\Gamma (E, w)\), we obtain \((u \circ \varepsilon)(\Gamma (E, w)) = (m \circ (S \otimes \text{id}) \circ \Delta)(\Gamma (E, w))\). Since \((u \circ \varepsilon)(\Gamma (E, w)) = 0\), this becomes

\[
0 = (m \circ (S \otimes \text{id}) \circ \Delta)(\Gamma (E, w)) = m((S \otimes \text{id})(\Delta(\Gamma (E, w))))
\]

\[
= m\left((S \otimes \text{id})\left(\sum_{(P,Q) \in \text{Adm}_E} \Gamma (E | P, w | P) \otimes \Gamma (E | Q, w | Q)\right)\right) \quad \text{(by (8))}
\]

\[
= \sum_{(P,Q) \in \text{Adm}_E} S(\Gamma (E | P, w | P)) \Gamma (E | Q, w | Q)
\]

\[
= S(\Gamma (E | E, w | E)) \Gamma (E | \varnothing, w | \varnothing) + \sum_{(P,Q) \in \text{Adm}_E; |P| < |E|} S(\Gamma (E | P, w | P)) \Gamma (E | Q, w | Q)
\]

\[
= S(\Gamma (E | E, w | E)) \Gamma (E | \varnothing, w | \varnothing) + \sum_{(P,Q) \in \text{Adm}_E; |P| < |E|} S(\Gamma ((P, <_1, <_2), w | P)) = (-1)^{|P|} \Gamma ((P, >_1, <_2), w | P) \quad \text{(by the induction hypothe-}
\]


sis). Hence, (14) rewrites as

\[ 0 = S \left( \Gamma \left( \begin{array}{c} E \\ w \\ \end{array} \right) \right) \left( \begin{array}{c} E \\ w \\ \end{array} \right) + \sum_{(P,Q) \in \text{Adm} E; |P| < |E|} (-1)^{|P|} \Gamma \left( \begin{array}{c} (P, >_1, <_2) \\ w \end{array} \right) \Gamma \left( \begin{array}{c} E \\ w \end{array} \right) \]

Thus,

\[ S \left( \Gamma \left( E, w \right) \right) = - \sum_{(P,Q) \in \text{Adm} E; |P| < |E|} (-1)^{|P|} \Gamma \left( \begin{array}{c} (P, >_1, <_2) \\ w \end{array} \right) \Gamma \left( \begin{array}{c} E \\ w \end{array} \right). \quad (15) \]

We shall now prove that

\[ 0 = \sum_{(P,Q) \in \text{Adm} E} (-1)^{|P|} \Gamma \left( \begin{array}{c} (P, >_1, <_2) \\ w \end{array} \right) \Gamma \left( \begin{array}{c} E \\ w \end{array} \right). \quad (16) \]

But first, let us explain how this will complete our proof. In fact, the only pair \((P,Q) \in \text{Adm} E\) satisfying \(|P| = |E|\) is \((E, \emptyset)\), whereas all other pairs \((P,Q) \in \text{Adm} E\) satisfy \(|P| < |E|\). Hence, if (16) is proven, then we can conclude that

\[ 0 = \sum_{(P,Q) \in \text{Adm} E} (-1)^{|P|} \Gamma \left( \begin{array}{c} (P, >_1, <_2) \\ w \end{array} \right) \Gamma \left( \begin{array}{c} E \\ w \end{array} \right). \]
so that

\[ (-1)^{|E|} \Gamma \left( (E, >_1, <_2), w \right) = - \sum_{(P, Q) \in \text{Adm } E; \ |P| < |E|} (-1)^{|P|} \Gamma \left( (P, >_1, <_2), w \mid_P \right) \Gamma \left( E \mid_Q, w \mid_Q \right) \]

\[ \quad = S \left( \Gamma \left( \left( \frac{E}{x = (E, >_1, <_2)}, w \right) \right) \right) \quad \text{(by (15))} \]

\[ \quad = S \left( \Gamma \left( (E, <_1, <_2), w \right) \right), \]

and thus \( S \left( \Gamma \left( (E, <_1, <_2), w \right) \right) = (-1)^{|E|} \Gamma \left( (E, >_1, <_2), w \right), \) which completes the induction step and thus the proof of Theorem 4.2. It thus remains to prove (16).

For every subset \( P \) of \( E \), we have

\[ \Gamma \left( (P, >_1, <_2), w \mid_P \right) = \sum_{\pi \text{ is a } (P, >_1, <_2)-\text{partition}} x_{\pi, w \mid_P} \]

(by the definition of \( \Gamma \left( (P, >_1, <_2), w \mid_P \right) \))

\[ \quad = \sum_{\sigma \text{ is a } (P, >_1, <_2)-\text{partition}} x_{\sigma, w \mid_P}. \quad \text{(17)} \]

For every subset \( Q \) of \( E \), we have

\[ \Gamma \left( \left( \frac{E}{x = (Q, <_1, <_2)}, w \mid_Q \right) \right) = \Gamma \left( ((Q, <_1, <_2), w \mid_Q) \right) \]

\[ \quad = \sum_{\pi \text{ is a } (Q, <_1, <_2)-\text{partition}} x_{\pi, w \mid_Q} \]

(by the definition of \( \Gamma \left( ((Q, <_1, <_2), w \mid_Q) \right) \))

\[ \quad = \sum_{\tau \text{ is a } (Q, <_1, <_2)-\text{partition}} x_{\tau, w \mid_Q}. \quad \text{(18)} \]
Double posets and the antipode of $\pi$ of Theorem 4.2), it therefore is enough to show that for every map $\pi$ of 1, 2, 3, \ldots, we have

$$\sum_{(P,Q) \in \text{Adm } E} (-1)^{|P|} \Gamma ((P, >_1, \leq_2), w | p) \sum_{\sigma \text{ is a } (P, >_1, \leq_2)\text{-partition} \text{ (by [17])}} \sum_{\tau \text{ is a } (Q, <_1, \leq_2)\text{-partition} \text{ (by [18])}} x_{\sigma, w | p} x_{\tau, w | Q} = \Gamma (E | Q, w | Q) \sum_{\sigma \text{ is a } (P, >_1, \leq_2)\text{-partition}} x_{\sigma, w | p} \sum_{\tau \text{ is a } (Q, <_1, \leq_2)\text{-partition}} x_{\tau, w | Q}$$

$$= \sum_{(P,Q) \in \text{Adm } E} (-1)^{|P|} \sum_{\sigma \text{ is a } (P, >_1, \leq_2)\text{-partition}} \sum_{\tau \text{ is a } (Q, <_1, \leq_2)\text{-partition}} x_{\sigma, w | p} x_{\tau, w | Q}$$

$$= \sum_{(P,Q) \in \text{Adm } E} (-1)^{|P|} \sum_{(\sigma, \tau)} x_{\sigma, w | p} x_{\tau, w | Q}$$

here, we have substituted $(\pi | p, \pi | Q)$ for $(\sigma, \tau)$ in the inner sum, since every pair $(\sigma, \tau)$ consisting of a map $\sigma : P \to \{1, 2, 3, \ldots\}$ and a map $\tau : Q \to \{1, 2, 3, \ldots\}$ can be written as $(\pi | p, \pi | Q)$ for a unique $\pi : E \to \{1, 2, 3, \ldots\}$ (namely, for the $\pi : E \to \{1, 2, 3, \ldots\}$ that is defined to send every $e \in P$ to $\sigma (e)$ and to send every $e \in Q$ to $\tau (e)$)

$$= \sum_{(P,Q) \in \text{Adm } E} (-1)^{|P|} \sum_{\pi : E \to \{1, 2, 3, \ldots\}; \pi | p \text{ is a } (P, >_1, \leq_2)\text{-partition}; \pi | Q \text{ is a } (Q, <_1, \leq_2)\text{-partition}} x_{\pi, w}$$

$$= \sum_{\pi : E \to \{1, 2, 3, \ldots\}} \left( \sum_{(P,Q) \in \text{Adm } E; \pi | p \text{ is a } (P, >_1, \leq_2)\text{-partition}; \pi | Q \text{ is a } (Q, <_1, \leq_2)\text{-partition}} (-1)^{|P|} x_{\pi, w} \right)$$

In order to prove that this sum is 0 (and thus to prove 16 and finish our proof of Theorem 4.2), it therefore is enough to show that for every map $\pi : E \to \{1, 2, 3, \ldots\}$, we have

$$\sum_{(P,Q) \in \text{Adm } E; \pi | p \text{ is a } (P, >_1, \leq_2)\text{-partition}; \pi | Q \text{ is a } (Q, <_1, \leq_2)\text{-partition}} (-1)^{|P|} = 0.$$
Hence, let us fix a map \( \pi : E \to \{1, 2, 3, \ldots \} \). Our goal is now to prove (19). To do so, we denote by \( Z \) the set of all \((P, Q) \in \text{Adm } E\) such that \( \pi \mid _P \) is a \((P, >_1, <_2)\)-partition and \( \pi \mid _Q \) is a \((Q, <_1, <_2)\)-partition. We are going to define an involution \( T : Z \to Z \) of the set \( Z \) having the property that, for any \((P, Q) \in Z\), if we write \( T((P, Q)) \) in the form \((P', Q')\), then \((-1)^{|P|} = -(-1)^{|P|}\). Once such an involution is found, it will be clear that it matches the addends on the left hand side of (19) into pairs of mutually cancelling addends\(^{21}\) and so (19) will follow and we will be done. It thus remains to find \( T \).

