SOLUTION SKETCHES TO SELECTED PROBLEMS FROM HOMEWORK 2

(3) $P = I$ in $PA = LU$ if and only if we do not need any row switches when we do elimination on A. Hence the same row operations turn each of the top left square submatrices into triangular form, with (non-zero) pivots on the diagonal, which shoews that all these submatrices are invertible. Conversley, assume all top left square submatrices A_k are invertible. Since $A_1 \neq 0$, there is a first pivot in A. After clearing the (2,1) entry of A, the (2,2) entry has to be non-zero (otherwise A_2 is non-invertible, since elimination would produce a 0 row), hence there is a second pivot and we may keep going.

(4) The trick is to rewrite $I = LDU$ as $L^{-1} = DU$. The inverse of a lower triangular matrix L with 1's along the diagonal has the same form. This can be seen by direct computation, but here is a better way to think about it: L encodes some row operations where one *subtracts* multiples of *previous* rows to a row. All these steps are reversible and moreover, the inverse steps involve *adding* multiples of *previous* rows to a row, hence the matrix L^{-1} has the claimed form. On the other hand DU is upper triangular with the entries of D along the diagonal. Therefore $L^{-1} = DU$ both upper and lower triangular with 1's on the diagonal (and on the other hand, with entries of D along the diagonal). Hence $L^{-1} = D = I$, which implies $L = D = U = I$.

(10) The angle between v and AA^Tv is acute if and only if $v \cdot AA^Tv > 0$. We rewrite this dot product as a matrix product as follows: $v \cdot AA^Tv = v^T(AA^Tv) = (v^TA)(A^Tv) = (A^Tv)^T(A^Tv) = A^Tv \cdot A^Tv = \| A^Tv \|^2$, which is positive because $A^Tv \neq 0$. (A is invertible, hence so is A^T, and $v \neq 0$.)