Effects of Delay in Multi-Agent Consensus and Oscillator Synchronization

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Abstract

The coordinated motion of multi-agent systems and oscillator synchronization are two important examples of networked control systems. In this paper, we consider what effect multiple, non-commensurate (heterogeneous) communication delays can have on the consensus properties of large-scale multi-agent systems endowed with nonlinear dynamics. We show that the structure of the delayed dynamics allows functionality to be retained for arbitrary communication delays, even for switching topologies under certain connectivity conditions. The results are extended to the related problem of oscillator synchronization.

I. INTRODUCTION

Understanding the collective behavior of systems that are formed by *arbitrary* (both in size and in structure) interconnections of smaller subsystems, is an important research area [1]. There are several approaches for analyzing the functionality of such networks, but also for *designing* control laws so that the networked system meets certain design objectives. Usually the interconnection topology and its size is not known *a priori*, or may even change with time – therefore the challenge is to ensure that the designed system has *scalable* properties. These could be robust stability or performance.

There are several examples of systems that exist in nature which possess these features. One example is *oscillator synchronization* [2], i.e. the way oscillating subsystems synchronize their frequencies or even lock their phases when coupled together. Synchronization has been used to describe many physical phenomena in which subsystems have the tendency to 'agree' to perform a common task when coupled together: the way pacemaker cells generate and

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pace the heartbeat, how fireflies flash in sync, etc. Beyond these are technological systems that we want to design for ‘agreement’ or ‘consensus’. One such example is coordinated motion of multivehicle systems for arbitrary interconnection topologies which, under certain conditions, can also be allowed to change [3]–[7]. The design procedure results in closed loop systems that have dynamics similar to the models used to describe synchronization in oscillator networks. Related examples are self-ordered particle motion [8], asynchronous distributed computation [9], agreement [10] and others [11], [12].

Here we investigate the effect of communication delays [13] in multi-agent consensus protocols and oscillator synchronization on large-scale networks described by nonlinear models. Time-delays can be used to model the effect of propagation of state information between interacting agents, but they are many times neglected to facilitate analysis. Even if they are small, it is well known that they can deteriorate the system’s performance or even destabilize it. Previous results have shown that coordination can be achieved for the discrete-time delayed system with linear dynamics [14], [15] and switching topologies [16]. In [17], [18] the authors used a frequency domain analysis for a linear, continuous time system to show stability independent of delays, while the authors in [10] used nonlinear undelayed dynamics with a linear control law and a contraction theorem to show consensus is independent of delay, which was also identified in [19]–[21]. The work in [22] extends the work in [4] to include time-delays, considering a discrete-time system. In [5] a delay-dependent condition is obtained for the case of continuous-time dynamics. Lastly, the work in [23] shows consensus reaching in multi-agent packet-switched networks, while [24] discusses output synchronization of nonlinear systems with communication time-delays. Here, we will build on our previous work [25], [26] on synchronization in oscillator networks to show that under certain connectivity assumptions, the consensus set is asymptotically attracting for nonlinear, continuous time dynamics and heterogeneous time-delays (Section II). Another problem that we will be investigating is whether coordination can be ensured for *switching* topologies within an admissible set even if delays are present in the system. This question is addressed using ideas from the stability analysis of systems with arbitrary switching with no chattering (Section III). In section IV we discuss the related problem of oscillator synchronization, concluding the paper in section V.

A. *Notation*

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $|\cdot|$ (the 2-norm unless otherwise stated). $C = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The norm on C is defined as $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Moreover, let $\rho \geq 0$ and $x \in C([-\tau, \rho], \mathbb{R}^n)$; then for any