The definition of \( T \) is simple (although it will take us a while to prove that it is well-defined): Let \( F \) be the subset of \( E \) consisting of those \( e \in E \) which have minimum \( \pi (e) \). Then, \( F \) is a nonempty subposet of the poset \((E, <_2)\), and hence has a minimal element \( f \) (that is, an element \( f \) such that no \( g \in F \) satisfies \( g <_2 f \)). Fix such an \( f \). Now, the map \( T \) sends a \((P, Q) \in Z\) to \( \begin{cases} (P \cup \{ f \}, Q \setminus \{ f \}), & \text{if } f \not\in P; \\ (P \setminus \{ f \}, Q \cup \{ f \}), & \text{if } f \in P. \end{cases} \)

In order to prove that the map \( T \) is well-defined, we need to prove that its output values all belong to \( Z \). In other words, we need to prove that

\[
\begin{cases} (P \cup \{ f \}, Q \setminus \{ f \}), & \text{if } f \not\in P; \\ (P \setminus \{ f \}, Q \cup \{ f \}), & \text{if } f \in P. \end{cases}
\]

for every \((P, Q) \in Z\).

Proof of (20): Fix \((P, Q) \in Z\). Thus, \((P, Q)\) is an element of \( \text{Adm } E\) with the property that \( \pi \mid _P \) is a \((P, >_1, <_2)\)-partition and \( \pi \mid _Q \) is a \((Q, <_1, <_2)\)-partition.

From \((P, Q) \in \text{Adm } E\), we see that \( P \cap Q = \emptyset \) and \( P \cup Q = E \), and furthermore that

\[
\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p.
\]

We know that \( f \) belongs to the set \( F \), which is the subset of \( E \) consisting of those \( e \in E \) which have minimum \( \pi (e) \). Thus,

\[
\pi (f) \leq \pi (h) \quad \text{for every } h \in E.
\]

Moreover,

\[
\pi (f) < \pi (h) \quad \text{for every } h \in E \text{ satisfying } h <_2 f
\]

\(21\)In fact, the \((-1)^{|P|} = -(-1)^{|P|}\) condition makes it clear that \( T \) has no fixed points. Therefore, to each addend on the left hand side of (19) corresponds an addend with opposite sign, which cancels it: Namely, for each \((A, B) \in (19)\), the addend for \((P, Q) = (A, B)\) is cancelled by the addend for \((P, Q) = T((A, B))\).

\(22\)Proof of (23): Let \( h \in E \) be such that \( h <_2 f \). We must prove (23). Indeed, assume the contrary. Thus, \( \pi (f) \geq \pi (h) \). Combined with (22), this shows that \( \pi (f) = \pi (h) \). Our definition of \( F \) shows that \( F \) is the subset of \( E \) consisting of those \( e \in E \) satisfying \( \pi (e) = \pi (f) \) (since \( f \in F \)). Therefore, \( h \in F \) (since \( \pi (h) = \pi (f) \)). But \( f \) is a minimal element of \( F \). In other words, no \( g \in F \) satisfies \( g <_2 f \). This contradicts the fact that \( h \in F \) satisfies \( h <_2 f \). This contradiction proves that our assumption was wrong, qed.
We need to prove (20). We are in one of the following two cases:

Case 1: We have \( f \in P \).

Case 2: We have \( f \notin P \).

Let us first consider Case 1. In this case, we have \( f \in P \).

Recall that \( P \cap Q = \emptyset \) and \( P \cup Q = E \). From this, we easily obtain \((P \setminus \{f\}) \cap (Q \cup \{f\}) = \emptyset\) and \((P \setminus \{f\}) \cup (Q \cup \{f\}) = E\).

Furthermore, there exist no \( p \in P \setminus \{f\} \) and \( q \in Q \cup \{f\} \) such that \( q \) is \( <_1 \)-covered by \( p \).\footnote{Proof. Assume the contrary. Thus, there exist \( p \in P \setminus \{f\} \) and \( q \in Q \cup \{f\} \) such that \( q \) is \( <_1 \)-covered by \( p \). Consider such \( p \) and \( q \).

We know that \( q \) is \( <_1 \)-covered by \( p \), and thus we have \( q <_1 p \). Also, \( p \in P \setminus \{f\} \subseteq P \). Hence, if we had \( q \in Q \), then we would obtain a contradiction to (21). Hence, we cannot have \( q \in Q \).

Therefore, \( q = f \) (since \( q \in Q \cup \{f\} \) but not \( q \in Q \)). Hence, \( f = q <_1 p \), so that \( p >_1 f \).

Therefore, \( \pi(p) \leq \pi(f) \) (since \( \pi |_p \) is a \((P,>_1,<_2)\)-partition, and since both \( p \) and \( f \) belong to \( P \)).

Now, recall that \( q \) is \( <_1 \)-covered by \( p \). Hence, \( q \) and \( p \) are \( <_2 \)-comparable (since \( E \) is tertispezial). In other words, \( f \) and \( p \) are \( <_2 \)-comparable (since \( q = f \)). In other words, either \( f <_2 p \) or \( f = p \) or \( p <_2 f \). But \( p <_2 f \) cannot hold (because if we had \( p <_2 f \), then \( h = p \) would lead to \( \pi(f) < \pi(p) \), which would contradict \( \pi(p) \leq \pi(f) \)), and \( f = p \) cannot hold either (since \( f <_1 p \)). Thus, we must have \( f <_2 p \).

Now, \( \pi |_p \) is a \((P,>_1,<_2)\)-partition. Hence, \( \pi(p) < \pi(f) \) (since \( p >_1 f \) and \( f <_2 p \), and since \( p \) and \( f \) both lie in \( P \)). But \( \pi(f) \) (applied to \( h = p \)) shows that \( \pi(f) \leq \pi(p) \). Hence, \( \pi(p) < \pi(f) \leq \pi(p) \), a contradiction. Thus, our assumption was wrong, qed.

Proof. In order to prove this, we need to verify the following two claims:

Claim 1: Every \( a \in Q \cup \{f\} \) and \( b \in Q \cup \{f\} \) satisfying \( a <_b b \) satisfy \( \pi(a) \leq \pi(b) \);

Claim 2: Every \( a \in Q \cup \{f\} \) and \( b \in Q \cup \{f\} \) satisfying \( a <_b b \) and \( b <_a a \) satisfy \( \pi(a) \leq \pi(b) \).

Proof of Claim 1: Let \( a \in Q \cup \{f\} \) and \( b \in Q \cup \{f\} \) be such that \( a <_b b \). We need to prove that \( \pi(a) \leq \pi(b) \).

If \( a = f \), then this follows immediately from (21) (applied to \( h = b \)). Hence, we WLOG assume that \( a \neq f \). Thus, \( a \in Q \) (since \( a \in Q \cup \{f\} \)). Now, if \( b \in P \), then \( a <_b b \) contradicts (21) (applied to \( p = b \) and \( q = a \)). Hence, we cannot have \( b \in P \). Therefore, \( b \in E \setminus P = Q \) (since \( P \cap Q = \emptyset \) and \( P \cup Q = E \)). Thus, \( \pi(a) \leq \pi(b) \) follows immediately from the fact that \( \pi |_Q \) is a \((Q,<_1,<_2)\)-partition (since \( a \in Q \) and \( b \in Q \)). This proves Claim 1.

Proof of Claim 2: Let \( a \in Q \cup \{f\} \) and \( b \in Q \cup \{f\} \) be such that \( a <_b b \) and \( b <_a a \). We need to prove that \( \pi(a) < \pi(b) \).

If \( a = f \), then this follows immediately from (23) (applied to \( h = b \)). Hence, we WLOG assume that \( a \neq f \). Thus, \( a \in Q \) (since \( a \in Q \cup \{f\} \)). Now, if \( b \in P \), then \( a <_b b \) contradicts (21) (applied to \( p = b \) and \( q = a \)). Hence, we cannot have \( b \in P \). Therefore, \( b \in E \setminus P = Q \) (since \( P \cap Q = \emptyset \) and \( P \cup Q = E \)). Thus, \( \pi(a) < \pi(b) \) follows immediately from the fact that \( \pi |_Q \) is a \((Q,<_1,<_2)\)-partition (since \( a \in Q \) and \( b \in Q \)). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven, and we are done.
other words, \((P \setminus \{f\}, Q \cup \{f\}) \in Z\) (by the definition of \(Z\)). Thus,

\[
\begin{cases}
(P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \not\in P; \\
(P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P
\end{cases}
\]

(since \(f \in P\))

\[\in Z.\]

Hence, (20) is proven in Case 1.

Let us next consider Case 2. In this case, we have \(f \not\in P\).

Recall that \(P \cap Q = \emptyset\) and \(P \cup Q = E\). From this, we easily obtain \((P \cup \{f\}) \cap (Q \setminus \{f\}) = \emptyset\) and \((P \setminus \{f\}) \cup (Q \setminus \{f\}) = E\).

Furthermore, there exist no \(p \in P \cup \{f\}\) and \(q \in Q \setminus \{f\}\) such that \(q\) is \(<_1\)-covered by \(p\). Hence, Lemma 6.1 (applied to \(P \cup \{f\}\) and \(Q \setminus \{f\}\) instead of \(P\) and \(Q\)) shows that \((P \cup \{f\}, Q \setminus \{f\}) \in \text{Adm} E\).

