

## DIAGONAL VECTORS OF SHIFTED YOUNG TABLEAUX

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ABSTRACT. We study vectors formed by entries on the diagonal of standard Young tableaux of shifted shapes. Such vectors are in bijection with integer lattice points of certain integral polytopes, which are Minkowski sums of simplices. We also describe vertices of these polytopes, and construct corresponding shifted Young tableaux.

## 1. SHIFTED YOUNG DIAGRAMS AND TABLEAUX

**Definition 1.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition with at most  $n$  parts. The *shifted Young diagram of shape  $\lambda$*  (or just  $\lambda$ -*shifted diagram*) is the set

$$D_\lambda = \{(i, j) \in \mathbb{R}^2 \mid 1 \leq j \leq n, j \leq i \leq n + \lambda_j\}.$$

We think of  $D_\lambda$  as a collection of boxes with  $n+1-i+\lambda_i$  boxes in row  $i$ , and such that the leftmost box of the  $i^{\text{th}}$  row is also in the  $i^{\text{th}}$  column. A *shifted standard Young tableau shape  $\lambda$*  (or just  $\lambda$ -*shifted tableau*) is a bijective map  $T : D_\lambda \rightarrow \{1, \dots, |D_\lambda|\}$  which is increasing along rows and down columns, i.e.  $T(i, j) < T(i, j+1)$  and  $T(i, j) < T(i+1, j)$  ( $|D_\lambda| = \binom{n+1}{2} + \lambda_1 + \dots + \lambda_n$  is the number of boxes in  $D_\lambda$ ). The *diagonal vector* of such a tableau  $T$  is  $\text{diag}(T) = (T(1, 1), T(2, 2), \dots, T(n, n))$ .

**Example 1.** The following is a shifted standard Young tableau for  $n = 4$ ,  $\lambda = (4, 2, 1, 0)$ . Its diagonal vector is  $(1, 4, 7, 17)$ .

1	2	3	5	8	9	12	13
	4	6	10	11	16		
		7	14	15			
			17				

We are interested in describing the possible diagonal vectors appearing in  $\lambda$ -shifted Young tableaux. The problem was solved in the case  $\lambda = (0, 0, \dots, 0)$  (the empty partition) by A. Postnikov, in [Pos, Section 15]. Specifically, it was shown that diagonal vectors of the shifted triangular shape  $D_\emptyset$  are in bijection with lattice points of the  $(n-1)$ -dimensional *associahedron*  $\text{Ass}_{n-1}$  (to be defined in section 2). Moreover, a simple explicit construction was given for the "extreme" diagonal vectors, i.e. the ones corresponding to the vertices of  $\text{Ass}_{n-1}$ .

In this article, we aim to generalize Postnikov's results to arbitrary shifted shapes. Specifically, in section 2 we will prove that diagonal vectors of shifted

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$\lambda$ -tableaux are in bijection with lattice points of a certain polytope  $\mathbf{P}_\lambda$ . This polytope is a Minkowski sum of simplices in  $\mathbb{R}^n$  and its combinatorial structure only depends on the length of the partition  $\lambda$ . In particular, if the length is  $n$ ,  $\mathbf{P}_\lambda$  turns out to be combinatorially equivalent to  $\text{Ass}_n$ . In section 3 we shall give an explicit construction of  $\lambda$ -shifted tableaux whose diagonal vectors corespond to the vertices of  $\mathbf{P}_\lambda$ .

For a non-negative integer vector  $(a_1, \dots, a_n)$ , let  $N_\lambda(a_1, \dots, a_n)$  be the number of standard  $\lambda$ -shifted tableaux  $T$  such that  $T(i+1, i+1) - T(i, i) - 1 = a_i$  for  $i = 1, \dots, n$ , where we set  $T(n+1, n+1) = \binom{n+1}{2} + \lambda_1 + \dots + \lambda_n + 1$ .

