On Recurrence of Graph Connectivity in Vicsek's Model of Motion Coordination for Mobile Autonomous Agents

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***Abstract*— In this paper we complete the analysis of Vicsek's model of distributed coordination among kinematic planar agents. The model is a simple discrete time heading update rule for a set of kinematic agents (or self-propelled particles as referred to by Vicsek) moving in a finite plane with periodic boundary conditions. Contrary to existing results in the literature, we do not make any assumptions on connectivity but instead prove that under the update scheme, the network of agents stays jointly connected infinitely often for almost all initial conditions, resulting in global heading alignment. Our main result is derived using a famous theorem of Hermann Weyl on equidistribution of fractional parts of sequences. We also show that the Vicsek update scheme is closely related to the Kuramoto model of coupled nonlinear oscillators.**

I. INTRODUCTION

Distributed motion coordination has been a major research focus in control theory and robotics over the past few years. From flocking, velocity alignment [1]–[3] and consensus [4]–[9] to distributed locational optimization and rendezvous problems [10], researchers have been trying to analyze and design distributed protocols for motion coordination that use local information and result in a global desired behavior.

A well-known example of such a model is Vicsek's model of alignment for self-propelled particles [11] introduced in the statistical physics literature. The model is a distributed iterative scheme in which a set of self-propelled particles moving with constant velocity update their headings based on the "average" of the headings of each particle and its neighbors. Vicsek's model was first analyzed in [3] in which the authors show that if the *proximity graphs* representing the interconnection of particles (hereafter denoted as agents) are infinitely-often *jointly connected*, then all headings converge to a common value asymptotically. This result was later extended to nonlinear update schemes and directed graphs in [12]. Update rules or protocols which result in agreement of a particular variable is often called a consensus scheme. Consensus and agreement protocols have an old history. Early instances of the problem have been addressed in the literature on Markov chains and distributed computation such as [4], [13]–[15].

Since the results of [3], there have been many generalizations and variations of this theme in the control theory literature. A non-exhaustive list of examples include distributed computation of averages and least squares [6], [8], [16], and rendezvous problems [10]. All these papers essentially present a condition on the evolution of the network's topology over time (more precisely, the condition of joint connectivity of the proximity graphs), under which a distributed iterative scheme defined over the network converges to a consensus.

Although joint connectivity over time is sufficient and in certain cases necessary [12] for reaching consensus over the network of agents, it is a difficult task to show when such conditions actually hold. One notable exception is the rendezvous protocol analyzed in [10], where connectivity is enforced by the protocol. In most other cases connectivity is merely an assumption. There are of course other scenarios where the authors assume no dependency between the evolution of the graph and the state value of the agents (e.g. the position). An example is when the evolution of the underlying proximity graph is not due to any motion model but is purely random [17], [18].

Therefore, the problem of reaching consensus (in velocity or any other variable) when the states of each agent (such as its position) affect the evolution of the network's topology, is still wide open. A particular example of such a case is Vicsek's model in where agents are mobile and neighboring relations are distance-dependent. Addressing this particular problem is precisely the goal of this paper.

We note that one important aspect of Vicsek's model and the corresponding simulations reported in [11] which was neglected before is that in the scheme proposed in [11], the agents move in a plane with *periodic boundary conditions* as opposed to an infinite plane. In other words, the state space in Vicsek's model is a flattened torus.

The main contribution of this paper is to prove that Vicsek's scheme for updating headings with a periodic boundary condition does indeed cause the agents to stay jointly connected infinitely often. As a result, all headings will be eventually aligned, and the agents will flock. Hence we provide an explanation for Vicsek's simulations. The key technical idea used in proving the main result is Hermann Weyl's theorem on equidistribution of sequences modulo 1, which is a cornerstone of analytic number theory and Diophantine approximation. A side contribution of this paper is that we also elaborate on connections between Vicsek's model for self-propelled particles and Kuramoto's model of coupled nonlinear oscillators [19]

This paper is organized as follows: In section II we present Vicsek's model of the motion of autonomous agents in full followed by investigating its link to the Kuramoto model in section III. Section IV describes the concept of joint

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connectivity of the network's proximity graphs over time, the well-known sufficient condition for reaching a consensus over the multiagent network. We present our main results in section V, proving the convergence of Vicsek's model to a consensus. This goal is achieved using Weyl's theorem. Section VI contains our simulations. Finally, the conclusions are presented in section VII.