Furthermore, \(\pi |_Q\) is a \((Q, <_1, <_2)\)-partition, and therefore \(\pi |_{Q \setminus \{f\}}\) is a \((Q \setminus \{f\}, <_1, <_2)\)-partition (since \(Q \setminus \{f\} \subseteq Q\)).

Furthermore, \(\pi |_{P \cup \{f\}}\) is a \((P \cup \{f\}, >_1, <_2)\)-partition.\(^{25}\)

---

25Proof. Assume the contrary. Thus, there exist \(p \in P \cup \{f\}\) and \(q \in Q \setminus \{f\}\) such that \(q\) is \(<_1\)-covered by \(p\). Consider such \(p\) and \(q\).

We have \(f \not\in P\) and thus \(f \in E \setminus P = Q\) (since \(P \cap Q = \emptyset\) and \(P \cup Q = E\)).

We know that \(q\) is \(<_1\)-covered by \(p\), and thus we have \(q < f\). Also, \(q \in Q \setminus \{f\} \subseteq Q\). Hence, if we had \(p \in P\), then we would obtain a contradiction to (21). Hence, we cannot have \(p \in P\). Therefore, \(p = f\) (since \(p \in P \cup \{f\}\) but \(p \not\in P\)). Hence, \(q < f\). Therefore, \(\pi(q) \leq \pi(f)\) (since \(q \in Q\) and \(f \in Q\), and since \(\pi |_Q\) is a \((Q, <_1, <_2)\)-partition). Thus, we cannot have \(q <_2 f\) (because if we had \(q <_2 f\), then (23) (applied to \(h = q\)) would show that \(\pi(f) < \pi(q)\), which would contradict \(\pi(q) \leq \pi(f)\)).

Now, recall that \(q\) is \(<_1\)-covered by \(p\). Hence, \(q\) and \(p\) are \(<_2\)-comparable (since \(E\) is tertiispécial). In other words, \(q\) and \(f\) are \(<_2\)-comparable (since \(p = f\)). In other words, either \(q <_2 f\) or \(q = f\). But we cannot have \(q <_2 f\) (as we have just shown), and we cannot have \(q = f\) either (since \(q <_1 f\)). Thus, we must have \(f <_2 q\).

From \(q < f\) and \(f <_2 q\), we conclude that \(\pi(q) < \pi(f)\) (since \(\pi |_Q\) is a \((Q, <_1, <_2)\)-partition, and since \(q \in Q\) and \(f \in Q\)). But (22) (applied to \(h = q\)) shows that \(\pi(f) \leq \pi(q)\). Hence, \(\pi(q) < \pi(f) \leq \pi(q)\), a contradiction. Thus, our assumption was wrong, qed.

26Proof. In order to prove this, we need to verify the following two claims:

Claim 1: Every \(a \in P \cup \{f\}\) and \(b \in P \cup \{f\}\) satisfying \(a >_1 b\) satisfy \(\pi(a) \leq \pi(b)\);

Claim 2: Every \(a \in P \cup \{f\}\) and \(b \in P \cup \{f\}\) satisfying \(a >_1 b\) and \(b <_2 a\) satisfy \(\pi(a) < \pi(b)\).

Proof of Claim 1: Let \(a \in P \cup \{f\}\) and \(b \in P \cup \{f\}\) be such that \(a >_1 b\). We need to prove that \(\pi(a) \leq \pi(b)\). If \(a = f\), then this follows immediately from (22) (applied to \(h = b\)). Hence, we WLOG assume that \(a \neq f\). Thus, \(a \in P\) (since \(a \in P \cup \{f\}\)). Now, if \(b \in Q\), then \(b <_1 a\) contradicts (21) (applied to \(p = a\) and \(q = b\)). Hence, we cannot have \(b \in Q\). Therefore, \(b \in E \setminus Q = P\) (since \(P \cap Q = \emptyset\) and \(P \cup Q = E\)). Thus, \(\pi(a) \leq \pi(b)\) follows immediately from the fact that \(\pi |_P\) is a \((P, >_1, <_2)\)-partition (since \(a \in P\) and \(b \in P\)). This proves Claim 1.

Proof of Claim 2: Let \(a \in P \cup \{f\}\) and \(b \in P \cup \{f\}\) be such that \(a >_1 b\) and \(b <_2 a\). We need to prove that \(\pi(a) < \pi(b)\). If \(a = f\), then this follows immediately from (23) (applied to \(h = b\)). Hence, we WLOG assume that \(a \neq f\). Thus, \(a \in P\) (since \(a \in P \cup \{f\}\)). Now, if \(b \in Q\), then \(b <_1 a\) contradicts (21) (applied to \(p = a\) and \(q = b\)). Hence, we cannot have \(b \in Q\). Therefore, \(b \in E \setminus Q = P\) (since \(P \cap Q = \emptyset\) and \(P \cup Q = E\)). Thus, \(\pi(a) < \pi(b)\) follows immediately from the fact that \(\pi |_P\) is a \((P, >_1, <_2)\)-partition (since \(a \in P\) and \(b \in P\)). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven, and we are done.
Altogether, we now know that \((P \cup \{f\}, Q \setminus \{f\}) \in \text{Adm } \mathcal{E}\), that \(\pi \mid_{P \cup \{f\}}\) is a 
\((P \cup \{f\}, >_1, <_2)\)-partition, and that \(\pi \mid_{Q \setminus \{f\}}\) is a 
\((Q \setminus \{f\}, >_1, <_2)\)-partition. In other words, \((P \cup \{f\}, Q \setminus \{f\}) \in Z\) (by the definition of \(Z\)). Thus,

\[
\begin{cases}
(P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\
(P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P
\end{cases}
\]

\(= (P \cup \{f\}, Q \setminus \{f\}) \in Z\). (since \(f \notin P\))

Hence, (20) is proven in Case 2.

We have now proven (20) in both Cases 1 and 2. Thus, (20) always holds. In other words, the map \(T\) is well-defined.

What the map \(T\) does to a pair \((P, Q) \in Z\) can be described as moving the element \(f\) from the set where it resides (either \(P\) or \(Q\)) to the other set. Clearly, doing this twice gives us the original pair back. Hence, the map \(T\) is an involution. Furthermore, for any \((P, Q) \in Z\), if we write \(T((P, Q))\) in the form \((P', Q')\), then

\((-1)^{|P|} = -(-1)^{|P|}\) (because \(P' = \begin{cases} P \cup \{f\}, & \text{if } f \notin P; \\
P \setminus \{f\}, & \text{if } f \in P\end{cases}\)).

As we have already explained, this proves (19). And this, in turn, completes the induction step of the proof of Theorem 4.2. \(\square\)

7. Proof of Theorem 4.6

Before we begin proving Theorem 4.6, we state a criterion for \(\mathcal{E}\)-partitions that is less wasteful (in the sense that it requires fewer verifications) than the definition:

**Lemma 7.1.** Let \(\mathcal{E} = (E, <_1, <_2)\) be a tertispecial double poset. Let \(\phi : E \to \{1, 2, 3, \ldots\}\) be a map. Assume that the following two conditions hold:

- **Condition 1:** If \(e \in E\) and \(f \in E\) are such that \(e\) is \(<_1\)-covered by \(f\), and if we have \(e <_2 f\), then \(\phi(e) \leq \phi(f)\).
- **Condition 2:** If \(e \in E\) and \(f \in E\) are such that \(e\) is \(<_1\)-covered by \(f\), and if we have \(f <_2 e\), then \(\phi(e) < \phi(f)\).

Then, \(\phi\) is an \(\mathcal{E}\)-partition.

**Proof of Lemma 7.1.** For any \(a \in E\) and \(b \in E\), we define a subset \([a, b]\) of \(E\) as in the proof of Lemma 6.1.

We need to show that \(\phi\) is an \(\mathcal{E}\)-partition. In other words, we need to prove the following two claims:

- **Claim 1:** Every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) satisfy \(\phi(e) \leq \phi(f)\).
- **Claim 2:** Every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) and \(f <_2 e\) satisfy \(\phi(e) < \phi(f)\).

**Proof of Claim 1:** Assume the contrary. Thus, there exists a pair \((e, f) \in E \times E\) satisfying \(e <_1 f\) but not \(\phi(e) \leq \phi(f)\). Such a pair will be called a **malrelation**. Fix a
malrelation \((u, v)\) for which the value \(|[u, v]|\) is minimum (such a \((u, v)\) exists, since there exists a malrelation). Thus, \(u \in E\) and \(v \in E\) and \(u \leq v\) but not \(\phi(u) \leq \phi(v)\).

If \(u\) was \(<_1\)-covered by \(v\), then we would obtain \(\phi(u) \leq \phi(v)\) \(^{27}\), which would contradict the assumption that we do not have \(\phi(u) \leq \phi(v)\). Hence, \(u\) is not \(<_1\)-covered by \(v\). Consequently, there exists a \(w \in E\) such that \(u \leq w \leq v\) (since \(u <_1 v\)). Consider this \(w\). Applying \((11)\) to \(a = u, c = w\) and \(b = v\), we see that both numbers \(|[u, w]|\) and \(|[w, v]|\) are smaller than \(|[u, v]|\), and therefore neither \((u, w)\) nor \((w, v)\) is a malrelation (since we picked \((u, v)\) to be a malrelation with minimum \(|[u, v]|\)). Therefore, we have \(\phi(u) \leq \phi(w)\) and \(\phi(w) \leq \phi(v)\) (since \(u \leq w\) and \(w \leq v\)). Combining these two inequalities, we obtain \(\phi(u) \leq \phi(v)\). This contradicts the assumption that we do not have \(\phi(u) \leq \phi(v)\). This contradiction concludes the proof of Claim 1.

