The purpose of this little note is to prove [2, Theorem 5.2] using the machinery of [1].

Let me add a few more definitions.

**Definition 0.1.** Let $n \in \mathbb{N}$. Let $u$ be a packed word of length $n$. Let $r = \max u$. Define $B_i = u_i^{-1}(\{i\})$ for every $i \in [r]$. (Thus, $(B_1, B_2, \ldots, B_r)$ is a set composition of $[n]$; it is what is called the “set composition of $[n]$ encoded by $u$” in [2].) Now, we define a polyhedral cone $K_u$ in $\mathbb{R}^n$ by

$$K_u = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{k} \sum_{i \in B_j} x_i \geq 0 \quad \text{for all } k = 1, 2, \ldots, r \right\}.$$ 

**Definition 0.2.** For any two sets $X$ and $Y$, let $\text{Map}(X,Y)$ denote the set of all maps from $X$ to $Y$. Define a $\mathbb{K}$-vector space $\mathfrak{M}$ by $\mathfrak{M} = \bigoplus_{n \geq 0} \text{Map}(\mathbb{R}^n, \mathbb{K})$ (where each $\text{Map}(\mathbb{R}^n, \mathbb{K})$ becomes a $\mathbb{K}$-vector space by pointwise addition and multiplication with scalars). We make $\mathfrak{M}$ into a $\mathbb{K}$-algebra, whose multiplication is defined as follows: For any $n \in \mathbb{N}$, any $m \in \mathbb{N}$, any $f \in \text{Map}(\mathbb{R}^n, \mathbb{K})$ and $g \in \text{Map}(\mathbb{R}^m, \mathbb{K})$, we define $fg$ to be the element of $\text{Map}(\mathbb{R}^{n+m}, \mathbb{K})$ which sends every $(x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ to $f(x_1, x_2, \ldots, x_n)\, g(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$.

**Definition 0.3.** For every $n \in \mathbb{N}$ and any subset $S$ of $\mathbb{R}^n$, we define a map $\mathbbm{1}_S \in \text{Map}(\mathbb{R}^n, \mathbb{K}) \subseteq \mathfrak{M}$ as the indicator function of $S$ (that is, the map which sends every $s \in S$ to 1 and every $s \in \mathbb{R}^n \setminus S$ to 0).

Our goal is to show:

**Theorem 0.4.** The map

$$\alpha : \text{WQSym} \to \mathfrak{M}, \quad u \mapsto (-1)^{\max u} \mathbbm{1}_{K_u}$$

is a $\mathbb{K}$-algebra homomorphism.
This is a stronger version of [2, Theorem 5.2] and a particular case of [2, Theorem 8.1].

We shall prove Theorem 0.4 using a detour via $H_T$. We first define a polyhedral cone for every $\mathcal{T} \in T$:

**Definition 0.5.** Let $n \in \mathbb{N}$ and $\mathcal{T} \in T_n$ (that is, let $\mathcal{T}$ be a topology on the set $[n] = \{1, 2, \ldots, n\}$). Then, we define a polyhedral cone $K_\mathcal{T}$ in $\mathbb{R}^n$ by

$$K_\mathcal{T} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in C} x_i \geq 0 \text{ for all closed sets } C \text{ of } \mathcal{T} \right\}.$$ 

The following follows from the definitions:

**Remark 0.6.** Let $u$ be a packed word. Then, $K_u = K_{\mathcal{T}_u}$, where $\mathcal{T}_u$ is as defined in [1, §2.1].

Let us define a few more things:

**Definition 0.7.** Let $X$ be a finite totally ordered set, and let $\mathcal{T}$ be a topology on $X$. We define $\mathcal{U}(\mathcal{T})$ to be the set of all $f \in \mathcal{P}(\mathcal{T})$ having the property that any two elements $i$ and $j$ of $X$ satisfying $i <_\mathcal{T} j$ must satisfy $f(i) < f(j)$. Notice that $\mathcal{L}(\mathcal{T}) \subseteq \mathcal{U}(\mathcal{T}) \subseteq \mathcal{P}(\mathcal{T})$. (We can call the elements of $\mathcal{U}(\mathcal{T})$ “strictly increasing packed words” for $\mathcal{T}$.)

**Definition 0.8.** We define a $\mathcal{K}$-linear map $U : H_T \to WQSym$ by

$$U(\mathcal{T}) = \sum_{f \in \mathcal{U}(\mathcal{T})} f \quad \text{for every } \mathcal{T} \in T.$$ 

**Remark 0.9.** This map $U$ is easily seen to be the map $\Gamma_{(0,0,1)}$ in the notation of [1, Proposition 14]. Thus, $U$ is a surjective Hopf algebra homomorphism.

Now, here is a rather trivial fact:

---

1Notice that [2, Theorem 5.2] talks not about our map $\alpha : WQSym \to \mathcal{M}$, but rather about a map $\mathcal{P} \to WQSym$ where $\mathcal{P}$ is a certain subquotient of $\mathcal{M}$ (namely, the subalgebra of $\mathcal{M}$ generated by $1_{K_u}$, taken modulo functions with measure-zero support). These two maps are “in some sense” inverse (allowing us to derive [2, Theorem 5.2] from Theorem 0.4). I find Theorem 0.4 the more natural statement.

Notice that [2] denotes by $(M_u)_u$ a packed word the basis of $WQSym$ that we call $(u)_u$ a packed word.

2At least, I suspect so – I have not checked all the details. I also suspect that the whole [2, Theorem 8.1] can be obtained in a similar way as we prove Theorem 0.4 below.
**Proposition 0.10.** The map
\[
\beta : H_T \to \mathcal{M},
\]
\[
T \mapsto (-1)^{|[n]/\sim_T|} 1_{K_T}
\]
is a \(\mathcal{K}\)-algebra homomorphism from \(H_T = (H_T, \cdot)\) to \(\mathcal{M}\).

**Proof of Proposition 0.10 (sketched).** The proof boils down to the observation that if \(n \in \mathbb{N}, m \in \mathbb{N}, T \in T_n\) and \(S \in T_m\), then
\[
K_{T,S} = \{ (x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m} \mid (x_1, x_2, \ldots, x_n) \in K_T
\]
and \((x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \in K_S\}.
\]

Now, we claim:

**Theorem 0.11.** The diagram
\[
\begin{array}{ccc}
H_T & \xrightarrow{U} & WQSym \\
\downarrow \beta & & \downarrow \alpha \\
\mathcal{M} & & 
\end{array}
\]
commutes. That is, we have \(\beta = \alpha \circ U\).

Before we prove this, we introduce some more notations.

**Definition 0.12.** We define a \(\mathcal{K}\)-linear map \(Z : H_T \to H_T\) by
\[
Z(T) = (-1)^{|[n]/\sim_T|} T \quad \text{for every } n \in \mathbb{N} \text{ and } T \in T_n.
\]
It is easy to see that \(Z\) is an involutive Hopf algebra isomorphism.

**Definition 0.13.** Let \(X\) be a finite totally ordered set, and let \(T\) be a topology on \(X\). Let \(a\) and \(b\) be two elements of \(X\). We define three new topologies \(T \leadsto (a \leq b), T \leadsto (a \geq b)\) and \(T \leadsto (a \sim b)\) on \(X\) as follows:
\[
T \leadsto (a \leq b) = \{ O \in T \mid (a \in O \implies b \in O) \};
\]
\[
T \leadsto (a \geq b) = \{ O \in T \mid (b \in O \implies a \in O) \};
\]
\[
T \leadsto (a \sim b) = \{ O \in T \mid (a \in O \iff b \in O) \}.
\]
(It is easy to check that these are actually topologies. Of course, \(T \leadsto (a \geq b) = T \leadsto (b \leq a)\).)

Here comes a collection of simple properties of these three new topologies:
**Lemma 0.14.** Let $X$ be a finite totally ordered set, and let $\mathcal{T}$ be a topology on $X$. Let $a$ and $b$ be two elements of $X$.

(a) We have
\[
(\mathcal{T} \ni (a \leq b)) \cap (\mathcal{T} \ni (a \geq b)) = \mathcal{T} \ni (a \sim b) \quad \text{and} \quad (1)
\]
\[
(\mathcal{T} \ni (a \leq b)) \cup (\mathcal{T} \ni (a \geq b)) = \mathcal{T}. \quad (2)
\]

(b) We have
\[
\mathcal{T} \ni (a \sim b) = (\mathcal{T} \ni (a \leq b)) \ni (a \geq b) = (\mathcal{T} \ni (a \geq b)) \ni (a \leq b).
\]

(c) If $a \leq_T b$, then $\mathcal{T} \ni (a \leq b) = \mathcal{T}$ and $\mathcal{T} \ni (a \sim b) = \mathcal{T} \ni (a \geq b)$.