II. Vicsek's Model

The system studied by Vicsek *et al.* in [11] consists of a group of n agents, labeled 1 through n, which are distributed uniformly over a two dimensional square of length 1 with periodic boundary conditions. By periodic boundary conditions we mean that if an agent hits the boundary of the square with velocity v and angle θ, it enters the square from the opposite boundary with the same velocity and angle. In fact, one can assume that the agents move over the *flattened unit torus*. The existence of periodic boundary conditions is a typical assumption in statistical physics models in order to avoid boundary effects in simulations. Furthermore, Vicsek assumes that all the agents move in this unit-area box with a common constant velocity v, but with different initial headings. The initial headings of the agents are drawn independently and uniformly from the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We denote the initial heading of agent i by $\theta_i(0)$. Each agent's heading is updated at discrete time steps $k \in \{1, 2, \cdots\}$ by a local rule, based on the headings of its neighbors. Agent i's *neighbors* at time k are those agents which are located in a circle of radius r, known as the interaction radius, and centered at agent i's location at time k.[1] Each agent is considered to be a neighbor of itself. Vicsek's local rule according to which agent i updates its heading θ_i, is given by

$$\theta_i(k+1) = \arctan \frac{\sum_{j\in\mathcal{N}_i(k)} \sin\theta_j(k)}{\sum_{j\in\mathcal{N}_i(k)} \cos\theta_j(k)}, \tag{1}$$

where $\mathcal{N}_i(k)$ denotes the set of neighbors of agent i at time k.[2] If we denote the location of agent i at time k by vector $\begin{bmatrix} x_i(k) \ y_i(k)\end{bmatrix}^T$, then it is related to agent i's heading through the following equations:

$$\begin{bmatrix} x_i(k+1) \\ y_i(k+1) \end{bmatrix} = \begin{bmatrix} \langle x_i(k) + v\cos\theta_i(k)\rangle \\ \langle y_i(k) + v\sin\theta_i(k)\rangle \end{bmatrix},$$

where $\theta_i(k)$ evolves based on (1) and $\langle a\rangle = a - \lfloor a \rfloor$ is the fractional part of real number a. Note that the fractional parts appear in the location equations because of the assumption of periodic boundary conditions. As a result, the location of agent i at time k in terms of its heading history is $[x_i(k) \quad y_i(k)] = [\langle \tilde{x}_i(k)\rangle \quad \langle \tilde{y}_i(k)\rangle]$, where

$$\begin{bmatrix} \tilde{x}_i(k) \\ \\ \tilde{y}_i(k) \end{bmatrix} = \begin{bmatrix} x_i(0) + v\sum_{m=0}^{k-1}\cos\theta_i(m) \\ \\ y_i(0) + v\sum_{m=0}^{k-1}\sin\theta_i(m) \end{bmatrix}. \tag{2}$$

[1] Note that since we are assuming that the agents move in a box with periodic boundary conditions, the neighborhood disks are also defined on the flattened unit torus.

[2] Vicsek's update also includes an additive noise which we ignore here.

Vicsek's extensive simulations show that if the agents update their headings based on (1) while moving on the flattened unit torus, the headings reach an eventual consensus and the agents get aligned. In other words, the heading vector $\theta(k) = \begin{bmatrix}\theta_1(k)\cdots\theta_n(k)\end{bmatrix}^T$ satisfies

$$\lim_{k\to\infty} \theta(k) = \theta^* \mathbf{1}$$

for some $\theta^* \in [0, 2\pi)$, where $\mathbf{1}$ is a vector with all entries equal to one. Our goal is to prove this observation analytically. But before presenting our main results, we want to emphasize that Vicsek's update scheme (1) and Kuramoto model of coupled nonlinear oscillators are closely related.

III. From Vicsek to Kuramoto

Before we proceed with the main results of the paper we first show an interesting connection between Vicsek's model and the Kuramoto model of phase coupled nonlinear oscillators. Specifically, we show that the update scheme (1) suggested by Vicsek in [11] is in fact a discrete-time splitting iteration scheme which solves for the equilibrium point of the dynamical system

$$\dot{\theta}_i = -\sum_{j\in\mathcal{N}_i} \sin(\theta_i - \theta_j) \tag{3}$$

known as the Kuramoto model of coupled nonlinear oscillators [19]–[21]. We note that other connections between the two models have also been shown recently in [22].

