RESEARCH STATEMENT

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1. Introduction

My main research interests lie in geometric and algebraic combinatorics and in particular in the geometry of convex polytopes. Studying classical invariants of convex polytopes, such as volumes, numbers of lattice points, f-vectors or the Ehrhart polynomials has been a central problem for much of the 20th century, not only in its own interest, but also because of connections of these invariants to algebraic geometry and representation theory. The classical *permutohedron* $P(x_1, \ldots, x_n)$ is the convex hull of the $n!$ points obtained by permuting $x_1, ..., x_n$. Permutohedra and their generalizations are appearing throughout the literature increasingly, and have been studied extensively, for example in [Pos, PRW]. Special cases of permutohedra appear as *graphical zonotopes* and *graph associahedra* in graph theory, as *moment polytopes* (and in particular *matroid polytopes*) in algebraic geometry, and as *weight polytopes* in representation theory. It is a general principle that the volumes, numbers of lattice points and other invariants of these polytopes should have alternative descriptions in terms of other objects such as trees, Young tableaux, degrees of toric varieties in projective spaces, elements of Coxeter groups with special properties and so on. For example, the *hypersimplex* $\Delta_{n+1,k} = P(1^k, 0^{n+1-k})$ (1^k means k ones), is the *matroid polytope* corresponding to the uniform matroid of rank k on $[n]$. It appears in algebraic geometry as the moment polytope of the toric variety $X_p = \overline{\mathbb{T}p}$ of a generic point $p \in Gr_{k,n}$ in the Grassmanian ([Ful]). As such, it is known that the normalized volume of $\Delta_{n+1,k}$ is the degree of X_p as a subvariety of $\mathbb{CP}^{\binom{n}{k}-1}$([GGMS]).

In my first research project at MIT, I have investigated volumes of permutohedra by computing the mixed volumes of hypersimplices, called the *mixed Eulerian numbers.* These numbers include many classical combinatorial numbers such as Catalan numbers, binomial coefficients and Eulerian numbers. I found various formulas that enable one to compute the mixed Eulerian numbers recursively, though a simple combinatorial explanation for these numbers is still to be found. Using geometric techniques, I generalized these results to general root systems. The mixed Eulerian numbers can be defined more generally, for arbitrary positive definite matrices; while computing them seems difficult, one could for instance, try to investigate when they are nonnegative. It would also be interesting to develop a similar theory for the numbers of lattice points of permutohedra (instead of their volumes). Another project that I worked on was studying patterns in Young tableaux of *shifted* shapes. I have shown that diagonal vectors of these tableaux are in bijection with lattice points of a certain polytope $\mathbf{P}_\lambda$, which is closely related to the *associahedron.* I also constructed *extremal* tableaux corresponding to the vertices of $\mathbf{P}_\lambda$, and plan to investigate similar questions for other classes of Young diagrams.

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2. Volumes and Mixed Eulerian numbers

It is a classical result that the normalized volume of the hypersimplex $\Delta_{n+1,k}$ is the *Eulerian number* $A_{n,k}$ - the number of permutations in S_n with $k-1$ descents. More recently, A. Postnikov has computed the volume of $P(x_1,\dots,x_{n+1})$ explicitly as a homogeneous polynomial of degree n in $x_1,\dots,x_{n+1}$ [Pos, Theorem 3.2]. Interestingly, the coefficients of this polynomial are signed numbers of permutations in S_n with given descent sets. However, this result does not seem to have a natural generalization to other root systems. Letting instead $u_1 = x_1 - x_2, \dots, u_n = x_n - x_{n+1}$, allows one to express the permutohedron $P(x_1,\dots,x_{n+1})$ as the Minkowski sum of the rescaled hypersimplices $u_i\Delta_{n+1,i}$. The classical Brunn-Minkowski theory then implies that its volume is a homogeneous polynomial in $u_1,...,u_n$ with nonnegative coefficients:

$$Vol\left(P(x_1,\dots,x_{n+1})\right) = \sum_{c_1+\ldots+c_n=n, c_i\geq 0} A_{c_1\ldots c_n}\frac{u_1^{c_1}}{c_1!}\cdots\frac{u_n^{c_n}}{c_n!},$$