$t \in [0, \rho]$, we define a segment $x_t \in C$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices $\mathcal{V} = \{v_i\}$, $i \in \mathcal{I} = \{1, \ldots, N\}$, and a set of edges $\mathcal{E} \subseteq \{(v_i, v_j) \mid v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$. If $v_i, v_j \in \mathcal{V}$ and $(v_i, v_j) \in \mathcal{E}$, then there is a directed arc from v_i to v_j and we say that v_i is the parent of v_j. A graph is said to be undirected if $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. The adjacency matrix A of a graph $\mathcal{G}$ is an $N \times N$ real matrix with $A_{ij} = 1 \Leftrightarrow (v_i, v_j) \in \mathcal{E}$ and $A_{ij} = 0$ otherwise. For undirected graphs, $A = A^T$. A directed path from vertex v_i to vertex v_j is a sequence of edges starting from v_i and ending at v_j so that consecutive edges belong to $\mathcal{E}$. A graph $\mathcal{G}$ is said to be strongly connected if there is a directed path between any two vertices in it. A directed tree is a directed graph for which every vertex v_j has exactly one v_i so that $(v_i, v_j) \in \mathcal{E}$ (i.e., v_j has exactly one parent) except the root of the tree. A spanning tree of a directed graph is a directed tree with the same vertex set but with an edge set which is perhaps a subset of the edge set of the directed graph.

II. Multi-agent system Consensus

Consider N agents, interacting over a network whose topology is given by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with adjacency matrix A. Agent j can send information to agent i only if $(v_j, v_i) \in \mathcal{E}$. The dynamics of agent i are given by

$$\dot{x}_i(t) = k_i \sum_{j=1}^{N} A_{ji} f_{ji} \left(x_j(t) - x_i(t) \right) \tag{1}$$

where $x_i \in \mathbb{R}$, $k_i > 0$ are constants and the functions f_{ij} are *locally passive*:

Definition 2.1: A C^1 function $f : \mathbb{R} \to \mathbb{R}$ is *locally passive*, if there exist $\sigma^- > 0$ and $\sigma^+ > 0$ such that $yf(y) > 0$ for all $y \in [-\sigma^-, \sigma^+] \subset \mathbb{R}$ apart from $y = 0$, where $f(0) = 0$.

Assumption 2.2: The functions $f_{ij} : \mathbb{R} \to \mathbb{R}$ are *locally passive* on $[-\sigma_{ij}^-, \sigma_{ij}^+]$ for some $\sigma_{ij}^- > 0$ and $\sigma_{ij}^+ > 0$, for all $i, j = 1, \ldots, N$.

We also define γ as:

$$\gamma = \min_{i,j=1,\ldots,N} \left\{ \sigma_{ij}^-, \sigma_{ij}^+ \right\} \tag{2}$$

An interesting equilibrium (set) of (1) is $x^* = c\mathbf{1}$ where c is a constant and $\mathbf{1}$ is the vector of ones of dimension N. Denote this set by $\mathcal{X}$:

$$\mathcal{X} = \{x \in \mathbb{R}^n | x = c\mathbf{1}, \text{ for } c \in \mathbb{R}\} \tag{3}$$

The exact value of c depends in a complicated way on the initial condition [27]. We emphasize that it may be possible that there are equilibria other than the consensus set depending on the structure of the topology, the allowable initial conditions etc. For example, see [28], where the ring topology with six vertices and $f_{ij}(y) = \sin(y)$ yields a second equilibrium point. However, one can restrict the set of initial conditions or the type of functions f_{ij} to ensure that there is only one equilibrium set, the agreement set.

We first show that if the interaction graph has a spanning tree, the consensus set is asymptotically attracting. This result will be used in the sequel. See also [20], [29].

Theorem 2.3: Consider the system given by (1) where the f_{ji}'s are locally passive. Let the initial condition be chosen in the set D defined by

$$D = \left\{ x \in \mathbb{R}^n \mid |x_i| \leq \frac{\gamma}{2} \right\}$$

where γ is given by Equation (2). If the interaction graph has a spanning tree then the consensus set $\mathcal{X}$ is asymptotically attracting.

Proof: It is not difficult to verify that region D is positively invariant, something which will be established for the delayed case later on. Therefore all solutions are bounded. Consider

$$V(x) = \max_i x_i \tag{4}$$

as a candidate Lyapunov function – note that this function is not differentiable, but it can still be used to conclude the attractivity properties of equilibria [29], [30]. In particular, the right hand side Dini derivative of $V(x)$ along the solutions of the system is defined as

$$\dot{V}(x(t)) = \lim \sup_{\tau \to 0^+} \frac{1}{\tau} \left[V(x(t+\tau)) - V(x(t)) \right]$$

Suppose I is the agent for which the maximum is achieved at time t. If there are many such agents, we choose the one that has maximum $\dot{x}_I$ and if there are still many such agents, we choose any one of those, but commit to that until a new agent holds the maximum value. Calculating the Dini derivative of the candidate Lyapunov function along (1) we get:

$$\dot{V}(x) = \dot{x}_I = k_I \sum_{j=1}^{N} A_{jI} f_{jI} \left(x_j(t) - x_I(t) \right)$$

We now argue that $\dot{V}(x) \leq 0$. Since $x_I = \max_{i=1,\ldots,N} x_i$ and $k_I > 0$, we immediately see that $\dot{x}_I \leq 0$ as the vector field is a positive summation of passive functions with non-positive arguments for $x \in D$. In conclusion, $\dot{V}(x) \leq 0$.