In order to show the connection, we use the change of variables $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} : \theta_i \mapsto z_i = \tan\theta_i$ suggested by Moreau in [12]. Expressed in terms of the new variables, the Kuramoto equation (3) becomes

$$\dot{z}_i = -\sum_{j\in\mathcal{N}_i} \sqrt{\frac{1+z_i^2}{1+z_j^2}}(z_i - z_j).$$

By defining the vector $z = [z_1, \cdots, z_n]^T$ the above dynamical system can be represented in the more compact form

$$\dot{z} = -L(z)z = -(D-A)\,z$$

where D and A are diagonal and nonnegative matrices respectively, with entries defined as

$$\begin{aligned} D_{ii} &= \sum_{j\in\mathcal{N}_i} \sqrt{\frac{1+z_i^2}{1+z_j^2}} \\ A_{ij} &= \begin{cases} \sqrt{\frac{1+z_i^2}{1+z_j^2}} & \text{if } j\in\mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

and $L(z)$ is a state dependent *Laplacian matrix* of the underlying weighted graph. The decomposition of the singular M-matrix Laplacian $L(z)$ as $D(z) - A(z)$ is called a *weak regular splitting* and is used for finding iterative methods to solve for singular linear systems [23], [24]. In fact, it is quit well-known that the iteration

$$z(k+1) = (D^{-1}A)z(k) \tag{4}$$

converges to the equilibrium point of the dynamical system $\dot{z} = -L(z)z$. In other words, the weighted averaging scheme in (4) is an iterative scheme that converges to elements in the kernel of $L(z)$ or set of vectors that satisfy $L(z)z = 0$.

The i-th entry of the update scheme (4) has the form

$$z_i(k+1) = \sum_{j\in\mathcal{N}_i} \frac{A_{ij}}{D_{ii}} z_j(k) = \sum_{j\in\mathcal{N}_i} \frac{\frac{z_j(k)}{\sqrt{1+z_j^2}}}{\sum\limits_{l\in\mathcal{N}_i(k)} \frac{1}{\sqrt{1+z_l^2}}}.$$

A careful examination of this weighted averaging scheme reveals that by undoing the variable change we did earlier, i.e. by setting $z_i(k) \mapsto \tan\theta_i(k)$, the above iteration becomes exactly identical to Vicsek's update scheme (1). In other words, Vicsek's update is nothing but a distributed weighted averaging scheme or implementation of a splitting algorithm for finding the fixed point of the Kuramoto model.

IV. Graph Theoretic Conditions For Reaching Consensus

we now review the existing results on reaching agreement among a set of agents with changing interconnection topology. The results are mainly based on a theorem first stated in [4] in the context of distributed computation and parallel processing and later in [3] in the context of motion coordination and flocking. The theorem provides a sufficient condition for convergence of updates schemes such as (1) to a consensus.

In order to be able to state the theorem, we need to define the *proximity graph* of the network of agents at a given time. The undirected time dependent graph $\mathbb{G}(k)$ is the proximity graph of the network at time k, if each of its vertices represents an agent and the unordered pair (i,j) is in its edge set if and only if agent j is a neighbor of agent i at time k, i.e. the distance between the two agents is less than the interaction radius r. We say the proximity graphs of the network are *jointly connected* over time interval $[k', k'']$, with $k', k'' \in \mathbb{N}$ and $k' \leq k''$, if the graph whose edge set is the union of the edge sets of the graphs in $\{\mathbb{G}(k'), \mathbb{G}(k'+1), \cdots, \mathbb{G}(k'')\}$ is connected. We denote such a graph, known as the *union graph*, by $\bigcup_{k=k'}^{k''} \mathbb{G}(k)$.

Given the above definitions, we can state the following theorem which was proved under slightly different assumptions in [3], [4], [6], [7], [12]. The theorem effectively guarantees convergence of distributed averaging schemes such as Vicsek's update scheme to a consensus under specific conditions on joint connectivity.

Theorem 1: Let $\mathbb{G}(k)$ be the proximity graph of the network of agents at time k. Assume that the initial heading values $\theta(0)$ are fixed and are updated over time based on the discrete-time update scheme (1). If there exists a sequence of integers $k_r, r = 1, 2, \cdots$ with $k_1 = 0$ such that the graphs $\bigcup_{k=k_r+1}^{k_{r+1}} \mathbb{G}(k)$ are connected, then

$$\lim_{k\to\infty} \theta(k) = \theta^* \mathbf{1},$$

where $\theta^* \in [0, 2\pi)$ is a constant depending on $\theta(0)$ and the sequence $\mathbb{G}(k)$.