(d) If $b \leq_T a$, then $\mathcal{T} \ni (a \geq b) = \mathcal{T}$ and $\mathcal{T} \ni (a \sim b) = \mathcal{T} \ni (a \leq b)$.

(e) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \leq b)} d$ holds if and only if
\[
(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)).
\]

(f) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[
(c \leq_T d \text{ or } (c \leq_T b \text{ and } a \leq_T d)).
\]

(g) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[
(c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d) \text{ or } (c \leq_T b \text{ and } a \leq_T d)).
\]

(h) If $c$ and $d$ are two elements of $X$, then $c \leq_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[
\left( c \leq_{\mathcal{T} \ni (a \leq b)} d \text{ or } c \leq_{\mathcal{T} \ni (a \geq b)} d \right).
\]

(i) If $c$ and $d$ are two elements of $X$, then $c \leq_T d$ holds if and only if
\[
\left( c \leq_{\mathcal{T} \ni (a \leq b)} d \text{ and } c \leq_{\mathcal{T} \ni (a \geq b)} d \right).
\]

(j) If $c$ and $d$ are two elements of $X$, then $c \sim_{\mathcal{T} \ni (a \leq b)} d$ holds if and only if
\[
(c \sim_T d \text{ or } (b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a)).
\]

(k) If $c$ and $d$ are two elements of $X$, and if we have neither $a \leq_T b$ nor $b \leq_T a$, then $c \sim_{\mathcal{T} \ni (a \sim b)} d$ holds if and only if
\[
(c \sim_T d \text{ or } (c \sim_T a \text{ and } d \sim_T b) \text{ or } (c \sim_T b \text{ and } d \sim_T a)).
\]

(l) We have
\[
\mathcal{P} (\mathcal{T} \ni (a \leq b)) \cap \mathcal{P} (\mathcal{T} \ni (a \geq b)) = \mathcal{P} (\mathcal{T} \ni (a \sim b)) \quad \text{and} \quad (3)
\]
\[
\mathcal{P} (\mathcal{T} \ni (a \leq b)) \cup \mathcal{P} (\mathcal{T} \ni (a \geq b)) = \mathcal{P} (\mathcal{T}). \quad (4)
\]

(m) Assume that neither $a \leq_T b$ nor $b \leq_T a$. Then, the three sets $\mathcal{U} (\mathcal{T} \ni (a \leq b))$, $\mathcal{U} (\mathcal{T} \ni (a \geq b))$ and $\mathcal{U} (\mathcal{T} \ni (a \sim b))$ are disjoint, and their union is $\mathcal{U} (\mathcal{T})$.

(n) Assume that neither $a \leq_T b$ nor $b \leq_T a$. Then,
\[
|X/ \sim_{\mathcal{T} \ni (a \leq b)}| = |X/ \sim_{\mathcal{T} \ni (a \geq b)}| = |X/ \sim_T| \quad \text{and} \quad (5)
\]
\[
|X/ \sim_{\mathcal{T} \ni (a \sim b)}| = |X/ \sim_T| - 1.
\]
Proof of Lemma 0.14 (sketched). Parts (a) and (b) are straightforward to check.

(c) Assume that \( a \leq_T b \). Then, every \( O \in \mathcal{T} \) satisfies \( (a \in \mathcal{T} \implies b \in \mathcal{T}) \). Hence, \( \mathcal{T} \mathcal{ \not\in } (a \leq b) = \mathcal{T} \) by the definition of \( \mathcal{T} \mathcal{ \not\in } (a \leq b) \). From Lemma 0.14 (b), we have \( \mathcal{T} \mathcal{ \not\in } (a \sim b) = (\mathcal{T} \mathcal{ \not\in } (a \leq b)) \mathcal{ \not\in } (a \geq b) = \mathcal{T} \mathcal{ \not\in } (a \geq b) \). Thus, \( c \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \) holds.

Lemma 0.14 (c) is proven.

(d) The proof of part (d) is similar to that of (c).

(e) \( \iff \): Assume that \( (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)) \). We need to check that \( c \leq_T (a \leq b) \). In other words, we need to check that every \( O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \) satisfying \( c \in O \) satisfies \( d \in O \). So let us fix an \( O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \) satisfying \( c \in O \). We must prove that \( d \in O \).

We have \( O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \subseteq \mathcal{T} \) (by the definition of \( \mathcal{T} \mathcal{ \not\in } (a \leq b) \)). Thus, if \( c \leq_T d \), then \( d \in O \). Hence, for the rest of this proof, WLOG assume that we don’t have \( c \leq_T d \). Thus, by assumption, we have \( c \leq_T a \) and \( b \leq_T d \). Therefore, \( a \in O \) (since \( c \in O \) and \( c \leq_T a \)). But \( O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \), and therefore \( (a \in O \implies b \in O) \) (by the definition of \( \mathcal{T} \mathcal{ \not\in } (a \leq b) \)), so that \( b \in O \) (since \( a \in O \)), and thus \( d \in O \) (since \( b \leq_T d \)). This completes the proof of the \( \iff \) direction of Lemma 0.14 (e).

\( \implies \): Assume that \( c \leq_T (a \leq b) \). We need to check that \( (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)) \). We can WLOG assume that we don’t have \( c \leq_T d \). Then, we must prove that \( (c \leq_T a \text{ and } b \leq_T d) \).

We don’t have \( c \leq_T d \). Hence, there exists a \( Q \in \mathcal{T} \) such that \( c \in Q \) but \( d \not\in Q \). Consider this \( Q \). If we had \( (a \in Q \implies b \in Q) \), then \( Q \) would belong to \( \mathcal{T} \mathcal{ \not\in } (a \leq b) \), which would yield \( d \in Q \) (since \( c \leq_T (a \leq b) \) \text{ and } \( c \in Q \)), which would contradict \( d \not\in Q \). Hence, we cannot have \( (a \in Q \implies b \in Q) \). Thus, \( a \in Q \) and \( b \not\in Q \).

Let \( O \in \mathcal{T} \) be such that \( c \in O \). We shall prove that \( a \in O \). Indeed, assume the contrary. Then, \( a \not\in O \). Thus, \( a \not\in Q \cap O \), so that \( (a \in Q \cap O \implies b \in Q \cap O) \). Since \( Q \cap O \in \mathcal{T} \) (because \( Q \in \mathcal{T} \) and \( O \in \mathcal{T} \), this yields \( Q \cap O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \). Since we also have \( c \in Q \cap O \) (since \( c \in Q \) and \( c \in O \)), this yields \( d \in Q \cap O \) (since \( c \leq_T (a \leq b) \)), which contradicts \( d \not\in Q \). This contradiction proves that our assumption was wrong. Hence, \( a \in O \) is proven. Forget now that we fixed \( O \). Thus we have shown that \( a \in O \) for every \( O \in \mathcal{T} \) which satisfies \( c \in O \). In other words, \( c \leq_T a \).

Let \( O \in \mathcal{T} \) be such that \( b \in O \). We shall prove that \( d \in O \). Indeed, assume the contrary. Then, \( d \not\in O \). Thus, \( d \not\in Q \cup O \) (since \( d \not\in Q \) and \( d \not\in O \)). But \( b \in O \subseteq Q \cup O \), so that \( (a \in Q \cup O \implies b \in Q \cup O) \). Since \( Q \cup O \in \mathcal{T} \) (because \( Q \in \mathcal{T} \) and \( O \in \mathcal{T} \), this yields \( Q \cup O \in \mathcal{T} \mathcal{ \not\in } (a \leq b) \). Since we also have \( c \in Q \cup O \) (since \( c \in Q \)), this yields \( d \in Q \cup O \) (since \( c \leq_T (a \leq b) \)), which contradicts \( d \not\in Q \cup O \). This contradiction proves that our assumption was wrong. Hence, \( d \in O \) is proven. Forget now that we fixed \( O \). Thus we have shown that \( d \in O \) for every \( O \in \mathcal{T} \) which satisfies \( b \in O \). In other words, \( b \leq_T d \).

We thus have shown that \( (c \leq_T a \text{ and } b \leq_T d) \). This completes the proof of the \( \implies \) direction of Lemma 0.14 (e).
(f) The proof of part (f) is analogous to that of (e).