Instead of Claim 2, we shall prove the following stronger claim:

**Claim 3:** Every \(e \in E\) and \(f \in E\) satisfying \(e <_1 f\) and not \(e <_2 f\) satisfy \(\phi(e) < \phi(f)\).

**Proof of Claim 3:** Assume the contrary. Thus, there exists a pair \((e, f) \in E \times E\) satisfying \(e <_1 f\) and not \(e <_2 f\) but not \(\phi(e) < \phi(f)\). Such a pair will be called a malrelation. Fix a malrelation \((u, v)\) for which the value \(|[u, v]|\) is minimum (such a \((u, v)\) exists, since there exists a malrelation). Thus, \(u \in E\) and \(v \in E\) and \(u <_1 v\) and not \(u <_2 v\) but not \(\phi(u) < \phi(v)\).

If \(u\) was \(<_1\)-covered by \(v\), then we would obtain \(\phi(u) < \phi(v)\) easily \(^{28}\), which would contradict the assumption that we do not have \(\phi(u) < \phi(v)\). Hence, \(u\) is not \(<_1\)-covered by \(v\). Consequently, there exists a \(w \in E\) such that \(u \leq w \leq v\) (since \(u <_1 v\)). Consider this \(w\). Applying \((11)\) to \(a = u, c = w\) and \(b = v\), we see that both numbers \(|[u, w]|\) and \(|[w, v]|\) are smaller than \(|[u, v]|\), and therefore neither \((u, w)\) nor \((w, v)\) is a malrelation (since we picked \((u, v)\) to be a malrelation with minimum \(|[u, v]|\)).

But \(\phi(v) \leq \phi(u)\) (since we do not have \(\phi(u) < \phi(v)\)). On the other hand, \(u \leq w\) and therefore \(\phi(u) \leq \phi(w)\) (by Claim 1). Furthermore, \(w \leq v\) and thus \(\phi(w) \leq \phi(v)\) (by Claim 1). The chain of inequalities \(\phi(v) \leq \phi(u) \leq \phi(w) \leq \phi(v)\) ends with the same term that it begins with; therefore, it must be a chain of equalities. In other words, we have \(\phi(v) = \phi(u) = \phi(w) = \phi(v)\).

Now, using \(\phi(w) = \phi(v)\), we can see that \(w <_2 v\) \(^{29}\). The same argument

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\(^{27}\)Proof. Assume that \(u\) is \(<_1\)-covered by \(v\). Thus, \(u\) and \(v\) are \(<_2\)-comparable (since the poset \(E\) is tertispecial). In other words, we have either \(u <_2 v\) or \(u = v\) or \(v <_2 u\). In the first of these three cases, we obtain \(\phi(u) \leq \phi(v)\) by applying Condition 1 to \(e = u\) and \(f = v\). In the third of these cases, we obtain \(\phi(u) < \phi(v)\) (and thus \(\phi(u) \leq \phi(v)\)) by applying Condition 2 to \(e = u\) and \(f = v\). The second of these cases cannot happen because \(u <_1 v\). Thus, we always have \(\phi(u) \leq \phi(v)\), qed.

\(^{28}\)Proof. Assume that \(u\) is \(<_1\)-covered by \(v\). Thus, \(u\) and \(v\) are \(<_2\)-comparable (since the poset \(E\) is tertispecial). In other words, we have either \(u <_2 v\) or \(u = v\) or \(v <_2 u\). Since neither \(u <_2 v\) nor \(u = v\) can hold (indeed, \(u <_2 v\) is ruled out by assumption, whereas \(u = v\) is ruled out by \(u <_1 v\)), we have \(v <_2 u\). Therefore, \(\phi(u) < \phi(v)\) by Condition 2 (applied to \(e = u\) and \(f = v\)), qed.

\(^{29}\)Proof. Assume the contrary. Thus, we do not have \(w <_2 v\). But \(\phi(w) = \phi(v)\) shows that we
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Proof of Claim 3: The condition \( f <_2 e \) is stronger than \( \text{not } e <_2 f \). Thus, Claim 2 follows from Claim 3.

Proof of Claim 2: The condition \( f <_2 e \) is stronger than \( \text{not } e <_2 f \). Thus, Claim 2 follows from Claim 3.

Claims 1 and 2 are now both proven, and so Lemma 7.1 follows.

Proof of Lemma 4.5. Consider the following three logical statements:

Statement 1: The orbit \( O \) is \( E \)-coeven.

Statement 2: All elements of \( O \) are \( E \)-coeven.

Statement 3: At least one element of \( O \) is \( E \)-coeven.

Statements 1 and 2 are equivalent (according to the definition of when an orbit is \( E \)-coeven). Our goal is to prove that Statements 1 and 3 are equivalent (because this is precisely what Lemma 4.5 says). Thus, it clearly suffices to show that Statements 2 and 3 are equivalent. Since Statement 2 obviously implies Statement 3, we therefore only need to show that Statement 3 implies Statement 2. Thus, assume that Statement 3 holds. We need to prove that Statement 2 holds.

There exists at least one \( E \)-coeven \( \phi \in O \) (because we assumed that Statement 3 holds). Consider this \( \phi \). Now, let \( \pi \in O \) be arbitrary. We shall show that \( \pi \) is \( E \)-coeven.

We know that \( \phi \) is \( E \)-coeven. In other words,

\[
every \ g \in G \text{ satisfying } g\phi = \phi \text{ is } E\text{-even.} \quad (24)
\]

Now, let \( g \in G \) be such that \( g\pi = \pi \). Since \( \phi \) belongs to the \( G \)-orbit \( O \), we have \( O = G\phi \). Now, \( \pi \in O = G\phi \). In other words, there exists some \( h \in G \) such that \( \pi = h\phi \). Consider this \( h \). We have \( g\pi = \pi \). Since \( \pi = h\phi \), this rewrites as \( gh\phi = h\phi \).

In other words, \( h^{-1}gh\phi = \phi \). Thus, (24) (applied to \( h^{-1}gh \) instead of \( g \)) shows that \( h^{-1}gh \) is \( E \)-even. In other words,

\[
the \text{ action of } h^{-1}gh \text{ on } E \text{ is an even permutation of } E. \quad (25)
\]

Now, let \( \varepsilon \) be the group homomorphism from \( G \) to \( \text{Aut } E \) which describes the \( G \)-action on \( E \). Then, \( \varepsilon (h^{-1}gh) \) is the action of \( h^{-1}gh \) on \( E \), and thus is an even permutation of \( E \) (by (25)).

But since \( \varepsilon \) is a group homomorphism, we have \( \varepsilon (h^{-1}gh) = \varepsilon (h)^{-1} \varepsilon (g) \varepsilon (h) \).

Thus, the permutations \( \varepsilon (h^{-1}gh) \) and \( \varepsilon (g) \) of \( E \) are conjugate. Since the permutation \( \varepsilon (h^{-1}gh) \) is even, this shows that the permutation \( \varepsilon (g) \) is even. In other words, the action of \( g \) on \( E \) is an even permutation of \( E \). In other words, \( g \) is \( E \)-even.

Now, let us forget that we fixed \( g \). We thus have shown that every \( g \in G \) satisfying \( g\pi = \pi \) is \( E \)-even. In other words, \( \pi \) is \( E \)-coeven.

\[\text{do not have } \phi (w) < \phi (v). \text{ Hence, } (w, v) \text{ is a malrelation (since } w <_1 v \text{ and not } w <_2 v \text{ but not } \phi (w) < \phi (v)). \text{ This contradicts the fact that } (w, v) \text{ is not a malrelation. This contradiction completes the proof.}\]
Let us now forget that we fixed \( \pi \). Thus, we have proven that every \( \pi \in O \) is \( E \)-coeven. In other words, Statement 2 holds. We have thus shown that Statement 3 implies Statement 2. Consequently, Statements 2 and 3 are equivalent, and so the proof of Lemma 4.5 is complete. \( \square \)

Next, we will show three simple properties of posets on which groups act.

**Proposition 7.2.** Let \( E \) be a set. Let \( < \) be a strict partial order relation on \( E \). Let \( G \) be a finite group which acts on \( E \). Assume that \( G \) preserves the relation \( < \).

Let \( g \in G \). Let \( E^g \) be the set of all orbits under the action of \( g \) on \( E \). Define a binary relation \( <^g_1 \) on \( E^g \) by

\[
(u <^g_1 v) \iff \text{(there exist } a \in u \text{ and } b \in v \text{ with } a < b) .
\]

Then, \( <^g_1 \) is a strict partial order relation.

**Proposition 7.2** is precisely [Joch13, Lemma 2.4], but let us outline the proof for the sake of completeness:

**Proof of Proposition 7.2.** Let us first show that the relation \( <^g_1 \) is irreflexive. Indeed, assume the contrary. Thus, there exists a \( u \in E^g \) such that \( u <^g_1 u \). Consider this \( u \). Since \( u <^g_1 u \), there exist \( a \in u \) and \( b \in u \) with \( a < b \). Consider these \( a \) and \( b \).

There exists a \( k \in \mathbb{N} \) such that \( b = g^k a \) (since \( a \) and \( b \) both lie in one and the same \( g \)-orbit \( u \)). Consider this \( k \).