Theorem 1 states that if joint connectivity occurs over infinitely many intervals, iterative averaging schemes such as (1) result in an eventual consensus over the network. As a result, all agents move in the same direction eventually if they update their headings based on the model described in section II. The question that remains to be answered is whether the assumption of joint connectivity over time actually holds, and if so, under what conditions. In the next section we prove that this is actually the case for Vicsek's model.

V. Main Result: Connectivity in Vicsek's Model

This section contains our main results, describing why the headings of all n agents eventually become the same if they follow Vicsek's update scheme (1) while moving on the flattened unit torus. Our results are based on theorem 1 regarding infinite often joint connectivity of the proximity graphs and a theorem by Hermann Weyl [25] on Diophantine approximations, which is a cornerstone of analytic number theory as well as convergence of exponential sums.

According to theorem 1 if there exists a path between any two vertices over time infinitely often, then all headings converge to a common value and the agents flock in the same direction. Therefore, all we need to show is that in Vicsek's setup the proximity graphs are indeed jointly connected over infinitely many time intervals. In fact, we prove that the assumption of having periodic boundary conditions guarantees this for almost all initial conditions. The following theorem formalized this argument.

Theorem 2: Consider a group of n agents moving on a unit length square with periodic boundary conditions. If the agents update their headings based on (1), then for almost all initial headings and locations there exists a sequence of integers $k_r, r = 1, 2, \cdots$ such that the graphs $\bigcup_{k=k_r+1}^{k_{r+1}} \mathbb{G}(k)$ are connected.

We use a contradiction argument to prove the above theorem. However, before doing so, we need to state some results from the field of Diophantine approximation in number theory, which will be later used in the proof. The theorem that we use here is a generalization of a theorem first stated and proved by Weyl in [25][3]. Nevertheless, we refer to it as Weyl's theorem. The theorem provides a sufficient condition for a sequence of real-valued r-dimensional vectors to be *equidistributed*[4] *mod 1*. Roughly speaking, the theorem states conditions under which the fraction of the fractional parts of the sequence terms which fall into a subset of the unit hypercube $[0,1]^r$ is equal to the Lebesgue measure of that subset.

Theorem 3 (Weyl's theorem): Let $\{\Lambda(k) : k \in \mathbb{N}\}$ be a sequence of entry-wise strictly monotone diagonal $r \times r$ matrices. If $c > 0$ and $\epsilon > 0$ exist such that for all $1 \leq q \leq r$, $\Lambda_{qq}(k)$ changes by at least c whenever k increases as much

[3] Weyl's article is in German. Refer to [26] for the statement of the theorem in English.

[4] Alternatively, in some papers it is referred to as uniformly distributed or *Gleichverteilung* as Weyl refers to it.

as $k(\log k)^{-1-\epsilon}$, then the points $\{\Lambda(k)\xi : k \in \mathbb{N}\}$ are equidistributed mod 1 for almost all values of $\xi \in \mathbb{R}^r$.

Theorem 3 states that for any sequence $\Lambda(k)$ whose entries grow fast enough, there exists a set $A \subseteq \mathbb{R}^r$ such that

$$\begin{aligned}
&\mu(A) = 0 \\
&\forall \xi \in A^c, \{\langle \Lambda(k)\xi \rangle : k \in \mathbb{N}\} \text{ is dense in } [0,1]^r.
\end{aligned}$$

Also it is worthwhile to mention that for $r = 1$ and $\Lambda(k) = k\xi$, Weyl's theorem reduces to the well-known Kronecker's theorem [27]. Kronecker's theorem states that given any irrational number ξ the fractional part of the sequence $k\xi$ is dense over the unit interval.

At this point one may realize the potential use of Weyl's theorem in the proof of theorem 2. In fact, using this theorem we show that the fractional parts of the distance between any two agents is dense. Therefore, the two agents get arbitrarily close to each other infinitely often, satisfying the conditions of theorem 1. We now formally state the proof.

Proof of theorem 2: Given a vertex h, define the set $\mathcal{V}_h$ to be the set of vertices l for which an increasing sequence of integers $k_r, r = 1, 2, \cdots$ exists such that there is a path from h to l in the graph $\bigcup_{k=k_r+1}^{k_{r+1}} \mathbb{G}(k)$. In other words, there exists a path from vertex h to any vertex in $\mathcal{V}_h$ over time infinitely often. Also assume that $h \in \mathcal{V}_h$. Theorem 1 implies that the headings of all agents in set $\mathcal{V}_h$ converge to a common value asymptotically.