(g) Let \( c \) and \( d \) be two elements of \( X \). Then, we have the following logical equivalence:

\[
\left( c \leq T \rightarrow (a \sim b) \right) d \\
\iff \left( c \leq (T \rightarrow (a \leq b)) \rightarrow (a \geq b) \right) d \quad \text{(by Lemma 0.14(b))} \\
\iff \left( c \leq T \rightarrow (a \leq b) \text{ or } \left( c \leq T \rightarrow (a \leq b) \text{ and } a \leq (T \rightarrow (a \leq b)) \right) \right) d \quad \text{(by Lemma 0.14(f))} \\
\iff \left( (c \leq T \text{ or } (c \leq T a \text{ and } b \leq T d)) \right) \text{ or } \left( ((c \leq T b \text{ or } (c \leq T a \text{ and } b \leq T b)) \text{ and } (a \leq T d \text{ or } (a \leq T a \text{ and } b \leq T d))) \right) \\
\iff \left( c \leq T d \text{ or } (c \leq T a \text{ and } b \leq T d) \right) \text{ or } (c \leq T b \text{ and } a \leq T d) \\
\text{(after simplifying using the transitivity and reflexivity of } \leq T). \\
\]

This proves Lemma 0.14(g).

(h) This is just a rewriting of Lemma 0.14(g) using parts (e) and (f).

(i) \( \iff \): This is clear.

\( \iff \): Assume that \( \left( c \leq T \rightarrow (a \leq b) \right) d \) and \( c \leq T \rightarrow (a \geq b) \). We need to show that \( c \leq T d \). Indeed, assume the contrary.

We have \( c \leq T \rightarrow (a \leq b) \). Thus, Lemma 0.14(e) yields \( c \leq T d \text{ or } (c \leq T a \text{ and } b \leq T d) \). Since we assumed that \( c \leq T d \) does not hold, this yields \( c \leq T a \text{ and } b \leq T d \). Similarly, \( c \leq T b \text{ and } a \leq T d \). Thus, \( c \leq T b \leq T d \), which contradicts our assumption that not \( c \leq T d \). This contradiction completes the proof.

(j) We have \( c \sim T \rightarrow (a \leq b) \) if and only if \( \left( c \leq T \rightarrow (a \leq b) \right) d \text{ and } d \leq T \rightarrow (a \leq b) \). We can rewrite each of the two statements \( c \leq T \rightarrow (a \leq b) \) and \( d \leq T \rightarrow (a \leq b) \) using Lemma 0.14(e), and then simplify the result; we end up with Lemma 0.14(j).

(k) Let \( c \) and \( d \) be two elements of \( X \). Assume that we have neither \( a \leq T b \) nor \( b \leq T a \). We have \( c \sim T \rightarrow (a \sim b) \) if and only if \( \left( c \leq T \rightarrow (a \leq b) \right) d \text{ and } d \leq T \rightarrow (a \sim b) \). We can rewrite each of the two statements \( c \leq T \rightarrow (a \sim b) \) and \( d \leq T \rightarrow (a \sim b) \) using Lemma 0.14(g), and then simplify the result (a disjunction with 9 cases, of which many can be ruled out due to the assumption that neither \( a \leq T b \) nor \( b \leq T a \)); we end up with Lemma 0.14(k).

(l) Proof of \( P(T \not\rightarrow (a \leq b)) \cap P(T \not\rightarrow (a > b)) = P(T \not\rightarrow (a \sim b)) \): Whenever \( f \) is a surjective map \( X \rightarrow [p] \) for some \( p \in \mathbb{N} \), we have the following
logical equivalence:

\[ (f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cap \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b))) \]

\[ \Longleftrightarrow \left( \begin{array}{c}
(f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b))) \\
\land \\
(f \in \mathcal{P}(\mathcal{T} \leftrightarrow (b \leq a)))
\end{array} \right)
\]

\[ \Longleftrightarrow \left( \begin{array}{c}
\text{(every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}(a \leq b)} d \text{ satisfy } f(c) \leq f(d)) \\
\land \\
\text{(every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}(b \leq a)} d \text{ satisfy } f(c) \leq f(d))
\end{array} \right)
\]

\[ \Longleftrightarrow \left( \begin{array}{c}
\text{(every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}(a \leq b)} d \text{ satisfy } f(c) \leq f(d)) \\
\land \\
\text{(every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}(b \leq a)} d \text{ satisfy } f(c) \leq f(d))
\end{array} \right)
\]

\[ \Longleftrightarrow \left( \begin{array}{c}
\text{every } c \in X \text{ and } d \in X \text{ satisfying } (c \leq_{\mathcal{T}(a \leq b)} d \text{ or } c \leq_{\mathcal{T}(b \leq a)} d)
\end{array} \right)
\]

\[ \Longleftrightarrow (f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \sim b))). \]

Thus, \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cap \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) = \mathcal{P}(\mathcal{T} \leftrightarrow (a \sim b)) \) is proven.

It remains to prove \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) = \mathcal{P}(\mathcal{T}) \). We shall achieve this by proving both inclusions separately:

Proof of \( \mathcal{P}(\mathcal{T}) \subseteq \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \): Let \( f \in \mathcal{P}(\mathcal{T}) \). We must prove that \( f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \).

We WLOG assume that \( f(a) \leq f(b) \). We shall now show that \( f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \).

This will yield that \( f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \), and thus complete this proof of \( \mathcal{P}(\mathcal{T}) \subseteq \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \).

Let \( c \in X \) and \( d \in X \) be such that \( c \leq_{\mathcal{T}(a \leq b)} d \). In order to prove that \( f \in \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \), we must now show that \( f(c) \leq f(d) \).

We have \( c \leq_{\mathcal{T}(a \leq b)} d \). Due to Lemma 0.14 (e), this yields that \( (c \leq d \text{ or } (c \leq a \text{ and } b \leq d)) \). In the first of these cases, \( f(c) \leq f(d) \) follows immediately from \( f \in \mathcal{P}(\mathcal{T}) \); thus, let us assume that we are in the second case. Thus, \( c \leq a \) and \( b \leq d \). From \( f \in \mathcal{P}(\mathcal{T}) \), we thus obtain \( f(c) \leq f(a) \) and \( f(b) \leq f(d) \). Hence, \( f(c) \leq f(a) \leq f(b) \leq f(d) \), qed.

Proof of \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \subseteq \mathcal{P}(\mathcal{T}) \): We now need to show that \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \subseteq \mathcal{P}(\mathcal{T}) \). To do so, it is clearly enough to prove \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \leq b)) \subseteq \mathcal{P}(\mathcal{T}) \) and \( \mathcal{P}(\mathcal{T} \leftrightarrow (a \geq b)) \subseteq \mathcal{P}(\mathcal{T}) \). We shall
only show the first of these two relations, as the second is analogous. Let \( f \in \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \). Then, every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} d \) satisfy \( f (c) \leq f (d) \). Hence, every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{\mathcal{T}} d \) satisfy \( f (c) \leq f (d) \) (since every \( c \in X \) and \( d \in X \) satisfying \( c \leq_{\mathcal{T}} d \) satisfy \( c \leq_{\mathcal{T}+\rightharpoonup (a \leq b)} d \) (due to Lemma 0.14 (e))). In other words, \( f \in \mathcal{P} (\mathcal{T}) \). Since this is proven for every \( f \in \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \), we thus conclude that \( \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \subseteq \mathcal{P} (\mathcal{T}) \).

The proof of Lemma 0.14 (l) is thus complete.

(m) It is clearly enough to prove the three equalities

\[
\mathcal{U} (\mathcal{T} \rightharpoonup (a \leq b)) = \{ f \in \mathcal{U} (\mathcal{T}) \mid f (a) < f (b) \}; \tag{3}
\]

\[
\mathcal{U} (\mathcal{T} \rightharpoonup (a \sim b)) = \{ f \in \mathcal{U} (\mathcal{T}) \mid f (a) = f (b) \}; \tag{4}
\]

\[
\mathcal{U} (\mathcal{T} \rightharpoonup (a \geq b)) = \{ f \in \mathcal{U} (\mathcal{T}) \mid f (a) > f (b) \}. \tag{5}
\]

We shall only check the first two of these three equalities (since the third one is analogous to the first).