The x_i for which $\dot{V} = 0$ is the set for which $\dot{x}_I = 0$, where $x_I = \max_{i=1,\ldots,N} x_i = \alpha$. This holds if and only if all the parents of node I also hold the maximum value and so do their respective parents, etc. Since we have assumed that the graph contains a spanning tree, the root of the spanning tree and all the nodes in a directed path to the node v_I – denote all these nodes by $\mathcal{R}$ – also satisfy $x_i = \alpha$, $i \in \mathcal{R}$. Therefore the set for which $\dot{V} = 0$ is the set for which all vertices v_i, $i \in \mathcal{R}$ hold the value α; nodes in $\mathcal{I} \setminus \mathcal{R}$ must hold a value less than or equal to α.

A similar argument can be used to show that the function $W = -\min_i x_i$ is also a Lyapunov function, and in this case the set for which $\dot{W} = 0$ is the set for which $\dot{x}_K = 0$, where $x_K = \min_{i=1,\ldots,N} x_i = \beta$. Following the same argument, the root of the (same) spanning tree and all the nodes in the path from this root to v_K (again, denote these nodes by $\mathcal{Q}$) have to hold a value β, and nodes in $\mathcal{I} \setminus \mathcal{Q}$ must hold a value more than or equal to β.

Since the root of the spanning tree holds both the minimum and the maximum value, $\alpha = \beta$. An invariant set in the set $\{\dot{V} = 0\} \cap \{\dot{W} = 0\}$ is the consensus set $\mathcal{X}$ (3), which is asymptotically attracting by the invariance principle in [30]. ■

Here we are concerned with delayed versions of nonlinear continuous-time models of these systems, for which we aim to prove that consensus is retained irrespective of the size of the heterogeneous delays. We will use the formulation as it was detailed above, i.e., the functions f_{ij} satisfy Assumption 2.2 and the node dynamics are given by

$$\dot{x}_i(t) = k_i \sum_{j=1}^{N} A_{ji} f_{ji} \left(x_j(t - \tau_{ji}) - x_i(t) \right). \tag{5}$$

Here $\tau_{ji} \geq 0$ are time-delays that model the propagation of state information from node v_j to v_i, for all $(v_j, v_i) \in \mathcal{E}$. The state-space is now infinite-dimensional, with the state $x_t \in C([-\tau, 0], \mathbb{R}^N)$ where $\tau = \max_{i,j=1,\ldots,N} \tau_{ji}$. The consensus set is defined by:

$$\mathcal{X}_\tau = \{x_t \in C | x(t+\theta) = c\mathbf{1}, c \in \mathbb{R} \text{ for all } \theta \in [-\tau, 0], t \geq 0\} \tag{6}$$

We first prove the following lemma:

Lemma 2.4: Consider (5) where the f_{ji}'s satisfy Assumption 2.2 and $\tau = \max_{i,j=1,\ldots,N} \tau_{ji}$. Define γ as in (2), and consider initial conditions ψ that satisfy

$$|\psi_i(\theta)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad \theta \in [-\tau, 0]. \tag{7}$$

Then $-\frac{\gamma}{2} \leq x_i(t) \leq \frac{\gamma}{2}$ for all $t \geq -\tau$.