The \( g \)-orbit \( u \) of \( a \) is finite (since \( g \) is finite). Thus, there exists a positive integer \( n \) such that \( g^n a = a \). Consider this \( n \). Notice that \( g^n a = (g^n)^p a = a \) for every \( p \in \mathbb{N} \) (since \( g^n a = a \)).

Now, \( a < b = g^k a \). Since \( G \) preserves the relation \( < \), this shows that \( h a < g g^k a \) for every \( h \in G \). Thus, \( g^{\ell + 1} a < g \ell g^k a \) for every \( \ell \in \mathbb{N} \). Hence, \( g^{\ell + 1} a < g^{\ell + 1} g^k a = g^{\ell + 1} g^k a \) for every \( \ell \in \mathbb{N} \). Consequently, \( g^{0} a < g^{1} a < g^{2} a < \cdots < g^{n} a \).

Thus, \( g^{0} a \) \( g^{nk} a = a \) (since \( g^{np} a = a \) for every \( p \in \mathbb{N} \)), which contradicts \( g^{0} a = 1 g a = a \). This contradiction proves that our assumption was wrong. Hence, the relation \( <^g_1 \) is irreflexive.

Let us next show that the relation \( <^g_1 \) is transitive. Indeed, let \( u, v \) and \( w \) be three elements of \( E^g \) such that \( u <^g_1 v \) and \( v <^g_1 w \). We must prove that \( u <^g_1 w \).

There exist \( a \in u \) and \( b \in v \) with \( a < b \) (since \( u <^g_1 v \)). Consider these \( a \) and \( b \).

There exist \( a' \in v \) and \( b' \in w \) with \( a' < b' \) (since \( v <^g_1 w \)). Consider these \( a' \) and \( b' \).

The elements \( b \) and \( a' \) lie in one and the same \( g \)-orbit (namely, in \( v \)). Hence, there exists some \( k \in \mathbb{N} \) such that \( a' = g^k b \). Consider this \( k \). We have \( a < b \) and thus \( g^k a < g^k b \) (since \( G \) preserves the relation \( <_1 \)). Hence, \( g^k a < g^k b = a' < b' \). Since \( g^k a \in u \) (because \( a \in u \)) and \( b' \in w \), this shows that \( u <^g_1 w \). We thus have proven that the relation \( <^g_1 \) is transitive.
Now, we know that the relation $<_1^S$ is irreflexive and transitive, and thus also antisymmetric (since every irreflexive and transitive binary relation is antisymmetric). In other words, $<_1^S$ is a strict partial order relation. This proves Proposition 7.2.

**Remark 7.3.** Proposition 7.2 can be generalized: Let $E$ be a set. Let $<_1$ be a strict partial order relation on $E$. Let $G$ be a finite group which acts on $E$. Assume that $G$ preserves the relation $<_1$. Let $H$ be a subgroup of $G$. Let $E^H$ be the set of all orbits under the action of $H$ on $E$. Define a binary relation $<_1^H$ on $E^H$ by

$$(u <_1^H v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then, $<_1^H$ is a strict partial order relation.

This result (whose proof is quite similar to that of Proposition 7.2) implicitly appears in [Stan84, p. 30].

**Proposition 7.4.** Let $E = (E, <_1, <_2)$ be a tertispecial double poset. Let $G$ be a finite group which acts on $E$. Assume that $G$ preserves both relations $<_1$ and $<_2$.

Let $g \in G$. Let $E^g$ be the set of all orbits under the action of $g$ on $E$. Define a binary relation $<_1^S$ on $E^g$ by

$$(u <_1^S v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Define a binary relation $<_2^S$ on $E^g$ by

$$(u <_2^S v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b).$$

Let $E^g$ be the triple $(E^g, <_1^S, <_2^S)$. Then, $E^g$ is a tertispecial double poset.

**Proof of Proposition 7.4.** Both relations $<_1$ and $<_2$ are strict partial order relations (since $E$ is a double poset). Proposition 7.2 shows that $<_1^S$ is a strict partial order relation. Proposition 7.2 (applied to $<_2$ and $<_2^S$ instead of $<_1$ and $<_1^S$) shows that $<_2^S$ is a strict partial order relation. Thus, $E^g = (E^g, <_1^S, <_2^S)$ is a double poset. It remains to show that this double poset $E^g$ is tertispecial.

Let $u$ and $v$ be two elements of $E^g$ such that $u$ is $<_1^S$-covered by $v$. We shall prove that $u$ and $v$ are $<_2^S$-comparable.

We have $u <_1^S v$ (since $u$ is $<_1^S$-covered by $v$). In other words, there exist $a \in u$ and $b \in v$ with $a <_1 b$. Consider these $a$ and $b$.

If there was a $c \in E^g$ satisfying $a <_1 c <_1 b$, then we would have $u <_1^S w <_1^S v$ with $w$ being the $g$-orbit of $c$, and this would contradict the condition that $u$ is $<_1^S$-covered by $v$. Hence, no such $c$ can exist. In other words, $a$ is $<_1$-covered by $b$. Thus, $a$ and $b$ are $<_2$-comparable (since the double poset $E$ is tertispecial). Consequently, $u$ and $v$ are $<_2^S$-comparable.

Now, let us forget that we fixed $u$ and $v$. We thus have shown that if $u$ and $v$ are two elements of $E^g$ such that $u$ is $<_1^S$-covered by $v$, then $u$ and $v$ are $<_2^S$-comparable.
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comparable. In other words, the double poset $E^g = (E^g, \prec_1^g, \prec_2^g)$ is tertispecial. This proves Proposition 7.4.

**Proposition 7.5.** Let $E = (E, \prec_1, \prec_2)$ be a tertispecial double poset. Let $G$ be a finite group which acts on $E$. Assume that $G$ preserves both relations $\prec_1$ and $\prec_2$.

Let $g \in G$. Define the set $E^g$, the relations $\prec_1^g$ and $\prec_2^g$ and the triple $E^g$ as in Proposition 7.4. Thus, $E^g$ is a tertispecial double poset (by Proposition 7.4).

There is a bijection $\Phi$ between

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the maps $\pi^g : E^g \to \{1, 2, 3, \ldots\}$.

Namely, this bijection $\Phi$ sends any map $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$ to the map $\pi^g : E^g \to \{1, 2, 3, \ldots\}$ defined by

$$\pi^g(u) = \pi(a) \quad \text{for every } u \in E^g \text{ and } a \in u.$$ (The well-definedness of this map $\pi^g$ is easy to see: Indeed, from $g\pi = \pi$, we can conclude that any two elements $a_1$ and $a_2$ of a given $g$-orbit $u$ satisfy $\pi(a_1) = \pi(a_2)$.)

Consider this bijection $\Phi$. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a map satisfying $g\pi = \pi$.

(a) If $\pi$ is an $E$-partition, then $\Phi(\pi)$ is an $E^g$-partition.

(b) If $\Phi(\pi)$ is an $E^g$-partition, then $\pi$ is an $E$-partition.

(c) Let $w : E \to \{1, 2, 3, \ldots\}$ be map. Define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by

$$w^g(u) = \sum_{a \in u} w(a) \quad \text{for every } u \in E^g.$$ Then, $x_{\Phi(\pi),w^g} = x_{\pi,w}$.

**Proof of Proposition 7.5 (sketched).** The definition of $\Phi$ shows that

$$(\Phi(\pi))(u) = \pi(a) \quad \text{for every } u \in E^g \text{ and } a \in u. \quad \text{(26)}$$

(a) Assume that $\pi$ is an $E$-partition. We want to show that $\Phi(\pi)$ is an $E^g$-partition. In order to do so, we can use Lemma 7.1 (applied to $E^g$, $(E^g, \prec_1^g, \prec_2^g)$ and $\Phi(\pi)$ instead of $E$, $(E, \prec_1, \prec_2)$ and $\phi$); we only need to check the following two conditions:

**Condition 1:** If $e \in E^g$ and $f \in E^g$ are such that $e$ is $\prec_1^g$-covered by $f$, and if we have $e \prec_2^g f$, then $(\Phi(\pi))(e) \leq (\Phi(\pi))(f)$.
Condition 2: If $e \in E^g$ and $f \in E^g$ are such that $e$ is $<^g_1$-covered by $f$, and if we have $f <^g_2 e$, then $(\Phi (\pi)) (e) < (\Phi (\pi)) (f)$.

Proof of Condition 1: Let $e \in E^g$ and $f \in E^g$ be such that $e$ is $<^g_2$-covered by $f$. Assume that we have $e <^g_2 f$.

We have $e <^g_1 f$ (because $e$ is $<^g_1$-covered by $f$). In other words, there exist $a \in e$ and $b \in f$ satisfying $a < b$. Consider these $a$ and $b$. Since $\pi$ is an $E$-partition, we have $\pi (a) \leq \pi (b)$ (since $a < b$). But the definition of $\Phi (\pi)$ shows that $(\Phi (\pi)) (e) = \pi (a)$ (since $a \in e$) and $(\Phi (\pi)) (f) = \pi (b)$ (since $b \in f$). Thus, $(\Phi (\pi)) (e) = \pi (a) \leq \pi (b) = (\Phi (\pi)) (f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E^g$ and $f \in E^g$ be such that $e$ is $<^g_2$-covered by $f$. Assume that we have $f <^g_2 e$.

We have $e <^g_1 f$ (because $e$ is $<^g_1$-covered by $f$). In other words, there exist $a \in e$ and $b \in f$ satisfying $a < b$. Consider these $a$ and $b$.