**Theorem 1.** *We have the following identity:*

$$\sum_{a_1, \dots, a_n \geq 0} N_\lambda(a_1, \dots, a_n) \frac{t_1^{a_1}}{a_1!} \cdots \frac{t_n^{a_n}}{a_n!} = \frac{1}{\prod_{i=1}^n (\lambda_i + n - i)!} \cdot \prod_{1 \leq i < j \leq n} (t_i + \dots + t_{j-1}) \cdot s_\lambda(t_1 + \dots + t_n, t_2 + \dots + t_n, \dots, t_n)$$

where  $s_\lambda$  denotes the Schur symmetric polynomial associated to  $\lambda$ .

*Proof.* Consider a vector  $\mathbf{x} = (x_1 > x_2 > \dots > x_n)$ . Define the polytope

$$P_\lambda(\mathbf{x}) = \{(p_{ij})_{(i,j) \in D_\lambda} \mid 0 \leq p_{ij} \leq p_{i(j+1)}, p_{ij} \geq p_{(i+1)j}, p_{ii} = x_i\}.$$

Thus  $P_\lambda(\mathbf{x})$  is the section of the order polytope of shape  $D_\lambda$  where the values along the main diagonal are  $x_1, \dots, x_n$ . If  $\lambda = \emptyset$ , this polytope is known as the *Gelfand-Tsetlin polytope*, which has important connections to finite-dimensional representations of  $\mathfrak{gl}_n \mathbb{C}$  (see [GT]). Our proof strategy is to compare two different formulas for the volume of  $P_\lambda(\mathbf{x})$ , one of which is more direct and the other is a summation over standard  $\lambda$ -shifted Young tableaux. By [BR, Proposition 12],

$$(1) \quad \text{vol}(P_\lambda(\mathbf{x})) = \frac{1}{\prod_{i=1}^n (\lambda_i + n - i)!} \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot s_\lambda(\mathbf{x}).$$

On the other hand, there is a natural map  $\phi$  from  $P_\lambda(\mathbf{x})$  (defined except on a set of measure 0), to the set of standard  $\lambda$ -shifted Young tableaux, given as follows: Let  $\mathbf{p} = (p_{ij})_{(i,j) \in D_\lambda} \in P_\lambda(\mathbf{x})$  be a point with distinct coordinates. Arrange the  $p_{ij}$ 's in decreasing order and define the tableau  $T = \phi(\mathbf{p})$  by writing  $k$  in box  $(i, j)$  if  $p_{ij}$  is the  $k^{\text{th}}$  element in the above list. By the definition of  $P_\lambda(\mathbf{x})$ , it is clear that  $T$  is a standard  $\lambda$ -shifted Young tableau. Given a standard  $\lambda$ -shifted tableau  $T$  with diagonal vector  $\text{diag}(T) = \{d_1, \dots, d_n\}$ , it is easy to see that  $\phi^{-1}(T)$  is isomorphic to the set

$$\{(y_i) \in \mathbb{R}^{|T|} \mid y_1 > y_2 > \dots > y_{|T|} > 0, y_{d_i} = x_i\}$$

which is a direct product of (inflated) simplices

$$\{x_1 = y_1 > y_2 > \dots > y_{d_2-1} > x_2\} \times \dots \times \{x_n = y_{d_n} > y_{d_n+1} > \dots > y_{|T|} > 0\}$$

Therefore,

$$\text{vol}(\phi^{-1}(T)) = \frac{(x_1 - x_2)^{a_1}}{a_1!} \cdots \frac{(x_{n-1} - x_n)^{a_{n-1}}}{a_{n-1}!} \cdot \frac{x_n^{a_n}}{a_n!}.$$

Summing over all  $T$ , we obtain

$$\begin{aligned} \text{vol}(P_\lambda(\mathbf{x})) &= \sum_T \text{vol}(\phi^{-1}(T)) \\ &= \sum_{a_1, \dots, a_n \geq 0} N_\lambda(a_1, \dots, a_n) \frac{(x_1 - x_2)^{a_1}}{a_1!} \cdots \frac{(x_{n-1} - x_n)^{a_{n-1}}}{a_{n-1}!} \cdot \frac{x_n^{a_n}}{a_n!}. \end{aligned}$$

Comparing the last formula to (1), and making the substitutions

$t_1 = x_1 - x_2, \dots, t_{n-1} = x_{n-1} - x_n, t_n = x_n$ , we obtain the identity in the theorem.  $\square$