Proof: From the bounds on the initial condition, we have $-\frac{\gamma}{2} \leq \psi_i(\theta) \leq \frac{\gamma}{2}$ for $\theta \in [-\tau, 0]$. Suppose this condition is violated at time t^*. When this happens, the following hold:

1) $-\frac{\gamma}{2} \leq x_i(t) \leq \frac{\gamma}{2}$ for $t \in [-\tau, t^*)$ for all $i = 1, \ldots, N$;
2) At t^* there is an $i \in \{1, \ldots, N\}$ such that we have either:

$$\left\{ x_i(t^*) = \frac{\gamma}{2} \text{ and } \dot{x}_i(t^*) > 0 \right\} \text{ or } \left\{ x_i(t^*) = -\frac{\gamma}{2} \text{ and } \dot{x}_i(t^*) < 0 \right\}$$

Suppose the first case holds. Recall the structure of the dynamics:

$$\dot{x}_i = k_i \sum_{j=1}^{N} A_{ji} f_{ji} (x_j(t - \tau_{ij}) - x_i(t))$$

Since $x_i(t^*) = \frac{\gamma}{2} \geq x_j(t^* - \tau_{ij})$ from the observations above, each term on the right hand side is non-positive as f_{ji} are locally passive functions; and so $\dot{x}_i(t^*) \leq 0$, contradiction. Similarly for the second case. We therefore have that $-\frac{\gamma}{2} \leq x_i(t) \leq \frac{\gamma}{2}$ for all $t \geq -\tau$. ■

The above Lemma has identified an invariant set. We now show that the consensus set $\mathcal{X}_\tau$ is asymptotically attracting using Theorem A-2. Here we consider

$$\overline{V}(\phi) = \max_{\theta \in [-\tau, 0]} V(\phi(\theta))$$

as a Lyapunov functional, where $V(\phi)$ is a 'Lyapunov-Razumikhin' function (see Appendix).

Theorem 2.5: Consider (5) with f_{ij} satisfying Assumption 2.2, where the digraph $\mathcal{G}$ contains a spanning tree. Define γ as in (2), let $\tau = \max_{i,j=1,\ldots,N} \tau_{ij}$ and consider initial conditions ψ in the set

$$\Omega = \left\{ \psi \in C([-\tau, 0], \mathbb{R}^N) \mid |\psi_i(\theta)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad \theta \in [-\tau, 0] \right\}.$$

Then the consensus set $\mathcal{X}_\tau$ is asymptotically attracting.

Proof: From Lemma 2.4, the set Ω is positively invariant, and hence solutions are bounded. Consider now the following as a candidate Lyapunov-Razumikhin function:

$$V(x(t)) = \max_i x_i(t) \tag{8}$$

Let I be the index for which the maximum at time t is achieved. If there are many such indices, pick the one which satisfies $\max_i \dot{x}_i$, and if there is still a choice, pick any one of them, but commit to that until another index achieves the maximum x_i.

In order to satisfy the first condition in Theorem A-2, we are interested in $V(\phi(0)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta))$, which translates to

$$\phi_I(0) \geq \phi_j(\theta), \quad \theta \in [-\tau, 0], \quad j = 1, \ldots, N. \tag{9}$$

Moreover, $\dot{V}(\phi) = \dot{\phi}_I(0)$. We want to ensure that while condition (9) holds, $\dot{V} \leq 0$. From Equation (9), we have that $\phi_j(\theta) - \phi_I(0) \leq 0$ for all $j = 1, \ldots, N$, $\theta \in [-\tau, 0]$. Since f_{jI} are locally passive, we have $f_{jI}(\phi_j(-\tau_{jI}) - \phi_I(0)) \leq 0$ for all j for which $(v_j, v_I) \in \mathcal{E}$ and $\phi \in \Omega$. Therefore $\dot{\phi}_I(0) \leq 0$. In conclusion, while (9) holds, we have $\dot{V} \leq 0$.

We now proceed to compute the sets E and L defined by Equations (22-23). First of all, since $\phi = 0$ is in L, this set is non-empty. Suppose $\phi \in E$, i.e., let $\phi \in \Omega$ be such that

$$\max_i \max_{-\tau \leq \theta \leq 0} x_i(\phi)(t + \theta) = \max_i \max_{-\tau \leq \theta \leq 0} \phi_i(\theta) \tag{10}$$

for all $t \geq 0$ and $\theta \in [-\tau, 0]$. For $\phi \in E$ satisfying (9), there exists a t^* for which we have $\dot{V}(x_{t^*}(\phi)) = 0$, as V attains a relative (local) maximum for such t^*. For such a t^*, we have:

$$\dot{V} = k_I \sum_{j=1}^{N} A_{jI} f_{jI} \left(x_j(t^* - \tau_{jI}) - x_I(t^*) \right) = 0$$

From (9) we have that $x_j(t^*+\theta) - x_I(t^*) \leq 0$ for all $\theta \in [-\tau, 0]$, and if the above condition is to hold, this means that $x_I(t^*) = x_j(t^* - \tau_{Ij})$ for all v_j that are parents of v_I. Continuing up a spanning tree we see that all nodes in the path from the root of the tree to v_I have to attain this value at some point in the past – call this vertex set $\mathcal{R}$, and define this value α. All other nodes can achieve a value less than or equal to α.