If there was a $c \in E$ satisfying $a < c < b$, then the $g$-orbit $w$ of this $c$ would satisfy $e <^g_1 w <^g_1 f$, which would contradict the fact that $e$ is $<^g_1$-covered by $f$. Hence, there exists no such $c$. In other words, $a$ is $<^g_1$-covered by $b$ (since $a < b$). Therefore, $a$ and $b$ are $<^g_2$-comparable (since $E$ is tertispecial). In other words, we have either $a <^g_2 b$ or $a = b$ or $b <^g_2 a$. Since $a <^g_2 b$ is impossible (because if we had $a <^g_2 b$, then we would have $e <^g_2 f$ (since $a \in e$ and $b \in f$), which would contradict $f <^g_2 e$ (since $<^g_2$ is a strict partial order relation)), and since $a = b$ is impossible (because $a < b$), we therefore must have $b <^g_2 a$. But since $\pi$ is an $E$-partition, we have $\pi (a) < \pi (b)$ (since $a < b$ and $b <^g_2 a$). But the definition of $\Phi (\pi)$ shows that $(\Phi (\pi)) (e) = \pi (a)$ (since $a \in e$) and $(\Phi (\pi)) (f) = \pi (b)$ (since $b \in f$). Thus, $(\Phi (\pi)) (e) = \pi (a) < \pi (b) = (\Phi (\pi)) (f)$. Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.5 (a) is proven.

(b) Assume that $\Phi (\pi)$ is an $E^g$-partition. We want to show that $\pi$ is an $E$-partition. In order to do so, we can use Lemma 7.1 (applied to $\phi = \pi$); we only need to check the following two conditions:

Condition 1: If $e \in E$ and $f \in E$ are such that $e$ is $<^g_1$-covered by $f$, and if we have $e <^g_2 f$, then $\pi (e) \leq \pi (f)$.

Condition 2: If $e \in E$ and $f \in E$ are such that $e$ is $<^g_1$-covered by $f$, and if we have $f <^g_2 e$, then $\pi (e) < \pi (f)$.

Proof of Condition 1: Let $e \in E$ and $f \in E$ be such that $e$ is $<^g_1$-covered by $f$. Assume that we have $e <^g_2 f$.

We have $e <^g_1 f$ (since $e$ is $<^g_1$-covered by $f$). Let $u$ and $v$ be the $g$-orbits of $e$ and $f$, respectively. Thus, $u$ and $v$ belong to $E^g$, and satisfy $u < v$ (since $e < f$). Hence, $(\Phi (\pi)) (u) \leq (\Phi (\pi)) (v)$ (since $\Phi (\pi)$ is an $E^g$-partition). But the definition of $\Phi (\pi)$ shows that $(\Phi (\pi)) (u) = \pi (e)$ (since $e \in u$) and $(\Phi (\pi)) (v) = \pi (f)$ (since $f \in v$). Thus, $\pi (e) = (\Phi (\pi)) (u) \leq (\Phi (\pi)) (v) = \pi (f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E$ and $f \in E$ be such that $e$ is $<^g_1$-covered by $f$. Assume that we have $f <^g_2 e$. 

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We have \( e <_1 f \) (since \( e <_1 \)-covered by \( f \)). Let \( u \) and \( v \) be the \( g \)-orbits of \( e \) and \( f \), respectively. Thus, \( u \) and \( v \) belong to \( E^g \), and satisfy \( u <_1 v \) (since \( e <_1 f \)) and \( v <_2 u \) (since \( f <_2 e \)). Hence, \((\Phi(\pi))(u) < (\Phi(\pi))(v)\) (since \( \Phi(\pi) \) is an \( E^g \)-partition). But the definition of \( \Phi(\pi) \) shows that \((\Phi(\pi))(u) = \pi(e)\) (since \( e \in u \)) and \((\Phi(\pi))(v) = \pi(f)\) (since \( f \in v \)). Thus, \( \pi(e) = (\Phi(\pi))(u) < (\Phi(\pi))(v) = \pi(f) \). Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.5 (b) is proven.

(c) The definition of \( x_{\Phi(\pi),w^g} \) shows that

\[
x_{\Phi(\pi),w^g} = \prod_{e \in E^g} x_{(\Phi(\pi))(e)}^{w^g(e)} = \prod_{u \in E^g} x_{(\Phi(\pi))(u)}^{w^g(u)} = \prod_{u \in E^g} \prod_{a \in u} x_{(\Phi(\pi))(u)}^{w(a)}
\]

(by the definition of \( x_{\pi,w} \)). This proves Proposition 7.5 (c).

Our next lemma is a standard argument in Pólya enumeration theory (compare it with the proof of Burnside’s lemma):

**Lemma 7.6.** Let \( G \) be a finite group. Let \( F \) be a finite \( G \)-set. Let \( O \) be a \( G \)-orbit on \( F \), and let \( \pi \in O \).

(a) We have

\[
\frac{1}{|O|} = \frac{1}{|G|} \sum_{g \in G; \ \bar{g}\pi = \pi} 1. \tag{27}
\]

(b) Let \( E \) be a further finite \( G \)-set. For every \( g \in G \), let \( \text{sign}_g \) denote the sign of the permutation of \( E \) that sends every \( e \in E \) to \( ge \). (Thus, \( g \in G \) is \( E \)-even if and only if \( \text{sign}_g = 1 \).) Then,

\[
\begin{cases}
\frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven}; \\
0, & \text{if } O \text{ is not } E\text{-coeven}
\end{cases} = \frac{1}{|G|} \sum_{g \in G; \ \bar{g}\pi = \pi} \text{sign}_g. \tag{28}
\]

**Proof of Lemma 7.6** Let \( \text{Stab}_G \pi \) denote the stabilizer of \( \pi \); this is the subgroup \( \{ g \in G \mid \bar{g}\pi = \pi \} \) of \( G \). The \( G \)-orbit of \( \pi \) is \( O \) (since \( O \) is a \( G \)-orbit on \( F \), and since \( \pi \in O \)). Hence,

\[
|O| = |G\pi| = \frac{|G|}{|\text{Stab}_G \pi|}
\]
(by the orbit-stabilizer theorem) and thus
\[ \frac{1}{|O|} = \frac{|\text{Stab}_G \pi|}{|G|}. \] (29)

(a) We have
\[ \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 = \left| \left\{ g \in G \mid g\pi = \pi \right\} \right| = |\text{Stab}_G \pi|. \]
Hence,
\[ \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 = \frac{1}{|G|} \frac{|\text{Stab}_G \pi|}{|G|} = \frac{1}{|G|} \frac{|O|}{|G|} \]
(by (29)). This proves Lemma 7.6 (a).

(b) We need to prove (28). Assume first that $O$ is $E$-coeven. Thus, $\pi$ is $E$-coeven (by the definition of what it means for $O$ to be $E$-coeven). This means that every $g \in G$ satisfying $g\pi = \pi$ is $E$-even. Hence, every $g \in G$ satisfying $g\pi = \pi$ satisfies $\text{sign}_E g = 1$. Thus,
\[ \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 = \frac{1}{|O|} \]
(by (27))
\[ = \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \]
(since $O$ is $E$-coeven).

Thus, we have proven (28) under the assumption that $O$ is $E$-coeven. We can therefore WLOG assume the opposite now. Thus, assume that $O$ is not $E$-coeven. Hence, no element of $O$ is $E$-coeven (due to the contrapositive of Lemma 4.5). In particular, $\pi$ is not $E$-coeven. In other words, not every $g \in G$ satisfying $g\pi = \pi$ is $E$-even. In other words, not every $g \in \text{Stab}_G \pi$ is $E$-even (since the elements $g \in G$ satisfying $g\pi = \pi$ are exactly the elements $g \in \text{Stab}_G \pi$). In other words, not every $g \in \text{Stab}_G \pi$ satisfies $\text{sign}_E g = 1$.

Now, the map
\[ \text{Stab}_G \pi \to \{1, -1\}, \quad g \mapsto \text{sign}_E g \]
is a group homomorphism (since the sign of a permutation is multiplicative) and is not the trivial homomorphism (since not every $g \in \text{Stab}_G \pi$ satisfies $\text{sign}_E g = 1$). Hence, it must send exactly half the elements of $\text{Stab}_G \pi$ to $1$ and the other half to $-1$. Therefore, the addends in the sum $\sum_{g \in \text{Stab}_G \pi} \text{sign}_E g$ cancel each other out (one
half of them are 1, and the others are $-1$). Therefore, 
\[ \frac{1}{|G|} \sum_{g \in \text{Stab}_G} \text{sign}_E g = 0, \]  
so that 
\[ \frac{1}{|G|^2} \sum_{g \in G} \sum_{g \pi = \pi} \text{sign}_E g = \begin{cases} 1, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \]  
(since $O$ is not $E$-coeven).

Thus, 
\[ \frac{1}{|G|} \sum_{g \in \text{Stab}_G} \text{sign}_E g = \frac{1}{|G|} \sum_{g \in G} \sum_{g \pi = \pi} \text{sign}_E g = \begin{cases} 1, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \]

This proves (28). Lemma 7.6 (b) is thus proven.

**Proof of Theorem 4.6 (sketched).** For every $g \in G$, define a tertispecial double poset $E^g = (E^g, <_1^g, <_2^g)$ as follows:

Let $E^g$ be the set of all orbits under the action of $g$ on $E$. Define a binary relation $<_1^g$ on $E^g$ by

\[ (u <_1^g v) \iff \text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b. \]

Similarly, define a strict partial order relation $<_2^g$ on $E^g$ by

\[ (u <_2^g v) \iff \text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b. \]

Finally, set $E^g = (E^g, <_1^g, <_2^g)$. Proposition 7.4 shows that this $E^g$ is a tertispecial double poset.