## 2. GENERALIZED PERMUTOHEDRA

In this section we recall the setup from [Pos, Section 6]. Let  $n \in \mathbb{N}$  and let  $e_1, \dots, e_n$  denote the standard basis of  $\mathbb{R}^n$ . For a subset  $I \in \{1, 2, \dots, n\}$ , let  $\Delta_I = \text{Conv}\{e_i \mid i \in I\}$ , which is an  $|I|$ -dimensional simplex. A large class of *generalized permutohedra* (cf. [Pos, Section 6]) is given by subsets of  $\mathbb{R}^n$  of the form

$$P_n^y(\{y_I\}) = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} y_I \Delta_I$$

i.e.  $P_n^y(\{y_I\})$  is the Minkowski sum of the simplices  $\Delta_I$  scaled by  $y_I \geq 0$ . It's not hard to see that if  $y_I = y_J$ , whenever  $|I| = |J|$ , then  $P_n^y(\{y_I\})$  is the usual permutohedron obtained by taking the convex hull of points  $(x_1, \dots, x_n)$  such that  $x_1, \dots, x_n$  is a permutation of the numbers

$$z_{[n]} = \sum_{I \subseteq [n]} y_I, \quad z_{[n-1]} = \sum_{I \subseteq [n-1]} y_I, \quad \dots, \quad z_{\{1\}} = y_{\{1\}}.$$

Generalized permutohedra have been studied extensively in [Pos]. One particular example of a generalized permutohedron, the *associahedron*, is defined as  $\text{Ass}_n = \sum_{1 \leq i \leq j \leq n} \Delta_{[i, j]}$ . It is also known as the *Stasheff polytope* and it first appeared in the work of Stasheff (cf. [Sta].)

**Proposition 1.** *For any subsets  $I_1, \dots, I_k \subseteq [n]$ , and any non-negative integers  $a_1, \dots, a_n$ , the coefficient of  $t_1^{a_1} \cdots t_n^{a_n}$  in*

$$(2) \quad \prod_{j=1}^k \left( \sum_{i \in I_j} t_i \right)$$

*is non-zero if and only if  $(a_1, \dots, a_n)$  is an integer lattice point of the polytope  $\sum_{j=1}^k \Delta_{I_j}$ .*

*Proof.* It's easy to see that the coefficient of  $t_1^{a_1} \cdots t_n^{a_n}$  in (2) is non-zero if and only if  $(a_1, \dots, a_n)$  can be written as a sum of vertices of the simplices  $\Delta_{I_1}, \dots, \Delta_{I_k}$ . By [Pos, Proposition 14.12], this happens if and only if  $(a_1, \dots, a_n)$  is a lattice point of  $\sum_{j=1}^k \Delta_{I_j}$ .  $\square$

**Proposition 2.** *The coefficient of  $t_1^{a_1} \cdots t_n^{a_n}$  in  $s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \dots, t_n)$  is non-zero if and only if  $(a_1, \dots, a_n)$  is a lattice point of the polytope  $\lambda_1 \Delta_{[1,n]} + \lambda_2 \Delta_{[2,n]} + \cdots + \lambda_n \Delta_{\{n\}}$ .*

*Proof.* Recall that

$$(3) \quad s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \dots, t_n) = \sum_T (t_1 + \cdots + t_n)^{w_1} \cdots t_n^{w_n},$$

where the sum ranges over all *semi-standard* Young tableaux  $T$  of shape  $\lambda$  and weight  $\mathbf{w} = (w_1, \dots, w_n)$ , i.e.  $w_i$  is the number of  $i$ 's appearing in  $T$  (see [St]). Let  $T$  be a SSYT of shape  $\lambda$  and weight  $\mathbf{w}$ . Then  $w_1 + \cdots + w_i \leq \lambda_1 + \cdots + \lambda_i$ ,  $\forall i = 1 \dots n$ . Indeed, if we consider the boxes containing the numbers  $1, 2, \dots, i$  in  $T$ , there can be no more than  $i$  of them in the same column. Hence the number of such boxes is at most the size of the first  $i$  rows of  $\lambda$ , which is  $\lambda_1 + \cdots + \lambda_i$ .