Now assume that there exist two vertices $i \neq j$ and only finitely many intervals $[k_r+1, k_{r+1}]$ such that there is a path between i and j in the union graphs $\bigcup_{k=k_r+1}^{k_{r+1}} \mathbb{G}(k)$. In other words, assume that $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$. By theorem 1,

$$\begin{aligned}
\forall l \in \mathcal{V}_i \qquad \lim_{k\to\infty} \theta_l(k) = \theta_i^*, \\
\forall l \in \mathcal{V}_j \qquad \lim_{k\to\infty} \theta_l(k) = \theta_j^*.
\end{aligned}$$

If $\theta_i^* = \theta_j^*$ the agents in the two sets will move in the same direction eventually. Therefore, we are only interested in the case that $\theta_i^* \neq \theta_j^*$. The l_1 distance between agents i and j satisfies

$$\begin{aligned}
d_{ij}(k) &= \min\{|x_i(k) - x_j(k)|, 1 - |x_i(k) - x_j(k)|\} \\
&\quad + \min\{|y_i(k) - y_j(k)|, 1 - |y_i(k) - y_j(k)|\} \\
&\leq \langle \tilde{x}_i(k) - \tilde{x}_j(k) \rangle + \langle \tilde{y}_i(k) - \tilde{y}_j(k) \rangle
\end{aligned}$$

Therefore, if we show the sequence of vectors $\left[\langle \tilde{x}_i(k) - \tilde{x}_j(k) \rangle \quad \langle \tilde{y}_i(k) - \tilde{y}_j(k) \rangle\right]^T$ is dense in the unit square $[0,1)^2$, it is straightforward to conclude that agents i and j get arbitrarily close to each other, and therefore, the assumption of $\mathcal{V}_i \cap \mathcal{V}_j =$ does not hold.

Define the 2×2 matrix $\Lambda(k) = diag\{\lambda_1(k), \lambda_2(k)\}$ where

$$\begin{aligned}
\lambda_1(k) &= \tilde{x}_i(k) - \tilde{x}_j(k) \\
\lambda_2(k) &= \tilde{y}_i(k) - \tilde{y}_j(k).
\end{aligned}$$

Since both $\theta_i(k)$ and $\theta_j(k)$ converge to a limit, both sequences above are monotonic for large enough k. For the same reason, there exists a constant $c > 0$ such that for large enough k, the two inequalities

$$\begin{aligned}
|\lambda_1(k+1) - \lambda_1(k)| &> c \\
|\lambda_2(k+1) - \lambda_2(k)| &> c
\end{aligned}$$

hold. Therefore, the conditions of theorem 3 are satisfied and as a result, there exists $A \subset \mathbb{R}^2$ with $\mu(A) = 0$, such that for every $\xi \in A^c$, the set $\{\langle \Lambda(k)\xi \rangle : k \in \mathbb{N}\}$ is dense in $[0,1]^2$. Therefore, our proof is complete once we show that in fact $(1,1) \in A^c$. Note that this is not necessarily the case for all initial conditions. In fact, one can even construct examples with two agents where such a statement is false. But, the key point is that since A^c is full measure and the update scheme is a measure preserving transformation (i.e. it maps generic sets to generic sets), we have $(1,1) \in A^c$ for almost all initial conditions.

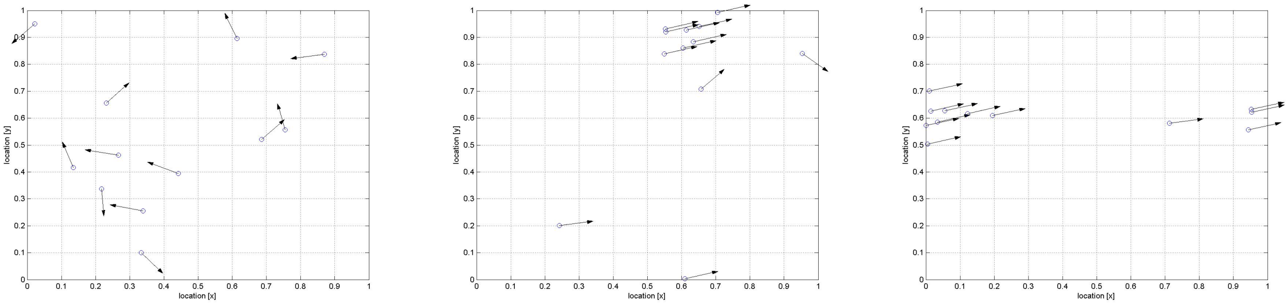
Hence, the set $\{\langle(\lambda_1(k), \lambda_2(k))\rangle : k \in \mathbb{N}\}$ is dense in $[0,1]^2$ for almost all initial conditions, and consequently, the agents i and j get arbitrarily close to each other eventually. In particular, the distance between i and j gets smaller than the interaction radius r, contradicting the assumption that there is no path between the two over time after a specific point in time. This finishes the proof. ■