Let us first check that \( a \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} b \). Indeed, it is clear from the definition of \( \mathcal{T} \rightharpoonup (a \leq b) \) that \( a \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} b \). Thus, in order to prove that \( a \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} b \), we must only show that we don’t have \( b \leq_{\mathcal{T}+\rightharpoonup (a \leq b)} a \). To achieve this, we assume the contrary. Lemma 0.14 (e) (applied to \( c = b \) and \( d = a \)) thus yields that \( (b \leq_{\mathcal{T} a} \text{ or } (b \leq_{\mathcal{T} a} \text{ and } b \leq_{\mathcal{T} a}) \). In either of these cases, we must have \( b \leq_{\mathcal{T} a} \), which contradicts the assumption that neither \( a \leq_{\mathcal{T} b} \) nor \( b \leq_{\mathcal{T} a} \). So \( a \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} b \) is proven.

Next, we are going to prove (3) by showing its two inclusions separately:

Proof of \( \mathcal{U} (\mathcal{T} \rightharpoonup (a \leq b)) \subseteq \{ f \in \mathcal{U} (\mathcal{T}) \mid f (a) < f (b) \} \): Let \( g \in \mathcal{U} (\mathcal{T} \rightharpoonup (a \leq b)) \).

Thus, \( g \in \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \), and every two elements \( i \) and \( j \) of \( X \) satisfying \( i \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} j \) must satisfy \( g (i) < g (j) \). Applying the latter fact to \( i = a \) and \( j = b \), we obtain \( g (a) < g (b) \) (since \( a \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} b \)).

Moreover, \( g \in \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \subseteq \mathcal{P} (\mathcal{T} \rightharpoonup (a \leq b)) \cup \mathcal{P} (\mathcal{T} \rightharpoonup (a \geq b)) = \mathcal{P} (\mathcal{T}) \) (by Lemma 0.14 (l)).

Let now \( i \) and \( j \) be any two elements of \( X \) satisfying \( i \prec_{\mathcal{T}} j \). We shall show that \( g (i) < g (j) \).

Indeed, \( i \prec_{\mathcal{T}} j \), thus \( i \leq_{\mathcal{T}} j \) and therefore \( i \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} j \) (due to Lemma 0.14 (e)). Assume (for the sake of contradiction) that \( j \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} i \). Then, \( i \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} j \), and thus (by Lemma 0.14 (j), applied to \( c = i \) and \( d = j \)) we have \((i \prec_{\mathcal{T}} j \text{ or } (b \leq_{\mathcal{T} i} \text{ and } b \leq_{\mathcal{T} j} \leq_{\mathcal{T} a}) \). But neither of these two cases can occur (since \( i \prec_{\mathcal{T}} j \) precludes \( i \prec_{\mathcal{T}} j \), and since \( b \leq_{\mathcal{T} i} \text{ and } b \leq_{\mathcal{T} j} \leq_{\mathcal{T} a} \)). Hence, we have our contradiction. Thus, our assumption (that \( j \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} i \) was false. We therefore have \( i \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} j \) but not \( j \leq_{\mathcal{T}-\rightharpoonup (a \leq b)} i \). In other words, \( i \prec_{\mathcal{T}-\rightharpoonup (a \leq b)} j \). Thus, \( g (i) < g (j) \) (since \( g \in \mathcal{U} (\mathcal{T} \rightharpoonup (a \leq b)) \)).

Now, let us forget that we fixed \( i \) and \( j \). We thus have shown that any two elements \( i \) and \( j \) of \( X \) satisfying \( i \prec_{\mathcal{T}} j \) satisfy \( g (i) < g (j) \). In other words, \( g \in \mathcal{U} (\mathcal{T}) \) (since we already know that \( g \in \mathcal{P} (\mathcal{T}) \)). Thus, \( g \) is an element of \( \mathcal{U} (\mathcal{T}) \) and satisfies \( g (a) < g (b) \). In other words, \( g \in \{ f \in \mathcal{U} (\mathcal{T}) \mid f (a) < f (b) \} \).
Since this is proven for every \( g \in \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \), we thus conclude that

\( \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \subseteq \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \).

Proof of \( \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \subseteq \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \): Let

\( g \in \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \). Then, \( g \in \mathcal{U}(\mathcal{T}) \) and \( g(a) < g(b) \). From

\( g \in \mathcal{U}(\mathcal{T}) \), we obtain \( g \in \mathcal{P}(\mathcal{T}) \).

Let now \( c \in X \) and \( d \in X \) be such that \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \). We now aim to show that \( g(c) \leq g(d) \).

Indeed, from \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \), we obtain \( (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)) \) (by Lemma 0.14(e)). In the first of these two cases, we obtain \( g(c) \leq g(d) \) immediately (since \( g \in \mathcal{P}(\mathcal{T}) \)), while in the second case we obtain

\[
g(c) \leq g(a) \quad \text{(since } c \leq_T a \text{ and } g \in \mathcal{P}(\mathcal{T}) \text{)}
\]

\[
g(b) \leq g(d) \quad \text{(since } b \leq_T d \text{ and } g \in \mathcal{P}(\mathcal{T}) \text{)}.
\]

Thus, \( g(c) \leq g(d) \) is proven in either case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) \leq g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \). In other words, \( g \in \mathcal{P}(\mathcal{T} \dashv (a \leq b)) \).

Now, let \( c \in X \) and \( d \in X \) be such that \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \). We now aim to show that \( g(c) < g(d) \).

Indeed, from \( c <_{\mathcal{T} \dashv (a \leq b)} d \), we obtain \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \), and thus

\( (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d)) \) (by Lemma 0.14(e)). In the second of these two cases, we have

\[
g(c) \leq g(a) \quad \text{(since } c \leq_T a \text{ and } g \in \mathcal{P}(\mathcal{T}) \text{)}
\]

\[
g(b) \leq g(d) \quad \text{(since } b \leq_T d \text{ and } g \in \mathcal{P}(\mathcal{T}) \text{)}.
\]

Thus, \( g(c) < g(d) \) is proven in the second case. We thus WLOG assume that we are in the first case. That is, we have \( c \leq_T d \). If \( c <_T d \), then we can immediately conclude that \( g(c) < g(d) \) (since \( g \in \mathcal{U}(\mathcal{T}) \)). Hence, we WLOG assume that we don’t have \( c <_{\mathcal{T} \dashv} d \). Thus, \( c \sim_T d \) (since \( c \leq_T d \)), so that \( d \leq_T c \). Hence, \( (d \leq_T c \text{ or } (d \leq_T a \text{ and } b \leq_T c)) \), so that Lemma 0.14(e) (applied to \( d \) and \( c \)) instead of \( c \) and \( d \) yields \( d \leq_{\mathcal{T} \dashv (a \leq b)} c \). But this contradicts \( c <_{\mathcal{T} \dashv (a \leq b)} d \). Thus, we have obtained a contradiction, and our proof of \( g(c) < g(d) \) is complete.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) < g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \leq_{\mathcal{T} \dashv (a \leq b)} d \). In other words, \( g \in \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \) (since \( g \in \mathcal{P}(\mathcal{T} \dashv (a \leq b)) \)). Since this is proven for every \( g \in \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \), we thus conclude that

\( \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \subseteq \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \).

Combining

\( \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \subseteq \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \)

with

\( \{ f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b) \} \subseteq \mathcal{U}(\mathcal{T} \dashv (a \leq b)) \),

we obtain 0.14(h). Let us next check that \( a \sim_{\mathcal{T} \dashv (a \leq b)} b \). Indeed, it is clear from the definition of \( \mathcal{T} \dashv (a \sim b) \) that \( a \leq_{\mathcal{T} \dashv (a \sim b)} b \) and that \( b \leq_{\mathcal{T} \dashv (a \sim b)} a \). Combining these, we obtain \( a \sim_{\mathcal{T} \dashv (a \sim b)} b \).
Next, we are going to prove \( \square \) by showing its two inclusions separately:

**Proof of** \( U (\mathcal{T} \mapsto (a \sim b)) \subseteq \{ f \in U (\mathcal{T}) \mid f (a) = f (b) \} \): Let \( g \in U (\mathcal{T} \mapsto (a \sim b)) \).

Thus, \( g \in \mathcal{P} (\mathcal{T} \mapsto (a \sim b)) \), and every two elements \( i \) and \( j \) of \( X \) satisfying \( i \sim_{\mathcal{T} \mapsto (a \sim b)} j \) must satisfy \( g (i) < g (j) \).

We have \( a \sim_{\mathcal{T} \mapsto (a \sim b)} b \) and \( g \in \mathcal{P} (\mathcal{T} \mapsto (a \sim b)) \); thus, \( g (a) = g (b) \).