where $A_{c_1\ldots c_n} = n!Vol(\Delta_{n+1,1}^{c_1},\dots,\Delta_{n+1,n}^{c_n})$ is the normalized *mixed volume* of c_1 copies of Δ_{n+1,i_1}, c_2 copies of Δ_{n+1,i_2} and so on. Mixed volumes of polytopes have important connections to algebraic geometry. For example, by a famous theorem of Bernstein, they count common zeroes of generic polynomials whose Newton polytopes are the given polytopes ([Ber]). The coefficients $A_{c_1\ldots c_n}$ are called the *mixed Eulerian numbers*. They are positive integers because hypersimplices are integer polytopes of full dimension ([Ful]). It is natural to ask whether these coefficients have a nice combinatorial interpretation. This is one of the main questions that I have explored in my research. The question becomes more relevant given the following list of (known) results.

Theorem 1. [Pos, ERS] *The mixed Eulerian numbers have the folowing properties:*
(1) $A_{c_1\ldots c_n} = A_{c_n\ldots c_1}$, $A_{k,0\ldots 0,n-k} = \binom{n}{k}$, $A_{0^{k-1},n,0^{n-k}} = A_{n,k}$.
(2) $A_{0^{k-1},n-i,i,0^{n-k-1}}$ *is the number of* $w\in S_{n+1}$ *with* k *descents and* $w_{n+1} = i+1$.
(3) If $c_1+\cdots+c_i \geq i$ *for* $i=1,\dots,n$ *then* $A_{c_1\ldots c_n} = 1^{c_1}2^{c_2}\ldots n^{c_n}$.
(4) Let $\sim$ *be the equivalence relation on sequences* $(c_1,\dots,c_n)$ *given by* $(c_1,\dots,c_n)\sim(d_1,\dots,d_n)$ *if and only if* $(c_1,\dots,c_n,0)$ *is a cyclic shift of* $(d_1,\dots,d_n,0)$. *Then the sum of mixed Eulerian numbers in each equivalence class is* $n!$.

Part (2) is the main result of [ERS]. The idea behind the proof is that $u_k\Delta_{n+1,k} + u_{k+1}\Delta_{n+1,k+1}$ (the volume of which yields the coefficients $A_{0^{k-1},n-i,i,0^{n-k-1}}, i=0,\dots,n$) turns out to be a slice of another cube, so it can be handled directly. However it's not clear how one could generalize their method to compute other mixed Eulerian numbers (for example of form $A_{0\ldots 0,a,b,c,0\ldots 0}$), because the Minkowski sum of (rescaled) hypersimplices is, in general, a very complicated polytope. Part (4) was conjectured by R. Stanley and proved by Postnikov in [Pos, Theorem 16.4]. It has an interesting geometric explanation in terms of alcoves of the affine Weyl group in type A.

Problem 2. It is known that there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ equivalence classes $\sim$ and each of them contains exactly one *Catalan sequence* $(c_1,\dots,c_n)$ such that $c_1+\cdots+c_i\geq i$. For each such sequence $(c_1,\dots,c_n)$, can we find some statistic on permutations whose distribution gives the mixed Eulerian numbers in its $\sim$-equivalence class? For example, by (2) of Theorem 1, the mixed Eulerian numbers in the $\sim$-class of $(n-i, i, 0^{n-2})$ are given by counting permutations $w \in S_{n+1}, w(n+1) = i+1$ with a fixed number of descents.

By using a formula of Postnikov for $\frac{\partial}{\partial x_i} VolP(x_1, ..., x_{n+1})$, I have showed

Proposition 3. [Cro] *Let $i = max\{j | c_j \neq 0\}$. Then*

$$A_{c_1 \dots c_n} = (n+1-i) \sum_{j \in S} \binom{n}{j-1} A_{c_1 \dots c_{j-1}} A_{c_{j+1} \dots c_{i-1}(c_i - 1) \dots c_n} + i A_{c_1 \dots c_{i-1}(c_i-1) \dots c_{n-1}}$$

Here and below, we define $A_{c_1 \dots c_n} = 0$ when $c_1 + ... + c_n \neq n$.