Similarly, the function $W = -\min x_i$ is non-increasing; let the value held by v_K for which $x_K = \min_i x_i$ and nodes from the root of the same spanning tree to node v_K at some points in the past be β. All other nodes can achieve a value more than or equal to β.

An invariant set in $\{\dot{V} = 0\} \cap \{\dot{W} = 0\}$ is the consensus set $\mathcal{X}_\tau$, for which $\alpha = \beta$, which is therefore asymptotically attracting for digraph topologies that contain a spanning tree. ■

We remark that the attractivity of the equilibrium set is retained *independent of the size of the delay*; we only require that f_{ji} be locally passive and that the digraph contains a spanning tree. To see why this is so, consider the simplest network, that of a bidirectional interaction of two agents with homogeneous delays and the simplest interconnection, $f_{ij} = \mathtt{id}$. If we write the dynamics of the state difference $x_{ij}(t) = x_i(t) - x_j(t)$ we get:

$$\dot{x}_{ij}(t) = -kx_{ij}(t) + kx_{ji}(t-\tau)$$

This system is *independent of delay stable*, i.e., it is stable for all k and finite τ [31].

Our result, however, allows for the functions f_{ji} to be different for each link, and moreover f_{ij} can be different from f_{ji}. However, these generalizations make it hard to say anything about convergence speed or the consensus value that is reached. Also, note that our results are independent of the network topology. We now consider the case of changing topologies.

III. Coordination of Multiagent Systems Under Switching Topologies

The issue of coordination under changing topologies is quite involved. It has been investigated in [3], [5], [15], [16], [22], for the case in which the system does not have any delays that make the state-space infinite-dimensional.

Switching arbitrarily among a set of possible topologies of size N is related to the problem of establishing stability for a switching system with an unknown switching rule between the subsystems. In this case, existence of a positive definite *common* Lyapunov function $\mathcal{V}$ such that the derivative of $\mathcal{V}$ along the solutions of any subsystems is negative definite guarantees asymptotic stability of the switched system under arbitrary switching. This is many times inconclusive, as this criterion is conservative – e.g., in [3] a different approach had to be taken to conclude coordination. Another issue is that an invariance principle needs to be invoked to conclude consensus, and the switching signal itself should satisfy certain conditions, as described in [32].

Consider a network of N agents interacting on a graph $\mathcal{G}^{(p)} = (\mathcal{V}, \mathcal{E}^{(p)})$, $p \in \mathcal{P} = \{1, \ldots, M\}$, each with an adjacency matrix $A^{(p)}$. The union graph is defined next.

Definition 3.1: Given a time interval $\Delta T_i = [T_i, T_{i+1}]$, denote by $\mathcal{P}(\Delta T_i) = \{p \in \mathcal{P} \mid \sigma(\theta) = p, \theta \in [T_i, T_{i+1}]\}$. Then the *union graph across* ΔT_i is defined by $\overline{\mathcal{G}}(\Delta T_i) = (\mathcal{V}, \bigcup_{p \in \mathcal{P}(\Delta T_i)} \mathcal{E}^{(p)})$.

When the topology is switching, vertex v_i has the following dynamics:

$$\dot{x}_i = k_i \sum_{j=1}^{N} A_{ij}^{(\sigma(t))} f_{ij}\left(x_j(t - \tau_{ij}) - x_i(t)\right), \tag{11}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is the piecewise constant switching signal with switching times $t_0 = 0 < t_1 < t_2 < \ldots$. Assume throughout that the functions f_{ji} are locally passive (Assumption 2.2). It is easy to construct an invariant region, even if the topologies change, in a similar way as it was done for the fixed topology case, as long as there is no chattering in the switching signal.