Furthermore, for every $g \in G$, define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by $w^g(u) = \sum_{a \in u} w(a)$. (Since $G$ preserves $w$, the numbers $w(a)$ for all $a \in u$ are equal (for given $u$), and thus $\sum_{a \in u} w(a)$ can be rewritten as $|u| \cdot w(b)$ for any particular $b \in u$.) Now,

\[ S \left( \Gamma \left( (E^g, <_1^g, <_2^g), w^g \right) \right) = (-1)^{|E^g|} \Gamma \left( (E^g, >_1^g, <_2^g), w^g \right) \]  
(by Theorem 4.2, applied to $((E^g, <_1^g, <_2^g), w^g)$ instead of $((E, <_1, <_2), w)$).

For every $g \in G$, we have

\[ \sum_{\pi \text{ is an } E\text{-partition}} x_{\pi, g \pi} = \Gamma (E^g, w^g) \]  
(31)

---

30**Proof of (31):** Let $g \in G$. In Proposition 7.5, we have introduced a bijection $\Phi$ between
It is clearly sufficient to prove Theorem 4.6 for $k = \mathbb{Z}$ (since all the power series that we are discussing are defined functorially in $k$, and thus any identity between these series that holds over $\mathbb{Z}$ must hold over any $k$). Therefore, it is sufficient to prove Theorem 4.6 for $k = \mathbb{Q}$ (since $\text{QSym}_\mathbb{Z}$ embeds into $\text{QSym}_\mathbb{Q}$). Thus, we WLOG assume that $k = \mathbb{Q}$. This will allow us to divide by positive integers.

Every $G$-orbit $O$ on $\text{Par}_E$ satisfies

$$\frac{1}{|O|} \sum_{\pi \in O} x_{\pi,\omega} = \frac{1}{|O|} \sum_{\pi \in O} x_{O,\omega} = \frac{1}{|O|} |O| x_{O,\omega} = x_{O,\omega}. \quad (32)$$

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g \pi = \pi$
- the maps $\pi : E^g \to \{1, 2, 3, \ldots\}$.

Parts (a) and (b) of Proposition 7.5 show that this bijection $\Phi$ restricts to a bijection between
- the $E$-partitions $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g \pi = \pi$
- the $E^g$-partitions $\pi : E^g \to \{1, 2, 3, \ldots\}$.

Hence,

$$\sum_{\pi \text{ is an } E^g\text{-partition}} x_{\pi,\omega} = \sum_{\pi \text{ is an } E\text{-partition}; \ g \pi = \pi} x_{\Phi(\pi),\omega} = \sum_{\pi \text{ is an } E\text{-partition}; \ g \pi = \pi} x_{\pi,\omega},$$

(by Proposition 7.5 (c))

whence

$$\sum_{\pi \text{ is an } E\text{-partition}; \ g \pi = \pi} x_{\pi,\omega} = \sum_{\pi \text{ is an } E^g\text{-partition}} x_{\pi,\omega} = \Gamma(E^g, \omega^g).$$

This proves (31).

31Here, we are using the notation $\text{QSym}_k$ for the Hopf algebra $\text{QSym}$ defined over a base ring $k$. 
Now,
\[
\Gamma (E, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \text{Par } E} x_{O, w} = \sum_{O \text{ is a } G\text{-orbit on } \text{Par } E} \frac{1}{|O|} \sum_{\pi \in O} x_{\pi, w}
\]
(by \(32\))

\[
= \sum_{O \text{ is a } G\text{-orbit on } \text{Par } E} \sum_{\pi \in O} \frac{1}{|O|} x_{\pi, w}
\]
(by \(27\), applied to \(F = \text{Par } E\))

\[
= \sum_{\pi \in \text{Par } E} \left( \frac{1}{|G|} \sum_{g \in G; g\pi = \pi} 1 \right) x_{\pi, w}
\]
(by \(30\), applied to \(F = \text{Par } E\))

\[
= \sum_{\pi \text{ is an } E\text{-partition}} \left( \frac{1}{|G|} \sum_{g \in G; g\pi = \pi} 1 \right) x_{\pi, w}
\]
(by \(31\))

\[
= \frac{1}{|G|} \sum_{g \in G, \pi \text{ is an } E\text{-partition; } g\pi = \pi} x_{\pi, w}
\]
(by \(33\))

\[
= \frac{1}{|G|} \sum_{g \in G} \Gamma (E^g, w^g) = \frac{1}{|G|} \sum_{g \in G} \Gamma (\{E^g, <_{1}^{g}, <_{2}^{g}\}, w^g). \tag{33}
\]

Hence, \(\Gamma (E, w, G) \in QSym\) (by Proposition \(3.5\)).

Applying the map \(S\) to both sides of the equality \(33\), we obtain

\[
S (\Gamma (E, w, G)) = \frac{1}{|G|} \sum_{g \in G} S \left( \gamma (\{E^g, <_{1}^{g}, <_{2}^{g}\}, w^g) \right)
= (-1)^{|E|} \Gamma (\{E^g, >_{1}^{g}, <_{2}^{g}\}, w^g)
\]
(by \(30\))

\[
= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E|} \Gamma (\{E^g, >_{1}^{g}, <_{2}^{g}\}, w^g). \tag{34}
\]
On the other hand, for every \( g \in G \), let \( \text{sign}_E g \) denote the sign of the permutation of \( E \) that sends every \( e \in E \) to \( ge \). Thus, \( g \in G \) is \( E \)-even if and only if \( \text{sign}_E g = 1 \). Now, every \( G \)-orbit \( O \) on \( \text{Par } E \) and every \( \pi \in O \) satisfy

\[
\begin{cases} 
1/|O|', & \text{if } O \text{ is } E\text{-coeven;} \\
0, & \text{if } O \text{ is not } E\text{-coeven}
\end{cases} = \frac{1}{|G|} \sum_{g \in G; \ g \pi = \pi} \text{sign}_E g \tag{35}
\]

(by (28), applied to \( F = \text{Par } E \)). Furthermore,

\[
\text{sign}_E g = (-1)^{|E| - |E^g|} \tag{36}
\]

for every \( g \in G \) \[32\].

\[32\text{Proof of (36): Let } g \in G. \text{ Recall that } \text{sign}_E g \text{ is the sign of the permutation of } E \text{ that sends every } e \in E \text{ to } ge. \text{ But if } \sigma \text{ is a permutation of a finite set } X, \text{ then the sign of } \sigma \text{ is } (-1)^{|X| - |X^\sigma|}, \text{ where } X^\sigma \text{ is the set of all cycles of } \sigma. \text{ Applying this to } X = E, \sigma = (\text{the permutation of } E \text{ that sends every } e \in E \text{ to } ge) \text{ and } X^\sigma = E^g, \text{ we see that the sign of the permutation of } E \text{ that sends every } e \in E \text{ to } ge \text{ is } (-1)^{|E| - |E^g|}. \text{ In other words, } \text{sign}_E g = (-1)^{|E| - |E^g|}, \text{ qed.} \]
Now, \( \Gamma^+ (E, w, G) \)
\[\begin{align*}
\Gamma^+ (E, w, G) &= \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \text{Par } E} x_{O, w} \\
&= \frac{1}{|G|} \sum_{\pi \in \text{Par } E} x_{\pi, w} \\
&= \sum_{O \text{ is a } G\text{-orbit on } \text{Par } E} \left\{ \begin{array}{ll}
\frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\
0, & \text{if } O \text{ is not } E\text{-coeven}
\end{array} \right\} x_{\pi, w}
\end{align*} \]
\[\text{here, we have extended the sum to all } G\text{-orbits on } \text{Par } E \text{ (not just the } E\text{-coeven ones); but all new addends are 0}
\text{ and therefore do not influence the value of the sum.}\]
\[\begin{align*}
&= \sum_{O \text{ is a } G\text{-orbit on } \text{Par } E} \sum_{\pi \in O} \left\{ \begin{array}{ll}
\frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\
0, & \text{if } O \text{ is not } E\text{-coeven}
\end{array} \right\} x_{\pi, w} \\
&= \frac{1}{|G|} \sum_{g \in G; g \pi = \pi} \sum_{\pi \in \text{Par } E} \text{sign}_E g x_{\pi, w}
\end{align*} \]
\[\begin{align*}
&= \sum_{\pi \in \text{Par } E} \sum_{\pi \in \text{Par } E} \left\{ \begin{array}{ll}
\frac{1}{|G|}, & \text{if } O \text{ is } E\text{-coeven;} \\
0, & \text{if } O \text{ is not } E\text{-coeven}
\end{array} \right\} x_{\pi, w} \\
&= \frac{1}{|G|} \sum_{\pi \in \text{Par } E} \sum_{\pi \in \text{Par } E} \text{sign}_E g x_{\pi, w}
\end{align*} \]
\[\begin{align*}
&= \Gamma (E^g, w^g) \\
&= \sum_{g \in G} \sum_{\pi \in \text{Par } E} \text{sign}_E g x_{\pi, w}
\end{align*} \]
\[\begin{align*}
&= (-1)^{|E| - |E^g|} \sum_{\pi \in \text{Par } E} \text{sign}_E g x_{\pi, w} \\
&= \Gamma (E^g, w^g) \\
&= \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma (E^g, w^g)
\end{align*} \]
\[\text{Hence, } \Gamma^+ (E, w, G) \in \text{QSym} \text{ (by Proposition 3.5).} \]
\[\text{The group } G \text{ preserves the relation } >_1 \text{ (since it preserves the relation } <_1). \text{ Hence,} \]
\[\text{(37)} \]
applying (37) to \((E, >_1, <_2)\) instead of \(E\), we obtain
\[
\Gamma^+ ((E, >_1, <_2), w, G) = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma ((E^g, >^g_1, <^g_2), w^g).
\]