Moreover,

\[
\begin{align*}
g & \in \mathcal{P} (\mathcal{T} \mapsto (a \sim b)) = \mathcal{P} (\mathcal{T} \mapsto (a \leq b)) \cap \mathcal{P} (\mathcal{T} \mapsto (a \geq b)) \\
& \subseteq \mathcal{P} (\mathcal{T} \mapsto (a \leq b)) \subseteq \mathcal{P} (\mathcal{T} \mapsto (a \leq b)) \cup \mathcal{P} (\mathcal{T} \mapsto (a \geq b)) = \mathcal{P} (\mathcal{T})
\end{align*}
\]

(by Lemma \[0.14\](l)).

Now, let \( i \) and \( j \) be any two elements of \( X \) satisfying \( i \sim_{\mathcal{T}} j \). We shall show that \( g (i) < g (j) \).

Indeed, \( i \sim_{\mathcal{T}} j \), thus \( i \leq_{\mathcal{T}} j \) and therefore \( i \leq_{\mathcal{T} \mapsto (a \sim b)} j \) (due to Lemma \[0.14\](g)). Assume (for the sake of contradiction) that \( j \leq_{\mathcal{T} \mapsto (a \sim b)} i \). Then, \( i \sim_{\mathcal{T} \mapsto (a \sim b)} j \), and thus (by Lemma \[0.14\](k), applied to \( c = i \) and \( d = j \)) we have \((i \sim_{\mathcal{T}} j \) or \((i \sim_{\mathcal{T}} a \) and \( j \sim_{\mathcal{T}} b \) or \((i \sim_{\mathcal{T}} b \) and \( j \sim_{\mathcal{T}} a \)). But neither of these three cases can occur. Hence, we have our contradiction. Thus, our assumption (that \( j \leq_{\mathcal{T} \mapsto (a \sim b)} i \)) was false. We therefore have \( i \sim_{\mathcal{T} \mapsto (a \sim b)} j \) but not \( j \leq_{\mathcal{T} \mapsto (a \sim b)} i \). In other words, \( i \not<_{\mathcal{T} \mapsto (a \sim b)} j \). Thus, \( g (i) < g (j) \) (since \( g \in U (\mathcal{T} \mapsto (a \sim b)) \)).

Now, let us forget that we fixed \( i \) and \( j \). We thus have shown that any two elements \( i \) and \( j \) of \( X \) satisfying \( i \sim_{\mathcal{T}} j \) satisfy \( g (i) < g (j) \). In other words, \( g \in U (\mathcal{T}) \) (since we already know that \( g \in \mathcal{P} (\mathcal{T}) \)). Thus, \( g \) is an element of \( U (\mathcal{T}) \) and satisfies \( g (a) = g (b) \). In other words, \( g \in \{ f \in U (\mathcal{T}) \mid f (a) = f (b) \} \).

Since this is proven for every \( g \in U (\mathcal{T} \mapsto (a \sim b)) \), we thus conclude that \( U (\mathcal{T} \mapsto (a \sim b)) \subseteq \{ f \in U (\mathcal{T}) \mid f (a) = f (b) \} \).

**Proof of** \( \{ f \in U (\mathcal{T}) \mid f (a) = f (b) \} \subseteq U (\mathcal{T} \mapsto (a \sim b)) \): Let \( g \in \{ f \in U (\mathcal{T}) \mid f (a) = f (b) \} \). Then, \( g \in U (\mathcal{T}) \) and \( g (a) = g (b) \). From \( g \in U (\mathcal{T}) \), we obtain \( g \in \mathcal{P} (\mathcal{T}) \).

Let now \( c \in X \) and \( d \in X \) be such that \( c \leq_{\mathcal{T} \mapsto (a \sim b)} d \). We now aim to show that \( g (c) \leq g (d) \).

Indeed, \( \square \), \( c \leq_{\mathcal{T} \mapsto (a \sim b)} d \), we obtain

\[
\begin{align*}
(c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d) \text{ or } (c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d)) \text{ (by Lemma \[0.14\](g)).}
\end{align*}
\]

In the first of these three cases, we obtain \( g (c) \leq g (d) \) immediately (since \( g \in \mathcal{P} (\mathcal{T}) \)). In the second case, we obtain

\[
\begin{align*}
g (c) & \leq g (a) \quad \text{(since } c \leq_{\mathcal{T}} a \text{ and } g \in \mathcal{P} (\mathcal{T}) \text{)} \\
& = g (b) \leq g (d) \quad \text{(since } b \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P} (\mathcal{T}) \text{)}.
\end{align*}
\]

Indeed, the first case \( (i \sim_{\mathcal{T}} j) \) is precluded by the fact that \( i \sim_{\mathcal{T}} j \). The second case \( (i \sim_{\mathcal{T}} a \) and \( j \sim_{\mathcal{T}} b \) cannot occur since it would lead to \( a \sim_{\mathcal{T}} i \leq_{\mathcal{T}} j \sim_{\mathcal{T}} b \), which would contradict the assumption that we have neither \( a \leq_{\mathcal{T}} b \) nor \( b \leq_{\mathcal{T}} a \). The third case \( (i \sim_{\mathcal{T}} b \) and \( j \sim_{\mathcal{T}} a \) cannot occur for a similar reason.
In the third case, we obtain
\[ g(c) \leq g(b) \quad (\text{since } c \leq_T b \text{ and } g \in \mathcal{P}(T)) \]
\[ = g(a) \leq g(d) \quad (\text{since } a \leq_T d \text{ and } g \in \mathcal{P}(T)). \]

Thus, \( g(c) \leq g(d) \) is proven in either case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) \leq g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c \leq_T (a \sim b) \). In other words, \( g \in \mathcal{P}(T \leftrightarrow (a \sim b)) \).

Now, let \( c \in X \) and \( d \in X \) be such that \( c <_{T \leftrightarrow (a \sim b)} d \). We now aim to show that \( g(c) < g(d) \).

Indeed, from \( c <_{T \leftrightarrow (a \sim b)} d \), we obtain \( c \leq_{T \leftrightarrow (a \sim b)} d \), and thus
\( (c \leq_T d \text{ or } (c \leq_T a \text{ and } b \leq_T d) \text{ or } (c \leq_T b \text{ and } a \leq_T d)) \) (by Lemma 0.14 (g)).

We study these three cases separately:

- Assume that we are in the first case, i.e., we have \( c \leq_T d \). Then, \( c <_{T \leftrightarrow (a \sim b)} d \) (since otherwise, we would have \( d \leq_T c \), and therefore \( d \leq_{T \leftrightarrow (a \sim b)} c \) (by Lemma 0.14 (g)), which would contradict \( c <_{T \leftrightarrow (a \sim b)} d \). Hence, \( g(c) < g(d) \) (since \( g \in \mathcal{U}(T) \)).

- Assume that we are in the second case, i.e., we have \( (c \leq_T a \text{ and } b \leq_T d) \). Then,
\[ g(c) \leq g(a) \quad (\text{since } c \leq_T a \text{ and } g \in \mathcal{P}(T)) \]
\[ = g(b) \leq g(d) \quad (\text{since } b \leq_T d \text{ and } g \in \mathcal{P}(T)). \]

If at least one of the strict inequalities \( c <_{T \leftrightarrow (a \sim b)} d \) holds, then we can strengthen this to a strict inequality \( g(c) < g(d) \) (because \( g \in \mathcal{U}(T) \)), and thus be done. Hence, we WLOG assume that none of the inequalities \( c <_{T \leftrightarrow (a \sim b)} d \) holds. Thus, \( c \sim_T a \text{ and } b \sim_T d \). Hence, \( c \sim_{T \leftrightarrow (a \sim b)} a \text{ and } b \sim_{T \leftrightarrow (a \sim b)} d \) (by Lemma 0.14 (k)), so that \( c \sim_{T \leftrightarrow (a \sim b)} a \sim_{T \leftrightarrow (a \sim b)} b \sim_{T \leftrightarrow (a \sim b)} d \), which contradicts \( c <_{T \leftrightarrow (a \sim b)} d \). Hence, we are done in the second case as well.

- The third case is similar to the second case.

Thus, our proof of \( g(c) < g(d) \) is complete in each case.

Now, let us forget that we fixed \( c \) and \( d \). We thus have proven that \( g(c) < g(d) \) for any \( c \in X \) and \( d \in X \) satisfying \( c <_{T \leftrightarrow (a \sim b)} d \). In other words, \( g \in \mathcal{U}(T \leftrightarrow (a \sim b)) \) (since \( g \in \mathcal{P}(T \leftrightarrow (a \sim b)) \)). Since this is proven for every \( g \in \{ f \in \mathcal{U}(T) \mid f(a) = f(b) \} \), we thus conclude that \( \{ f \in \mathcal{U}(T) \mid f(a) = f(b) \} \subseteq \mathcal{U}(T \leftrightarrow (a \sim b)) \).