It follows that any monomial  $t_1^{a_1} \cdots t_n^{a_n}$  appearing in  $(t_1 + \cdots + t_n)^{w_1} \cdots t_n^{w_n}$  also appears in  $(t_1 + \cdots + t_n)^{\lambda_1} \cdots t_n^{\lambda_n}$ . On the other hand,  $(t_1 + \cdots + t_n)^{\lambda_1} \cdots t_n^{\lambda_n}$  does appear in the right side of (3) as the term corresponding to the tableau  $T$  with 1's in the first row, 2's in the second row, etc. Therefore, the coefficient of  $t_1^{a_1} \cdots t_n^{a_n}$  in  $s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \dots, t_n)$  is non-zero if and only if it is non-zero in  $(t_1 + \cdots + t_n)^{\lambda_1} \cdots t_n^{\lambda_n}$ , which by Proposition 1, is non-zero if and only if  $(a_1, \dots, a_n)$  is a lattice point of  $\lambda_1 \Delta_{[1,n]} + \lambda_2 \Delta_{[2,n]} + \cdots + \lambda_n \Delta_{\{n\}}$ .  $\square$

**Theorem 2.** *The number of (distinct) diagonal vectors of  $\lambda$ -shifted Young tableaux is equal to the number of lattice points of the polytope*

$$\mathbf{P}_\lambda := \sum_{1 \leq i \leq j \leq n-1} \Delta_{[i,j]} + \lambda_1 \Delta_{[1,n]} + \lambda_2 \Delta_{[2,n]} + \cdots + \lambda_n \Delta_{\{n\}}.$$

*Proof.* By Theorem 1, and Propositions 1, 2 it follows that  $N_\lambda(a_1, \dots, a_n) \neq 0$  if and only if  $(a_1, \dots, a_n)$  is an integer lattice point of the polytope

$$\sum_{1 \leq i \leq j \leq n-1} \Delta_{[i,j]} + \lambda_1 \Delta_{[1,n]} + \lambda_2 \Delta_{[2,n]} + \cdots + \lambda_n \Delta_{\{n\}}.$$

$\square$

In particular, if  $\lambda$  has  $n$  parts (i.e.  $\lambda_n > 0$ ), we see that  $\mathbf{P}_\lambda$  is combinatorially equivalent to  $\text{Ass}_n$ .

### 3. VERTICES OF $\mathbf{P}_\lambda$

In what follows we describe the vertices  $\mathbf{P}_\lambda$  by using techniques developed in [Pos]. Given a generalized permutohedron  $P_n^y(\{y_I\}) = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} y_I \Delta_I$ , assume that its *building set*  $B = \{I \subseteq [n] \mid y_I > 0\}$  satisfies the following conditions:

- (1) If  $I, J \in B$  and  $I \cap J \neq \emptyset$ , then  $I \cup J \in B$ .
- (2)  $B$  contains all singletons  $\{i\}$ , for  $i \in [n]$ .

A *B-forest* is a rooted forest  $F$  on the vertex set  $[n]$  such that

- (1) For any  $i$ ,  $\text{desc}(i, F) \in B$  ( $\text{desc}(i, F)$  is the set of descendants of  $i$  in  $F$ ).
- (2) There are no  $k \geq 2$  distinct incomparable nodes  $i_1, \dots, i_k$  in  $F$  such that  $\bigcup_{j=1}^k \text{desc}(i_j, F) \in B$ .
- (3)  $\{\text{desc}(i, F) \mid i \text{ - root of } F\} = \{I \in B \mid I \text{ - maximal}\}$ .

We will need the following result of Postnikov:

**Proposition 3.** [Pos, Proposition 7.9] *Vertices of  $P_n^y(\{y_I\})$  are in bijection with  $B$ -forests. More precisely, the vertex  $v_F = (t_1, \dots, t_n)$  of  $P_n^y(\{y_I\})$  associated with a  $B$ -forest  $F$  is given by  $t_i = \sum_{J \in B: i \in J \subseteq \text{desc}(i, F)} y_J$ , for  $i \in [n]$ .*

*Remark.* It's not hard to see that Proposition 3 remains true even if we allow the building set  $B$  not to contain the singletons  $\{i\}$ . We will make use of this later on.