Then, one can show that the Lyapunov-Razumikhin functions $V(x(t)) = \max_i x_i(t)$ and $W(x(t)) = -\min_i x_i(t)$ in the proof of Theorem 2.5 can be used to construct a common Lyapunov-Krasovskii functional $\mathcal{V}$ for all the systems indexed by $p \in \mathcal{P}$. Consider the functional $\mathcal{V} : C([-\tau, 0], \mathbb{R}^N) \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{V}(x_t) &= \max_{\eta \in [-\tau, 0]} V(x(t+\eta)) - \max_{\mu \in [-\tau, 0]} W(x(t+\mu)) \\ &= \max_{\eta \in [-\tau, 0]} \max_i x_i(t+\eta) - \min_{\mu \in [-\tau, 0]} \min_j x_j(t+\mu).\end{aligned} \tag{12}$$

Clearly, $\mathcal{V}$ is continuous because all states x_i are continuous and we have continuous initial conditions. However, $\mathcal{V}$ is not differentiable at several points. These non-differentiable points result from one of the following three cases: (i) a switching in the indexes i, j of the max- and min-functions, (ii) a switching in the time η, μ in the max- and min-functions, or (iii) non-differentiable state trajectories x_i or x_j because of the switching topology. We denote all non-differentiable points of $\mathcal{V}$ because of (i) switching indexes i, j or (ii) switching times η, μ by $h_i, i = 1, 2, \ldots$ with $h_i < h_{i+1}$. Accordingly, all non-differentiable points of $\mathcal{V}$ because of (iii) switching topologies are denoted $\overline{h}_i, i = 1, 2, \ldots$ with $\overline{h}_i < \overline{h}_{i+1}$. Note that all these non-differentiable points $\overline{h}_i$ are related to one of the switching times t_k by $\overline{h}_i = t_k + \tau$ for some $k \in \mathbb{N}$. That is, a topology switch affects x_i directly but $\mathcal{V}$ with a delay of τ. Then, we have the following assumption on the switching signal σ:

Assumption 3.2: The switching signal $\sigma : [0, \infty) \to \mathcal{P}$ is piecewise constant, continuous from the right, and non-chattering. The switching times $t_0 = 0 < t_1 < t_2 < \ldots$ are positively divergent and $\inf_k(t_{k+1} - t_k) \geq h$ where $h > 0$. In addition, σ is such that the union graph $\overline{\mathcal{G}}([t, t+\mathcal{T}])$ contains a spanning tree for some $\mathcal{T} > 0$ and for all $t \geq 0$. Finally, the switching signal is such that, for all $i \in \mathbb{N}$, we have $h_k - \overline{h}_i \geq \tilde{h}$ for all $h_k > \overline{h}_i, k \in \mathbb{N}$, where $\tilde{h} > 0$. Also, there exists a $\mathcal{T} > 0$ such that the union graph $\overline{\mathcal{G}}([t, t+\mathcal{T}])$ contains a spanning tree for all $t \geq 0$.

The technical condition on the times h_k and $\overline{h}_i$ means that switches in graph topologies cannot all be followed immediately by a switching in the index. Under these assumptions, we have the following result:

Theorem 3.3: Consider a piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ satisfying Assumption 3.2 where $\mathcal{P} = \{1, \ldots, M\}$. Define γ as in (2), and suppose the initial conditions ϕ belong to the set

$$\Omega = \left\{ \phi \in C([-\tau, 0], \mathbb{R}^N) \mid |\phi_i(s)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad s \in [-\tau, 0]. \right\} \tag{13}$$

where $\tau = \max_{i,j=1,\ldots,N} \tau_{ij}$. Then, all solutions of the MAS (11) satisfy $-\frac{\gamma}{2} \leq x_i(t) \leq \frac{\gamma}{2}$ for all $t \geq -\tau$. Moreover, the consensus equilibrium set is asymptotically attracting.

Proof: For a fixed topology, the positive invariance of Ω has already been shown in the proof of Lemma 2.4. When the topologies switch, a similar argument can be used to show that indeed the region is invariant as long as chattering is avoided, as during switching, the next subsystem will have an initial condition that satisfies Equation (13).