Combining \( \mathcal{U}(T \leftrightarrow (a \sim b)) \subseteq \{ f \in \mathcal{U}(T) \mid f(a) = f(b) \} \) with \( \{ f \in \mathcal{U}(T) \mid f(a) = f(b) \} \subseteq \mathcal{U}(T \leftrightarrow (a \sim b)) \), we obtain (4).

Now, our proof of Lemma 0.14 (m) is complete.
If $c$ and $d$ are two elements of $X$, then $c \sim_{T+\rho(a \leq b)} d$ holds if and only if

$$(c \sim_{T} d \text{ or } (b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a))$$

(according to Lemma 0.14 (j)). Since $(b \leq_T c \leq_T a \text{ and } b \leq_T d \leq_T a)$ cannot hold (because of our assumption that not $b \leq_T a$), this simplifies as follows: If $c$ and $d$ are two elements of $X$, then $c \sim_{T+\rho(a \leq b)} d$ holds if and only if $c \sim_{T} d$. Thus, the equivalence relation $\sim_{T+\rho(a \leq b)}$ is identical to $\sim_{T}$. Hence, $\left| X/ \sim_{T+\rho(a \leq b)} \right| = |X/ \sim_{T}|$. Similarly, $\left| X/ \sim_{T+\rho(a \geq b)} \right| = |X/ \sim_{T}|$. Thus, $\left| X/ \sim_{T+\rho(a \leq b)} \right| = \left| X/ \sim_{T+\rho(a \geq b)} \right| = |X/ \sim_{T}| - 1$.

Lemma 0.14 (k) yields the following: If $c$ and $d$ are two elements of $X$, then $c \sim_{T+\rho(a \sim b)} d$ holds if and only if

$$(c \sim_{T} d \text{ or } (c \sim_{T} a \text{ and } d \sim_{T} b) \text{ or } (c \sim_{T} b \text{ and } d \sim_{T} a)).$$

In other words, two elements of $X$ are equivalent under the equivalence relation $\sim_{T+\rho(a \sim b)}$ if and only if either they are equivalent under $\sim_{T}$, or one of them is in the $\sim_{T}$-class of $a$ while the other is in the $\sim_{T}$-class of $b$. Thus, when passing from the equivalence relation $\sim_{T}$ to $\sim_{T+\rho(a \sim b)}$, the equivalence classes of $a$ and $b$ get merged (and these two classes used to be separate for $\sim_{T}$, because of our assumption that neither $a \leq_T b$ nor $b \leq_T a$), while all other equivalence classes stay as they were. Thus, the total number of equivalence classes decreases by 1. In other words, $\left| X/ \sim_{T+\rho(a \sim b)} \right| = |X/ \sim_{T}| - 1$. This completes the proof of Lemma 0.14 (n).

**Lemma 0.15.** Let $n \in \mathbb{N}$ and $T \in T_n$. Let $a$ and $b$ be two elements of $[n]$. Then,

$$1_{K_T} = 1_{K_{T+\rho(a \leq b)}} + 1_{K_{T+\rho(a \geq b)}} - 1_{K_{T+\rho(a \sim b)}}.$$

**Proof of Lemma 0.15.** It is clearly enough to prove that

$$K_T = K_{T+\rho(a \leq b)} \cap K_{T+\rho(a \geq b)} \quad (6)$$

and

$$K_{T+\rho(a \sim b)} = K_{T+\rho(a \leq b)} \cup K_{T+\rho(a \geq b)} \quad (7)$$

Before we start proving these statements, let us rewrite the definition of $K_S$ for any topology $S$ on $[n]$. Namely, if $O$ is a subset of $[n]$, then we define a subset $K_O$ of $\mathbb{R}^n$ by

$$K_O = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus O} x_i \geq 0 \right\}.$$
It is now clear that any topology $\mathcal{S}$ on $[n]$ satisfies
\[
K_\mathcal{S} = \bigcap_{O \in \mathcal{S}} K_O.
\] (8)

(Indeed, this is just a restatement of the definition of $K_\mathcal{S}$, since the closed sets of $\mathcal{S}$ are the sets of the form $[n] \setminus O$ with $O$ being an open set of $\mathcal{S}$).

**Proof of (6):** From (8), we obtain $K_T = \bigcap_{O \in T} K_O$, $K_{T+\varphi(a \leq b)} = \bigcap_{O \in T+\varphi(a \leq b)} K_O$ and $K_{T+\varphi(a \geq b)} = \bigcap_{O \in T+\varphi(a \geq b)} K_O$. Thus,
\[
K_{T+\varphi(a \leq b)} \cap K_{T+\varphi(a \geq b)} = \left( \bigcap_{O \in T+\varphi(a \leq b)} K_O \right) \cap \left( \bigcap_{O \in T+\varphi(a \geq b)} K_O \right) \\
= \bigcap_{O \in (T+\varphi(a \leq b)) \cup (T+\varphi(a \geq b))} K_O = \bigcap_{O \in T} K_O \quad \text{(by (2))} \\
= K_T.
\]

This proves (6).

**Proof of (7):** It is easy to see that $K_{T+\varphi(a \leq b)} \subseteq K_{T+\varphi(a \sim b)}$ and similarly $K_{T+\varphi(a \leq b)} \subseteq K_{T+\varphi(a \sim b)}$. Combining these two relations, we obtain $K_{T+\varphi(a \leq b)} \cup K_{T+\varphi(a \geq b)} \subseteq K_{T+\varphi(a \sim b)}$. Hence, in order to prove (7), it remains to show that $K_{T+\varphi(a \sim b)} \subseteq K_{T+\varphi(a \sim b)} \cup K_{T+\varphi(a \sim b)}$. So let us do this now.

Let $y \in K_{T+\varphi(a \sim b)}$. Our goal is to show that $y \in K_{T+\varphi(a \sim b)} \cup K_{T+\varphi(a \sim b)}$. In fact, assume the contrary. Then, $y \notin K_{T+\varphi(a \sim b)}$ and $y \notin K_{T+\varphi(a \sim b)}$.

We have $y \notin K_{T+\varphi(a \sim b)} = \bigcap_{O \in T+\varphi(a \leq b)} K_O$ (by (8)). Hence, there exists a $P \in T \leftarrow (a \leq b)$ such that $y \notin K_P$. Similarly, using $y \notin K_{T+\varphi(a \sim b)}$, we can see that there exists a $Q \in T \leftarrow (a \geq b)$ such that $y \notin K_Q$. Consider these $P$ and $Q$.

We have $P \in T \leftarrow (a \leq b) = \{O \in T \mid (a \in O \implies b \in O)\}$. Thus, $P \in T$ and $(a \in P \implies b \in P)$. But we do not have $(b \in P \implies a \in P)$.

Since $a \notin P$ and $b \in P$ (since $(a \in P \implies b \in P)$ but not $(b \in P \implies a \in P)$).

4Proof. Indeed, (1) yields $(T \leftarrow (a \leq b)) \cap (T \leftarrow (a \geq b)) = T \leftarrow (a \sim b)$, so that $T \leftarrow (a \sim b) \subseteq T \leftarrow (a \leq b)$. Now, from (2), we obtain $K_{T+\varphi(a \leq b)} = \bigcap_{O \in T+\varphi(a \leq b)} K_O$ and $K_{T+\varphi(a \sim b)} = \bigcap_{O \in T+\varphi(a \sim b)} K_O$. Thus,
\[
K_{T+\varphi(a \sim b)} = \bigcap_{O \in T+\varphi(a \sim b)} K_O \subseteq \bigcap_{O \in T+\varphi(a \sim b)} K_O \quad \text{(since $T \leftarrow (a \sim b) \subseteq T \leftarrow (a \leq b)$)} \\
= K_{T+\varphi(a \sim b)},
\]

qed.