The combinatorial structure of  $\mathbf{P}_\lambda$  clearly depends only on its building set, i.e. the number of non-zero parts of the partition  $\lambda$ . Assume  $\lambda_1, \dots, \lambda_k > 0$ ,  $\lambda_{k+1} = \dots = \lambda_n = 0$ , so that the building set of  $\mathbf{P}_\lambda$  is

$$B_k = \{[i, j] \mid 1 \leq i \leq j \leq n-1\} \cup \{[i, n] \mid 1 \leq i \leq k\}.$$

We first deal with the case  $k = n$ . Let  $T$  be a plane binary tree on  $n$  nodes. For a node  $v$  of  $T$ , denote by  $L_v, R_v$  the left and right branches at  $v$ . There is a unique way to label the nodes of  $T$  such that for any node  $v$ , its label is greater than all labels in  $L_v$  and smaller than all labels in  $R_v$ . This labelling is called the *binary search labelling* of  $T$ .

**Proposition 4.** [Pos, Proposition 8.1] *The  $B_n$ -forests are exactly plane binary trees on  $n$  nodes with the binary search labeling.*

Let  $T$  be a  $B_n$ -forest. It's easy to see that  $\text{desc}(x, T)$  has form  $[a, n]$  if and only if the path from the root to  $x$  always goes to the right. In this case,  $\text{desc}(x, T) = [n - |L_x|, n]$  and  $n - |L_x|$  is maximal when  $x$  is the right-most node in  $T$ , i.e.  $x = n$ . It follows that  $\{\text{desc}(x, T) \mid x \in [n]\} \subseteq B_k \subseteq B_n \Leftrightarrow |L_n| \geq n - k$ . This argument together with Proposition 4 implies

**Proposition 5.** *The  $B_k$ -forests are exactly plane binary trees on  $n$  nodes with the binary search labeling and such that  $|L_n| \geq n - k$ , i.e. such that the (left) subtree of the right-most node in  $T$  has size at least  $n - k$ .*

**Corollary 1.** *The number of vertices of  $\mathbf{P}_\lambda$  is*

$$C_1 C_{n-1} + C_2 C_{n-2} + \dots + C_k C_{n-k}$$

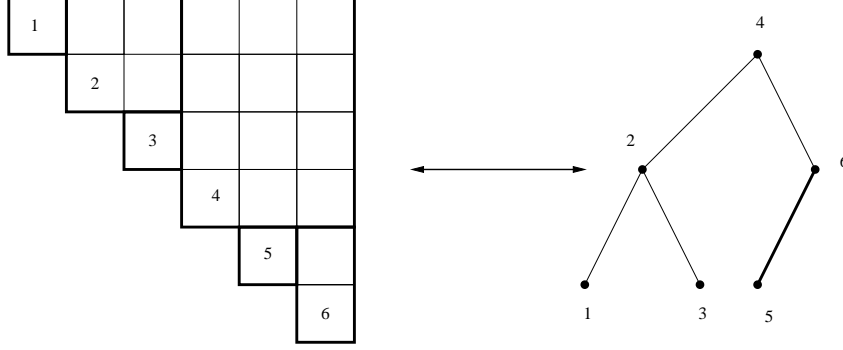
where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  denotes the  $n^{\text{th}}$  Catalan number.

*Proof.* By Propositions 3 and 5, the number of vertices of  $\mathbf{P}_\lambda$  is equal to the number of plane binary trees  $T$  on  $n$  nodes such that left subtree  $L$  of the right-most node in  $T$  has size at least  $n - k$ . If  $|L| = n - i$ , then there are  $C_{n-i}$  ways to choose  $L$  and  $C_i$  ways to choose the tree  $T \setminus L$ . Summing over  $i = 1, \dots, k$  yields the desired formula.  $\square$

To describe the vertices of  $\mathbf{P}_\lambda$ , recall that plane binary trees  $T$  on  $n$  nodes are in bijective correspondence with the  $C_n$  subdivisions of the shifted Young diagram  $D_\emptyset$  into  $n$  rectangles. This can be defined inductively as follows: Let  $i$  be the root of  $T$  (in the binary search labeling). Then draw an  $(|L_i| + 1) \times (|R_i| + 1)$  rectangle. Then attach the subdivisions corresponding to the binary trees  $L_i, R_i$  to the left and, respectively, bottom of the rectangle.