This proposition provides new simple proofs for almost all older results on the Mixed Eulerian numbers, and it completely determines them. It is convenient to consider the recursion of Proposition 3 as follows: Each weak composition $(c_1, \dots, c_n)$ of n corresponds naturally to a lattice path L in $\mathbb{Z}^2$ from $(1,1)$ to $(n+1, n+1)$. Then S of indices j for which the summand is non-zero is the set of x-coordinates where L crosses the diagonal $x = y$ horizontally.

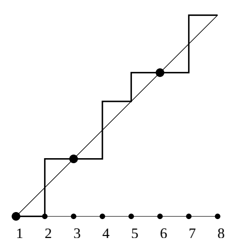

FIGURE 2.1. The path L corresponding to $(0,2,0,1,2,0,2)$. Here $S = \{1,3,6\}$.

The mixed Eulerian numbers include the factorials, binomial coefficients, numbers of permutations with various restrictions, numbers of the form $1^{c_1} ... n^{c_n}$. While finding a closed formula for $A_{c_1 \dots c_n}$ is unlikely (there is already no such formula for $A_{0 \dots 0,k,n-k,0..0}$), it seems reasonble to try to find way to label the n vertical segments of the path L with numbers $1, \dots, n$ with certain order restrictions depending on how L behaves (e.g. how L crosses the diagonal $x = y$), such that the number of labelings is $A_{c_1 \dots, c_n}$.

3. Generalizations to other root systems and beyond

The above setup can be generalized as follows. Let Φ be a root system of rank n spanning a real vector space V. Fix a choice of simple roots $\alpha_1, \dots, \alpha_n$ in Φ, consistent with the Dynkin labelling of the nodes of Φ. The roots are scaled so that the volume of the box with edges $\alpha_1, \dots, \alpha_n$ is 1. Let W_Φ be the Weyl group, and $\lambda_1, \dots, \lambda_n$ - the fundamental dominant weights of Φ. For a point $x = u_1\lambda_1 + ... + u_n\lambda_n \in V$, the *weight polytope* $P_{W_\Phi}(x)$ is the convex hull in V of the orbit $W_\Phi x$. The volume of this polytope, denoted by $V_\Phi(u_1, ..., u_n)$, is a homogeneous degree n polynomial in $u_1, ..., u_n$:

$$V_\Phi = \sum_{c_1, \dots, c_n \geq 0, c_1 + \dots + c_n = n} A^\Phi_{c_1 \dots c_n} \frac{u_1^{c_1}}{c_1!} \cdots \frac{u_n^{c_n}}{c_n!},$$

and the coefficients

$$A^\Phi_{c_1 \dots c_n} = n! \cdot Vol\left(P_{W_\Phi}(\lambda_1)^{c_1}, \dots, P_{W_\Phi}(\lambda_n)^{c_n}\right).$$

are called *the mixed Φ-Eulerian numbers.* When $\Phi = \mathrm{A}_n$, one can show that these are the usual mixed Eulerian numbers. Although the vertices of the Φ-*hypersimplices* $P_{W_\Phi}(\lambda_i)$ don't always lie in the root lattice of Φ, it turns out that

Theorem 4. [Cro] *For any Φ, the mixed Φ-Eulerian numbers are positive integers.*

Problem 5. The last result reinforces the idea that the Φ-mixed Eulerian numbers should have a combinatorial description. Do they count numbers of Weyl group elements with certain restrictions on lengths, descents, inversions, etc? Such a description would have to take into account root lengths, because the dual root systems B_n and C_n have the same Weyl group, but different mixed Eulerian coefficients. In fact finding the volumes of the Φ-*hypersimplices* $P_{W_\Phi}(\lambda_i)$ (i.e. $\frac{1}{n!}A^{\Phi}_{0\ldots n\ldots 0}$) and studying their various triangulations has been the goal of an ongoing project of Postnikov and Lam.

The following is the first result than enables one to compute all the mixed Φ-Eulerian numbers quickly. Given a weak composition $(c_1, ..., c_n)$ of n, we identify it with the labelling $w : i \mapsto c_i$ of the Dynkin diagram. We write A^{Φ}_w for $\frac{1}{(n+1)!}A^{\Phi}_{c_1\ldots c_n}$. Let $w_{i\to j}$ be the labelling which is identical to w except $w_{i\to j}(i) = w(i) - 1, w_{i\to j}(j) = w(j) + 1$.