5Proof. Assume the contrary. Then, $(b \in P \implies a \in P)$. Hence, $P \in T \leftarrow (a \sim b)$ (by the definition of $T \leftarrow (a \sim b)$). Now, $y \in K_{T+\varphi(a \sim b)} = \bigcap_{O \in T+\varphi(a \sim b)} K_O$ (by (8)). But $\bigcap_{O \in T+\varphi(a \sim b)} K_O \subseteq K_P$ (since $P \in T \leftarrow (a \sim b)$), so that $y \in \bigcap_{O \in T+\varphi(a \sim b)} K_O \subseteq K_P$, which contradicts $y \notin K_P$. This contradiction proves that our assumption was wrong, qed.
We have thus shown that $P \in \mathcal{T}$, $a \notin P$ and $b \in P$. Similarly, we find that $Q \in \mathcal{T}$, $b \notin Q$ and $a \in Q$. Now, it is easy to see that $P \cap Q \in \mathcal{T} \iff (a \sim b) \oplus$ and $P \cup Q \in \mathcal{T} \iff (a \sim b) \ominus$

Let us write $y \in \mathbb{R}^n$ in the form $y = (y_1, y_2, \ldots, y_n)$. We have $(y_1, y_2, \ldots, y_n) = y \notin K_P = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus P} x_i \geq 0 \right\}$. Hence, $\sum_{i \in [n] \setminus P} y_i < 0$. Similarly, from $y \notin K_Q$, we obtain $\sum_{i \in [n] \setminus Q} y_i < 0$.

We have
\[
(y_1, y_2, \ldots, y_n) = y \in K_{\mathcal{T} \setminus P(a \sim b)} \cap K_{O} \quad \text{(by (8))}
\]
\[
\subseteq K_{P \cap Q} \quad \text{(since $P \cap Q \in \mathcal{T} \iff (a \sim b)$)}
\]
\[
= \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus (P \cap Q)} x_i \geq 0 \right\},
\]
so that $\sum_{i \in [n] \setminus (P \cap Q)} y_i \geq 0$. The same argument can be applied to $P \cup Q$ instead of $P \cap Q$, and leads to $\sum_{i \in [n] \setminus (P \cup Q)} y_i \geq 0$.

But any two subsets $A$ and $B$ of $[n]$ satisfy $\sum_{i \in A} y_i + \sum_{i \in B} y_i = \sum_{i \in [n] \setminus A} y_i + \sum_{i \in [n] \setminus B} y_i$. Applying this to $A = [n] \setminus P$ and $B = [n] \setminus Q$, we obtain
\[
\sum_{i \in [n] \setminus P} y_i + \sum_{i \in [n] \setminus Q} y_i = \sum_{i \in ([n] \setminus P) \cap ([n] \setminus Q)} y_i + \sum_{i \in ([n] \setminus P) \cup ([n] \setminus Q)} y_i
\]
\[
= \sum_{i \in [n] \setminus (P \cap Q)} y_i + \sum_{i \in [n] \setminus (P \cup Q)} y_i,
\]
(since $([n] \setminus P) \cup ([n] \setminus Q) = [n] \setminus (P \cap Q)$ and $([n] \setminus P) \cap ([n] \setminus Q) = [n] \setminus (P \cup Q)$). Thus,
\[
\sum_{i \in [n] \setminus (P \cap Q)} y_i + \sum_{i \in [n] \setminus (P \cup Q)} y_i = \sum_{i \in [n] \setminus P} y_i + \sum_{i \in [n] \setminus Q} y_i < 0.
\]

\textit{Proof.} From $P \in \mathcal{T}$ and $Q \in \mathcal{T}$, we infer that $P \cap Q \in \mathcal{T}$. Also, $a \notin P \cap Q$ (since $a \notin P$), so that $(a \in P \cap Q \implies b \in P \cap Q)$. Moreover, $b \notin P \cap Q$ (since $b \notin Q$), and thus $(b \in P \cap Q \implies a \in P \cap Q)$. Combined with $(a \in P \cap Q \implies b \in P \cap Q)$, this yields $(a \in P \cap Q \iff b \in P \cap Q)$. Thus, $P \cap Q$ is an element of $\mathcal{T}$ satisfying $(a \in P \cap Q \iff b \in P \cap Q)$. Hence, $P \cap Q \in \{O \in \mathcal{T} \mid (a \in O \iff b \in O)\} = \mathcal{T} \iff (a \sim b)$, qed.

\textit{Proof.} From $P \in \mathcal{T}$ and $Q \in \mathcal{T}$, we infer that $P \cup Q \in \mathcal{T}$. Also, $b \in P \cup Q$ (since $b \in P$), so that $(a \in P \cup Q \implies b \in P \cup Q)$. Moreover, $a \in P \cup Q$ (since $a \in Q$), and thus $(b \in P \cup Q \implies a \in P \cup Q)$. Combined with $(a \in P \cup Q \implies b \in P \cup Q)$, this yields $(a \in P \cup Q \iff b \in P \cup Q)$. Thus, $P \cup Q$ is an element of $\mathcal{T}$ satisfying $(a \in P \cup Q \iff b \in P \cup Q)$. Hence, $P \cup Q \in \{O \in \mathcal{T} \mid (a \in O \iff b \in O)\} = \mathcal{T} \iff (a \sim b)$, qed.
This contradicts
\[
\sum_{i \in [n] \setminus (P \cap Q)} y_i + \sum_{i \in [n] \setminus (P \cup Q)} y_i \geq 0.
\]

This contradiction proves that our assumption was wrong. Hence, \( y \in K_{T \leftarrow P(a \leq b)} \cup K_{T \leftarrow P(a \geq b)} \). Since we have proven this for every \( y \in K_{T \leftarrow P(a \sim b)} \), we thus conclude that \( K_{T \leftarrow P(a \sim b)} \subseteq K_{T \leftarrow P(a \leq b)} \cup K_{T \leftarrow P(a \geq b)} \). This finishes the proof of (7).

Now that both (6) and (7) are proven, Lemma 0.15 easily follows. 

**Definition 0.16.** Let \( V \) be a \( K \)-vector space. A \( K \)-linear map \( f : H_T \rightarrow V \) is said to be \( T \)-additive if and only if every \( n \in \mathbb{N} \), every \( T \in T_n \) and every two distinct elements \( a \) and \( b \) of \( [n] \) satisfy

\[
f(T) = f(T \leftarrow a \leq b) + f(T \leftarrow a \geq b) - f(T \leftarrow a \sim b).\]

\[
(9)
\]

**Proposition 0.17.** Let \( V \) be a \( K \)-vector space. Let \( f \) and \( g \) be two \( T \)-additive \( K \)-linear maps \( H_T \rightarrow V \). Assume that \( f(T_u) = g(T_u) \) for every packed word \( u \). Then, \( f = g \).

**Proof of Proposition 0.17.** It is clearly enough to show that

\[
f(T) = g(T) \quad \text{for every } T \in T.
\]

\[
(10)
\]

For any topology \( T \) on a finite set \( X \), we let \( h(T) \) denote the nonnegative integer \( \# \{ (x, y) \in X^2 \mid \text{ neither } x \leq_T y \text{ nor } y \leq_T x \} \). We shall prove (10) by strong induction over \( h(T) \). So we fix some \( T \in T \), and we want to prove (10), assuming that every \( S \in T \) satisfying \( h(S) < h(T) \) satisfies

\[
f(S) = g(S).
\]

\[
(11)
\]

Let \( n \in \mathbb{N} \) be such that \( T \in T_n \). If there exist no two elements \( a \) and \( b \) of \( [n] \) satisfying neither \( a \leq_T b \) nor \( b \leq_T a \), then we have \( T = T_u \) for some packed word \( u \), and this \( u \) satisfies \( f(T_u) = g(T_u) \) (due to the assumption of the proposition); thus, (10) follows immediately (since \( T = T_u \)). Hence, we can WLOG assume that such two elements \( a \) and \( b \) exist. Consider these two elements. Of course, \( a \) and \( b \) are distinct.

If \( S \) is any of the three posets \( T \leftarrow a \leq b) \), \( T \leftarrow a \geq b) \) and \( T \leftarrow a \sim b) \), then \( h(S) < h(T) \). Hence, we can apply (11) to each of these three posets. We obtain

\[
f(T \leftarrow a \leq b) = g(T \leftarrow a \leq b);
\]

\[
f(T \leftarrow a \geq b) = g(T \leftarrow a \geq b);
\]

\[
f(T \leftarrow a \sim b) = g(T \leftarrow a \sim b).
\]

\[
(8)
\]

\[
(9)
\]

This is because \( \{ (x, y) \in X^2 \mid \text{ neither } x \leq_T y \text{ nor } y \leq_T x \} \) is a proper subset of \( \{ (x, y) \in X^2 \mid \text{ neither } x \leq_S y \text{ nor } y \leq_S x \} \). (Proper because \( (a, b) \) or \( (b, a) \) belongs to the latter but not to the former.)
But since $f$ is $T$-additive, we have
\[
\begin{align*}
f(T) &= g(T \leftrightarrow (a \leq b)) + g(T \leftrightarrow (a \geq b)) - g(T \leftrightarrow (a \sim b)) \\
&= g(T \leftrightarrow (a \leq b)) + g(T \leftrightarrow (a \geq b)) - g(T \leftrightarrow (a \sim b)) = g(T)
\end{align*}
\]
(since $g$ is $T$-additive). Thus, (10) is proven, and the induction step is complete. \hfill \square

Proof of Theorem 0.11 (sketched). We need to show that $\beta = \alpha \circ U$.