For a subdivision  $\Xi$  of  $D_\emptyset$  into  $n$  rectangles, the  $i^{\text{th}}$  rectangle is the rectangle containing the  $i^{\text{th}}$  diagonal box of  $D_\emptyset$ . If  $T$  is the binary tree corresponding to  $\Xi$ , then the  $i^{\text{th}}$  rectangle of  $\Xi$  has size  $(|L_i| + 1) \times (|R_i| + 1)$ . In particular,  $|L_n| + 1$  is the length of the (bottom-right) vertical strip of the subdivision  $\Xi$ .

**Example 2.** Here is a subdivision of  $D_\emptyset$  and the corresponding binary tree with the binary search labeling when  $n = 4$ .



We are finally in a position to prove the main result of this paper.

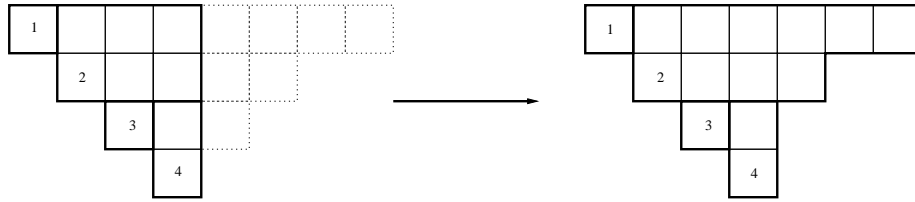
**Theorem 3.** *Vertices of  $\mathbf{P}_\lambda$  are in bijection with subdivisions of the shifted diagram  $D_\emptyset$  into  $n$  rectangles such that the bottom-right vertical strip of the subdivision has at least  $n - k + 1$  boxes. Specifically, let  $\Xi$  be such a subdivision. Then we can get a subdivision  $\Xi^*$  of  $D_{\lambda - \langle 1^k \rangle}$  by merging the rectangles in  $\Xi$  with the rows of the Young diagram of  $\lambda - \langle 1^k \rangle$  that they border. Then the corresponding vertex of  $\mathbf{P}_\lambda$  is  $v_\Xi = (t_1, \dots, t_n)$ , where  $t_i$  is the number of boxes in the  $i^{\text{th}}$  region of  $\Xi^*$ .*

*Proof.* The first part of the theorem follows from Proposition 5 and the discussion preceding the theorem. To prove the second part, we use Proposition 3. Recall that the building set of  $\mathbf{P}_\lambda$  is  $B_k = \{[i, j] \mid 1 \leq i \leq j \leq n\} \cup \{[i, n] \mid 1 \leq i \leq k\}$ , and  $\mathbf{P}_\lambda = \sum_{[i, j] \in B_k} y_{ij} \Delta_{[i, j]}$  where  $y_{ij} = 1$  if  $j \neq 1$  and  $y_{in} = \lambda_i$ . Let  $T$  be a  $B_k$ -forest, i.e. a binary tree on  $n$  nodes with the binary search labeling such that  $|L_n| \geq n - k$  (cf. Proposition 5.) Note that  $\text{desc}(i, T) = [i - |L_i|, i + |R_i|]$ . Now Proposition 3 implies that the corresponding vertex  $v_T = (t_1, \dots, t_n)$  of  $\mathbf{P}_\lambda$  is given by

$$\begin{aligned} t_i &= \sum_{J \in B_k, i \in J \subseteq \text{desc}(i, T)} y_J = \sum_{[k, l] \in B_k, i - |L_i| \leq k \leq i \leq i + |R_i|} y_{kl} \\ &= (|L_i| + 1) \cdot |R_i| + \sum_{k=i-|L_i|}^i y_{k(i+|R_i|)}. \end{aligned}$$