Proposition 6. [Cro] *Fix i such that $w(i) > 0$. Then*

$$2A^{\Phi}_w - \sum_{i,j-\text{connected}} A^{\Phi}_{w_{i\to j}} = A^{\Phi-\{i\}}_{w|_{\Phi-\{i\}}} \tag{3.1}$$

In type A, this gives a simpler (but equivalent) set of recurrences than the ones from Proposition 3. Note also how apparent the root system structure is in equation 3.1.

Problem 7. Looking at these recursions, it seems that one could generalize the mixed Eulerian numbers to arbitrary graphs. However, they overdetermine A^{Φ}_w, and one can show that the only simple (connected) graphs which admit such coefficients come from the Dynkin diagrams. Still, it seems plausible that A^{Φ}_w could count the number of labellings of the Dynkin diagram Φ with certain local properties (depending on w).

Part (4) of Theorem 1 also has a generalization to the following cyclic formula for the volumes of generalized permutohedra:

Theorem 8. [Cro]*Let $\overline{\Phi}_i$ denote the root system in V spanned by $\{\alpha_j | j \neq i\}$. Then*

$$\sum_{i=1}^{n+1} \frac{m_i}{|W_{\overline{\Phi}_i}|} V_{\overline{\Phi}_i}(u_1, ..., \hat{u}_i, ..., u_{n+1}) = \frac{|\det(\alpha_1 \ \ldots \ \alpha_n)|^{-2}}{n! m_1 \cdots m_n} \left(\sum_{i=1}^{n+1} \frac{m_i (\alpha_i, \alpha_i)}{2} u_i \right)^n$$

Here α_{n+1} denotes a *highest root* of Φ, $m_1, ..., m_n$ are given from $\alpha_{n+1} = m_1\alpha_1 + \cdots + m_n\alpha_n$, and $m_{n+1} = 1$. The idea behind this result is that if we take a point $x = u_1\lambda_1 + ... + u_n\lambda_n$ inside the *fundamental alcove* of the affine Coxeter arrangement of Φ, then its images under the affine Weyl group give a subdivision of the space into generalized permutohedra. This gives an alternative way to compute the simple volume of the alcove.

The polynomials $V_\Phi(u_1, ..., u_n)$ are, essentially, dependent only on the Cartan matrix A_Φ. Indeed, there is a recurrent formula (which has a simple geometric explanation) that expresses V_Φ in terms of the similar polynomials associated to all root subsystems of Φ.

This allows us to generalize the polynomials V_Φ to arbitrary positive definite matrices A as follows: We define the homogeneous polynomials $P_A(u_1,...,u_n)$ by the following recursion:

$$P_A(u_1,...,u_n) = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} A^{-1} \begin{bmatrix} P_{A_{11}}(u_2,...,u_n) \\ \vdots \\ P_{A_{nn}}(u_1,...,u_{n-1}) \end{bmatrix}, \ P_{[a]}(u) = \frac{u}{a}.$$

Proposition 9. [Cro] *Let Φ be a root system, and A_Φ its Cartan matrix. Then*

$$P_{A_\Phi}(u_1,...,u_n) = \frac{n!}{|W_\Phi|} V_\Phi(u_1,...,u_n)$$

The main geometric idea behind this result is that the weight polytope P_{W_Φ} can be triangulated into rectangular pyramids with bases consisting of $P_{W_{\Phi_j}}$ (Φ_j is Φ with node j removed).

The polynomials P_A satisfy various nice properties. For example, if A is a block diagonal matrix, then P_A can be easily expressed as a product of the polynomials corresponding to the blocks of A. Curiously, the coefficient of $u_1...u_n$ in P_A is $\frac{n!}{\det A}$, which in particular implies that for any root system, the mixed volume of the n Φ-hypersimplices is $\frac{|W_\Phi|}{n! \det A_\Phi}$.

By using the recursive definition of P_A, it is possible to derive an explicit (but complicated) formula for P_A as a signed sum of products and quotients of minors of A. However, in the case of a Cartan matrix $A = A_\Phi$, this formula simplifies, because one can expresse the minors of A_Φ in terms of weighted paths in the Dynkin diagram of Φ: There is a natural weight function wt on edges of Φ (in types A, D, E all edges have weight 1/2), and the weight of a path is the product of weights of its edges.