We notice that every topology $S$ on $[n]$ satisfies
\[
(\beta \circ Z)(S) = \beta \left( \frac{Z(S)}{=-1|[^n]/\sim S|} \right) = (-1)^{|[^n]/\sim S|} \frac{\beta(S)}{=(1-1)|[^n]/\sim S|1_{K_S}}
\]
(by the definition of $Z$)
\[
= (-1)^{|[^n]/\sim S|} \left( 1 - (-1)^{|[^n]/\sim S|} \right) 1_{K_S}
\]
\[
= 1_{K_S}
\]
\hfill (12)

and
\[
(\alpha \circ U \circ Z)(S) = \alpha \left( U \left( \frac{Z(S)}{=-1|[^n]/\sim S|} \right) \right) = (-1)^{|[^n]/\sim S|} \alpha \left( \sum_{f \in U(S)} f \right).
\]
\hfill (13)

We shall now show that both maps $\beta \circ Z : H_T \rightarrow WQSym$ and $\alpha \circ U \circ Z : H_T \rightarrow WQSym$ are $T$-additive.

Proof that the map $\beta \circ Z$ is $T$-additive: Let $n \in \mathbb{N}$. Let $T \in T_n$. Let $a$ and $b$ be two distinct elements of $[n]$. In order to show that $\beta \circ Z$ is $T$-additive, we must prove that
\[
(\beta \circ Z)(T) = (\beta \circ Z)(T \leftrightarrow (a \leq b)) + (\beta \circ Z)(T \leftrightarrow (a \geq b)) - (\beta \circ Z)(T \leftrightarrow (a \sim b)).
\]
\hfill (14)

This rewrites as follows:
\[
1_{K_T} = 1_{K_{T \leftrightarrow (a \leq b)}} + 1_{K_{T \leftrightarrow (a \geq b)}} - 1_{K_{T \leftrightarrow (a \sim b)}}
\]
(because of (12)). But this is precisely the claim of Lemma 0.15. Hence, (14) is proven. We thus have shown that the map $\beta \circ Z$ is $T$-additive.

Proof that the map $\alpha \circ U \circ Z$ is $T$-additive: Let $n \in \mathbb{N}$. Let $T \in T_n$. Let $a$ and $b$ be two distinct elements of $[n]$. In order to show that $\alpha \circ U \circ Z$ is $T$-additive, we must prove that

$$(a \circ U \circ Z)(T) = (a \circ U \circ Z)(T \curvearrowright \{a \leq b\}) + (a \circ U \circ Z)(T \curvearrowright \{a > b\}) - (a \circ U \circ Z)(T \curvearrowright \{a \sim b\}).$$

This is rather obvious if $a \leq_T b$. Hence, for the rest of this proof, we WLOG assume that we don’t have $a \leq_T b$. Similarly, we WLOG assume that we don’t have $b \leq_T a$. Now, using (13), we can rewrite the equality (15) as follows:

$$(-1)^{|[n]/\sim_T|} \sum_{f \in U(T)} \alpha(f)$$

$$= (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a \leq b))} \alpha(f) + (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a > b))} \alpha(f)$$

$$- (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a \sim b))} \alpha(f).$$

This can be rewritten further as

$$(-1)^{|[n]/\sim_T|} \sum_{f \in U(T)} \alpha(f)$$

$$= (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a \leq b))} \alpha(f) + (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a > b))} \alpha(f)$$

$$- (-1)^{|[n]/\sim_T|} \sum_{f \in U(T \rightharpoonup (a \sim b))} \alpha(f)$$

(because Lemma 0.14 (n) (applied to $X = [n]$) yields $|[n]/\sim_T| = |[n]/\sim_T \triangleright (a \leq b)| = |[n]/\sim_T \triangleright (a > b)| = |[n]/\sim_T| - 1$). Upon cancelling $(-1)^{|[n]/\sim_T|}$, this simplifies to

$$\sum_{f \in U(T)} \alpha(f) = \sum_{f \in U(T \rightharpoonup (a \leq b))} \alpha(f) + \sum_{f \in U(T \rightharpoonup (a > b))} \alpha(f) + \sum_{f \in U(T \rightharpoonup (a \sim b))} \alpha(f).$$

Proof. Assume that $a \leq_T b$. Then, Lemma 0.14 (c) yields $T \rightharpoonup \{a \leq b\} = T$ and $T \rightharpoonup \{a \sim b\} = T \rightharpoonup \{a > b\}$. Hence, (15) rewrites as

$$(a \circ U \circ Z)(T) = (a \circ U \circ Z)(T) + (a \circ U \circ Z)(T \rightharpoonup \{a > b\}) - (a \circ U \circ Z)(T \rightharpoonup \{a > b\}).$$

But this is obvious.
But this follows immediately from Lemma 0.14 (applied to $X = [n]$). Thus, (15) is proven. We have thus shown that $\alpha \circ U \circ Z$ is $T$-additive.

Now, it is easy to see that $(\beta \circ Z) (T_u) = (\alpha \circ U \circ Z) (T_u)$ for every packed word $u$. Hence, Proposition 0.17 (applied to $V = \mathfrak{M}$, $f = \beta \circ Z$ and $g = \alpha \circ U \circ Z$) yields $\beta \circ Z = \alpha \circ U \circ Z$. Since $Z$ is an isomorphism, we can cancel $Z$ from this equality, and obtain $\beta = \alpha \circ U$. This proves Theorem 0.11. \qed

**Proof of Theorem 0.4** Theorem 0.11 yields $\beta = \alpha \circ U$. Since both $\beta$ and $U$ are $\mathbb{K}$-algebra homomorphisms, and since $U$ is surjective, this easily yields that $\alpha$ is a $\mathbb{K}$-algebra homomorphism. (Indeed, let $p \in \text{WQSym}$ and $q \in \text{WQSym}$. Then, thanks to the surjectivity of $U$, there exist $P \in H_T$ and $Q \in H_T$ satisfying $p = U(P)$ and $q = U(Q)$. Consider these $P$ and $Q$. Since $U$ is a $\mathbb{K}$-algebra homomorphism, we have $U(P, Q) = U(P) U(Q) = pq$. Now,

$$\alpha \left( \begin{array}{c} p \\ U(P) \end{array} \right) \cdot \alpha \left( \begin{array}{c} q \\ U(Q) \end{array} \right) = \alpha \left( \begin{array}{c} \alpha \circ U(P) \\ = (\alpha \circ U)(P) \end{array} \right) \cdot \alpha \left( \begin{array}{c} \alpha \circ U(Q) \\ = (\alpha \circ U)(Q) \end{array} \right) = \beta \cdot \beta = (P, Q)$$

(since $\beta$ is a $\mathbb{K}$-algebra homomorphism)

$$= (\alpha \circ U)(P, Q) = \alpha \left( \begin{array}{c} U(P, Q) \\ = pq \end{array} \right) = \alpha(pq),$$

and this shows that $\alpha$ is a $\mathbb{K}$-algebra homomorphism.) Theorem 0.4 is proven. \qed

---

**Proof.** Let $u$ be a packed word. Applying (12) to $S = T_u$, we obtain $(\beta \circ Z) (T_u) = 1_{K_T} = 1_{K_u}$ (since Remark 0.6 yields $K_T = K_u$). But applying (13) to $S = T_u$ leads to

$$(\alpha \circ U \circ Z)(T_u) = \frac{(-1)^{|n|/\sim T_u}}{(-1)^{\max_u}} \sum_{f \in U(T_u)} \alpha(f)$$

(since $|n|/\sim T_u = \max_u$)

$$= (-1)^{\max_u} \alpha(u) = (-1)^{\max_u} (-1)^{\max_u} 1_{K_u} = 1_{K_u}$$

(by the definition of $\alpha$)

$$= (\beta \circ Z)(T_u),$$

qed.
References

http://arxiv.org/abs/1407.0476v2

http://arxiv.org/abs/1109.1634v2