Multiplying both sides of this equality by \((-1)^{|E|}\), we transform it into
\[
(-1)^{|E|} \Gamma^+ ((E, >_1, <_2), w, G) = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} (-1)^{|E^g|} \Gamma ((E^g, >^g_1, <^g_2), w^g)
\]
\[
= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E^g|} \Gamma ((E^g, >^g_1, <^g_2), w^g)
\]
\[
= S (\Gamma (E, w, G)) \quad (\text{by (34)}).
\]

This proves Theorem 4.6. \(\square\)

8. Application: Jochemko’s theorem

We shall now demonstrate an application of Theorem 4.6, namely, we will use it to provide an alternative proof of [Joch13, Theorem 2.13]. The way we derive [Joch13, Theorem 2.13] from Theorem 4.6 is classical, and in fact was what originally motivated the discovery of Theorem 4.6 (although, of course, it cannot be conversely derived from [Joch13, Theorem 2.13], so it is an actual generalization).

An intermediate step between [Joch13, Theorem 2.13] and Theorem 4.6 will be the following fact:

**Corollary 8.1.** Let \(E = (E, <_1, <_2)\) be a tertispecial double poset. Let \(w : E \to \{1, 2, 3, \ldots\}\). Let \(G\) be a finite group which acts on \(E\). Assume that \(G\) preserves both relations \(<_1\) and \(<_2\), and also preserves \(w\). For every \(q \in \mathbb{N}\), let \(\text{Par}_q E\) denote the set of all \(E\)-partitions whose image is contained in \(\{1, 2, \ldots, q\}\). Then, the group \(G\) also acts on \(\text{Par}_q E\); namely, \(\text{Par}_q E\) is a \(G\)-subset of the \(G\)-set \(\{1, 2, \ldots, q\}^E\) (see Definition 4.4 (d) for the definition of the latter).

(a) There exists a unique polynomial \(\Omega_{E,G} \in \mathbb{Q}[X]\) such that every \(q \in \mathbb{N}\) satisfies
\[
\Omega_{E,G} (q) = (\text{the number of all } G\text{-orbits on } \text{Par}_q E).
\] (38)

(b) This polynomial satisfies
\[
\Omega_{E,G} (-q)
= (-1)^{|E|} (\text{the number of all even } G\text{-orbits on } \text{Par}_q (E, >_1, <_2))
= (-1)^{|E|} (\text{the number of all even } G\text{-orbits on } \text{Par}_q (E, <_1, >_2))
\]
for all \(q \in \mathbb{N}\).
Proof of Corollary 8.1 (sketched). Set $k = \mathbb{Q}$. For any $f \in \text{QSym}$ and any $q \in \mathbb{N}$, we define an element $ps^1(f)(q) \in \mathbb{Q}$ by

$$ps^1(f)(q) = f \left( \frac{1, 1, \ldots, 1, 0, 0, 0, \ldots}{q \text{ times}} \right)$$

(that is, $ps^1(f)(q)$ is the result of substituting 1 for $x_1, x_2, \ldots, x_q$ and 0 for $x_{q+1}, x_{q+2}, x_{q+3}, \ldots$ in the power series $f$).

(a) Consider the elements $\Gamma(E, w, G)$ and $\Gamma^+(E, w, G)$ of $\text{QSym}$ defined in Theorem 4.6. Observe that $\text{Par}_q E$ is a $G$-subset of $\text{Par} E$.

Now, [GriRei14, Proposition 7.7 (i)] shows that, for any given $f \in \text{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $ps^1(f)(q)$. Applying this to $f = \Gamma(E, w, G)$, we conclude that there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $ps^1(\Gamma(E, w, G))(q)$. But since every $q \in \mathbb{N}$ satisfies

$$ps^1(\Gamma(E, w, G))(q) = \left( \frac{\Gamma(E, w, G)}{O \text{ is a } G\text{-orbit on } \text{Par} E} \right) \left( \frac{1, 1, \ldots, 1, 0, 0, 0, \ldots}{q \text{ times}} \right)$$

$$= \sum_{O \text{ is a } G\text{-orbit on } \text{Par} E} x_{O,w} \left( \frac{1, 1, \ldots, 1, 0, 0, 0, \ldots}{q \text{ times}} \right)$$

$$= \begin{cases} 1, & \text{if } O \subseteq \text{Par}_q E; \\ 0, & \text{if } O \not\subseteq \text{Par}_q E \end{cases}$$

$$= \sum_{O \text{ is a } G\text{-orbit on } \text{Par} E} 1 = \text{(the number of all } G\text{-orbits on } \text{Par}_q E)$$

(40)

this rewrites as follows: There exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals (the number of all $G$-orbits on $\text{Par}_q E$). This proves Corollary 8.1 (a).

(b) [GriRei14, Proposition 7.7 (i)] shows that, for any given $f \in \text{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $ps^1(f)(q)$. This polynomial is denoted by $ps^1(f)$ in [GriRei14, Proposition 7.7]. From our above proof of Corollary 8.1 (a), we see that

$$\Omega_{E,G} = ps^1(\Gamma(E, w, G))$$
But [GriRei14, Proposition 7.7 (iii)] shows that, for any \( f \in \text{QSym} \) and \( m \in \mathbb{N} \), we have \( \text{ps}^1 (S(f)) (m) = \text{ps}^1 (f) (-m) \). Applying this to \( f = \Gamma (E, w, G) \), we obtain
\[
\text{ps}^1 (S(\Gamma(E, w, G))) (m) = \text{ps}^1 (\Gamma(E, w, G)) (-m) = \Omega_{E,G} (-m)
\]
for any \( m \in \mathbb{N} \). Thus, any \( m \in \mathbb{N} \) satisfies
\[
\Omega_{E,G} (-m) = \text{ps}^1 \left( S(\Gamma(E, w, G)) \right) (m) = (-1)^{|E|} \text{ps}^1 (\Gamma^+ ((E, >_{>1}, <_2), w, G)) (m)
\]
\[
= (-1)^{|E|} \text{ps}^1 (\Gamma^+ ((E, >_{>1}, <_2), w, G)) (m).
\]
Renaming \( m \) as \( q \) in this equality, we see that every \( q \in \mathbb{N} \) satisfies
\[
\Omega_{E,G} (-q) = (-1)^{|E|} \text{ps}^1 (\Gamma^+ ((E, >_{>1}, <_2), w, G)) (q). \quad (41)
\]
But just as we proved (40), we can show that every \( q \in \mathbb{N} \) satisfies
\[
\text{ps}^1 (\Gamma^+ (E, w, G)) (q) = \text{(the number of all even } G\text{-orbits on } \text{Par}_q E\text{)}.
\]
Applying this to \( (E, >_{>1}, <_2) \) instead of \( E \), we obtain
\[
\text{ps}^1 (\Gamma^+ ((E, >_{>1}, <_2), w, G)) (q)
\]
\[
= \text{(the number of all even } G\text{-orbits on } \text{Par}_q (E, >_{>1}, <_2)\text{)}.
\]
Now, (41) becomes
\[
\Omega_{E,G} (-q) = (-1)^{|E|} \text{ps}^1 (\Gamma^+ ((E, >_{>1}, <_2), w, G)) (q)
\]
\[
= (\text{the number of all even } G\text{-orbits on } \text{Par}_q (E, >_{>1}, <_2)).
\]
In order to prove Corollary 8.1 (b), it thus remains to show that
\[
(\text{the number of all even } G\text{-orbits on } \text{Par}_q (E, >_{>1}, <_2))
\]
\[
= (\text{the number of all even } G\text{-orbits on } \text{Par}_q (E, <_{<1}, >_2)) \quad (42)
\]
for every \( q \in \mathbb{N} \).

Proof of (42): Let \( q \in \mathbb{N} \). Let \( w_0 : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, q\} \) be the map sending each \( i \in \{1, 2, \ldots, q\} \) to \( q + 1 - i \). Then, the map
\[
\text{Par}_q (E, >_{>1}, <_2) \to \text{Par}_q (E, <_{<1}, >_2), \quad \pi \mapsto w_0 \circ \pi
\]
is an isomorphism of \( G \)-sets (this is easy to check). Thus, \( \text{Par}_q (E, >_{>1}, <_2) \cong \text{Par}_q (E, <_{<1}, >_2) \) as \( G \)-sets. From this, (42) follows (by functoriality, if one wishes).

The proof of Corollary 8.1 (b) is now complete. \qed
Now, the second formula of [Joch13, Theorem 2.13] follows from our (39), applied to \( E = (P, \prec, <_\omega) \) (where \( <_\omega \) is the partial order on \( P \) given by \( (p <_\omega q) \iff (\omega(p) < \omega(q)) \)). The first formula of [Joch13, Theorem 2.13] can also be derived from our above arguments. We leave the details to the reader.

References


Double posets and the antipode of QSym


http://dedekind.mit.edu/~rstan/pubs/pubfiles/60.pdf