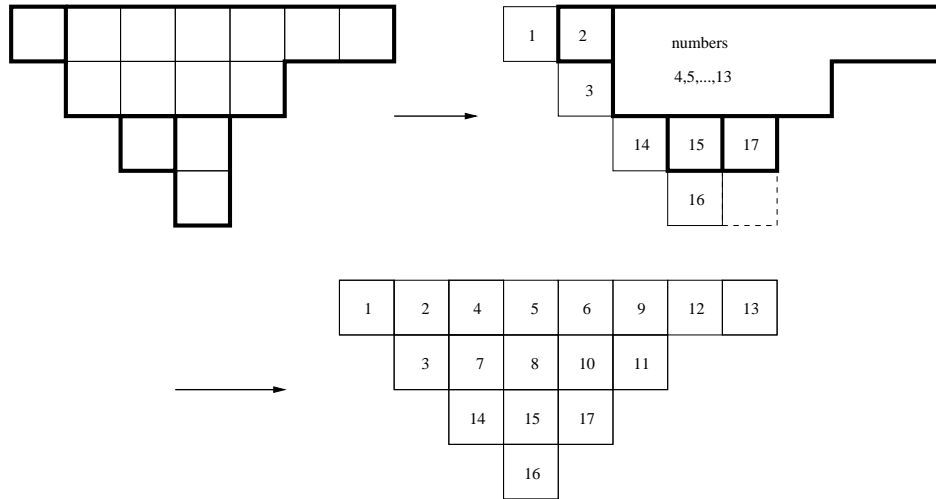
If the  $i^{\text{th}}$  rectangle of  $\Xi$  borders the right edge of  $D_\emptyset$  (i.e.  $n \in \text{desc}(i, T)$ ), then  $t_i = (|L_i| + 1) \cdot |R_i| + \sum_{k=i-|L_i|}^i \lambda_k$ . Otherwise,  $t_i = (|L_i| + 1) \cdot (|R_i| + 1)$ . In each case,  $t_i$  is the number boxes in the  $i^{\text{th}}$  region of  $\Xi^*$ .  $\square$

**Example 3.** Let  $n = 4$ ,  $\lambda = (4, 2, 1, 0)$ ,  $k = 3$ . The figure shows how a subdivision  $\Xi$  of  $D_\emptyset$  yields the subdivision  $\Xi^*$  of  $D_{\lambda - \langle 1^k \rangle} = D_{(3, 1, 0)}$ . The corresponding vertex of  $\mathbf{P}_\lambda$  is given by counting boxes in the regions of  $\Xi^*$ :  $v_{\Xi^*} = (1, 10, 1, 2)$ . It follows that there is a  $(4, 2, 1, 0)$ -shifted Young tableau  $T$  whose diagonal vector is  $\text{diag}(T) = (1, 1 + 1 + 1, 1 + 1 + 1 + 10 + 1, 1 + 1 + 1 + 10 + 1 + 2) = (1, 3, 14, 16)$ .



On the other hand, one can directly construct  $\lambda$ -shifted Young tableaux with diagonal vector  $v_{\Xi^*} = (c_1, c_2, \dots, c_n)$  by using the subdivision  $\Xi^*$ . Indeed, we know what the diagonal vector of the tableau  $(a_1, \dots, a_n)$  should be. Consider again the subdivision  $\Xi^*$  of  $D_{\lambda-(1^k)}$ . We can extend the diagram  $D_{\lambda-(1^k)}$  to  $D_\lambda$  by first adding a box to the left of each row of  $D_{\lambda-(1^k)}$ , and then, by deleting the last  $n - k$  boxes in the  $n^{\text{th}}$  column of  $D_{\lambda-(1^k)}$ . Now, we start by putting  $a_1, \dots, a_n$  in the diagonal boxes of  $D_\lambda$ . The remaining part of  $D_\lambda$  is divided into  $n$  regions by  $\Xi^*$ . Finally, for each  $i = 1, \dots, n$ , put the  $c_i$  numbers  $a_i + 1, \dots, a_{i+1} - 1$  in the  $i^{\text{th}}$  region of  $\Xi^*$  in a standard way, i.e. such that entries increase along rows and down columns (as before, we set  $a_{n+1} = |D_\lambda| + 1$ .) In this way we obtain a  $\lambda$ -shifted tableau  $T$  such that  $\text{diag}(T) = (a_1, \dots, a_n)$ .

We illustrate the above procedure for the subdivision in Example 3.



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