As a result of theorem 2, for almost all initial conditions there is a path between any two agents in the network infinitely often, or equivalently, there exists the sequence of integers $k_r, r = 1, 2, \cdots$ such that the graphs $\bigcup_{k=k_r+1}^{k_{r+1}} \mathbb{G}(k)$ are connected. At this point, theorem 1 implies the eventual consensus in the network, describing in full Vicsek's results for coordination of mobile autonomous agents with heading update scheme (1). Note that the above theorem can be used to achieve similar results for any measure preserving consensus scheme, as long as we assume periodic boundary conditions.

As a final remark we would like to emphasize that it is the periodic boundary condition assumption in Vicsek's model that imposes asymptotic convergence to the consensus. The agents are not guaranteed to have such a behavior while moving on an infinite plane or with reflective boundary conditions, as is the case in real world.

VI. Simulations

In this section, we numerically verify the convergence of Vicsek's update scheme to a heading consensus for a group of 10 mobile autonomous agents. In figure 1(a) the initial positions and headings of the agents in the unit box with periodic boundary conditions are shown. The agents are placed uniformly in the box and their initial headings are drawn randomly from the uniform distribution over interval $[0, 2\pi]$. The headings of each agents are updated based on the local rule (1), in which the interaction radius is $r = \frac{1}{10}$. We assumed the same constant speed $v = \frac{1}{20}$ for all agents. Figures 1(b) and 1(c) demonstrate the headings and locations at time steps $k = 50$ and $k = 100$ respectively.



(a) (b) (c)

Fig. 1. (a) The initial headings and locations of 10 agents in the unit box with periodic boundary conditions. (b) The headings and locations of the agents after 50 time steps. (c) The headings and locations of the agents after 100 time steps.

We also define an *order parameter* as

$$\alpha(k) = \frac{1}{n \cdot v} \left| \sum_{i=1}^{n} v_i(k) \right|,$$

where $v_i(k) = \begin{bmatrix} v\cos\theta_i(k) & v\sin\theta_i(k) \end{bmatrix}^T$ represents the speed of agent i at time step k. The order parameter is approximately zero if the headings of the agents are randomly distributed, while it is a number close to one if all the agents are aligned. Therefore, we can use it as a parameter to measure the level of alignment of the agents [11]. The variations of the order parameter over time are demonstrated in figure 2. Clearly, it is a non-decreasing function and converges to one, as expected.

VII. Conclusions

In this paper, we showed how the model suggested by Vicsek for a network of mobile autonomous agents is related to Kuramoto model of coupled nonlinear oscillators. Furthermore, using a theorem of Weyl, we also proved that if the agents move on a unit square with periodic boundary condition and update their headings based on Vicsek's local update rule, for almost all initial conditions the proximity graphs become jointly connected over infinitely many time intervals, and therefore, all the agents move with the same heading eventually.

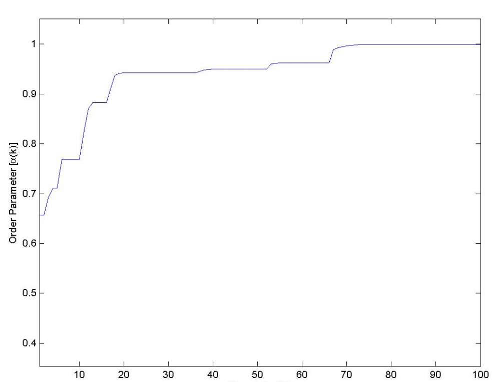


Fig. 2. The order parameter versus the number of updates. It is a non-decreasing function which converges to 1 as expected.

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