Theorem 10. [Cro] *We have*

$$A^\Phi_{c_1...c_n} = \frac{|W|c_1!...c_n!}{2^n n!} \sum_{\pi=\pi_1...\pi_n \in S_n} \sum_{P_1,...,P_n} wt(P_1)...wt(P_n)$$

where the sum is over all paths $P_i : \sigma_i \to \pi_i$ in the Dynkin diagram such that P_i avoids $\pi_1, ..., \pi_{i-1}$, and such that exactly c_j of these paths start at j.

It's interesting to note that similar expressions appear in the famous work on *quasi-determinants* by Gelfand and Retakh ([GR]).

Problem 11. What are the coefficients of u_i^n in P_A? (for $A = A_\Phi$ they are the volumes of the Φ-hypersimplices) When are they nonnegative integers? What can be said about other coefficients of P_A?

4. Diagonal Vectors of Shifted Young Tableaux

One other problem that I studied is describing diagonal vectors of Young tableaux of *shifted shapes.* These shapes are similar to regular Young diagrams: given a partition $\lambda = (\lambda_1, ..., \lambda_n)$ with at most n parts, we add $n - i + 1$ boxes to the left of row i. For example, the tableau in Figure 4.1 has diagonal vector $(1, 4, 7, 17)$.

1	2	3	5	8	9	12	13
	4	6	10	11	16		
		7	14	15			
			17				

FIGURE 4.1. A shifted Young tableau of shape $\lambda = (4, 2, 1, 0)$. Entries are increasing along rows and down the columns.

The problem was solved in the case $\lambda = (0^n)$ by A. Postnikov, in [Pos]. Specifically, it was shown that diagonal vectors of the shifted triangular shape $D_\emptyset$ are in bijection with lattice points of a famous polytope, the $(n-1)$-dimensional *associahedron* Ass_{n-1}. The main idea of the proof is that shifted Young tableaux give a way to triangulate a certain *Gelfand-Tsetlin*-type polytope, and so they can be counted by computing the volume of this polytope in two ways. Using similar ideas, I have shown

Theorem 12. [Cro] *There is a bijection between diagonal vectors of λ-shifted Young tableaux is equal to the number of lattice points of the polytope*

$$\mathbf{P}_\lambda := \sum_{1\le i\le j\le n-1} \Delta_{[i,j]} + \lambda_1\Delta_{[1,n]} + \lambda_2\Delta_{[2,n]} + \cdots + \lambda_n\Delta_{\{n\}}.$$

It's interesting to note that if λ has n parts, then $\mathbf{P}_\lambda$ is combinatorially equivalent to $\sum_{1\le i\le j\le n} \Delta_{[i,j]} = \mathrm{Ass}_{n-1}$. In particular, it has $C_n = \frac{1}{n+1}\binom{2n}{n}$ vertices. In the same paper I also show an explicit way to construct the "extremal" Young tableaux, whose diagonal vectors correspond to vertices of $\mathbf{P}_\lambda$. These vertices are in bijection with binary trees on $[n]$, with certain restrictions on the size of the branches at $[n]$ depending on the number of parts of λ. The corresponding shifted Tableaux can essentially be read from the *binary search labelling* of these trees.

Problem 13. The normalized volume is in some sense "dual" to the number of lattice points of a polytope. Are there any combinatorial objects (vectors of Young tableaux, trees, etc) which give a triangulation of the polytope $\mathbf{P}_\lambda$? Can we at least interpret the normalized volume of $\mathbf{P}_\lambda$ combinatorially?

Problem 14. The diagonal vectors of shifted tableaux *are* precisely the lattice points of an integer polytope (not $\mathbf{P}_\lambda$, but its image under an element of SL_n). One can explore the similar question for other classes of Young diagrams. For example, given a partition $\lambda = (\lambda_1, ..., \lambda_n)$, one can ask if the set of vectors arising as reading words of standard Young λ- tableaux are exactly the lattice points of a convex integer polytope? While this is not true in general, it is true that the *first row* vectors arising from Young tableaux come as lattice points of an integer polytope. In this case one can ask if they are in bijection with other objects such as trees, parking functions, etc.

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