Consider now the common Lyapunov-Krasovskii functional $\mathcal{V}$ as defined in (12) and its right-hand Dini derivative $\dot{\mathcal{V}}$. Following the proof of Theorem 2.5, we see that $\mathcal{V}$ is non-increasing for any $p \in \mathcal{P}$. Remember that $V(x(t))$ and $W(x(t))$ are both non-increasing if $\max_i x_i(t) \geq x_j(t+\theta), \theta \in [-\tau, 0]$, see (8) and (9). This argument holds for any $p \in \mathcal{P}$, i.e. also for switching topologies. Thus, $\dot{\mathcal{V}} \leq 0$. Remember that $\dot{\mathcal{V}}$ is piecewise continuous with discontinuities at h_i and $\overline{h}_i$.

We now proceed to show that even if topologies change, the consensus set $\mathcal{X}_\tau$ is asymptotically attracting. We first show that $\lim_{t\to\infty} \dot{\mathcal{V}} = 0$ using a Barbalat-like argument. So far, we know that $\mathcal{V}$ is non-increasing and bounded from below by 0. Therefore, $\lim_{t\to\infty} \int_0^t \dot{\mathcal{V}}(\eta) d\eta$ is finite. However, Barbalat's lemma is not directly applicable because $\dot{\mathcal{V}}$ is not continuous. Also, the Barbalat-like argument in [32] for hybrid systems does not apply directly because we cannot assume a dwell time between all discontinuity points h_i, i.e. when either the indexes i, j or the time η, μ change. Here we first investigate the influence of the different discontinuities on $\dot{\mathcal{V}}$ and then show that $\dot{\mathcal{V}} \to 0$.

Consider first any time interval Δh where $\dot{\mathcal{V}}$ is continuous, i.e. $h_i, \overline{h}_i \notin \Delta h$ for all i. Note first that $|\dot{x}_i(t)|$ is uniformly bounded because $|x_j(t-\tau_{ij}) - x_i(t)| \leq \mathcal{V}(0)$ for all i, j and all

$t \geq 0$ and because f_{ji} are continuous, see (11). Therefore, $|\ddot{x}_i(t)|$ is bounded for any interval Δh where $\dot{\mathcal{V}}$ is continuous because

$$\ddot{x}_i = k_i \sum_{j=1}^{N} A_{ij}^{(\sigma(t))} f'_{ij} \left(x_j(t-\tau_{ij}) - x_i(t)\right) \left(\dot{x}_j(t-\tau_{ij}) - \dot{x}_i(t)\right), \tag{14}$$

where $f'_{ij} = \frac{df_{ij}(y)}{dy}$ is also bounded because $x_j(t-\tau_{ij}) - x_i(t) \leq \mathcal{V}(0)$. Hence, $\dot{\mathcal{V}}$ is uniformly continuous in any interval Δh where $\dot{\mathcal{V}}$ is continuous. In particular, $\dot{\mathcal{V}}$ decreases at most linearly with time.

Consider now a time instant h_i where the index i, j or the times η, μ switch. For this instant, we know that $\mathcal{V}$ does not decrease faster right after this instant h_i than right before, i.e. $\lim_{\eta\to 0, \eta<0} \dot{\mathcal{V}}(h_i+\eta) \leq \lim_{\eta\to 0, \eta>0} \dot{\mathcal{V}}(h_i+\eta)$. This is due to the fact that the indexes change because the states $x_i(t+\eta)$ and $x_j(t+\mu)$ of the corresponding agents of the new indexes contract "slower" than the corresponding states of the agents with the old index. Summarizing, we know that $\dot{\mathcal{V}}$ is non-positive, piecewise continuous, decreases at most linearly in time, and increases at discontinuities where either the index i, j or the time η, μ changes.

It remains to investigate the instants $\overline{h}_i$ where $\dot{\mathcal{V}}$ is discontinuous because of the switching topology. At these discontinuities h_i, $\dot{\mathcal{V}}$ can increase or decrease. However, there are dwell times $h > 0, \tilde{h} > 0$ between those times $\overline{h}_i$ and the consecutive instants $\overline{h}_{i+1}$ and h_k, see Assumption 3.2.

Next, we show that $\lim_{t\to\infty} \dot{\mathcal{V}} = 0$. Assume this is not the case, i.e. there exists an infinite sequence of times $\chi_1, \chi_2, \ldots$ such that $\chi_i \to \infty$ as $i \to \infty$ and $\dot{\mathcal{V}}(\chi_i) \leq -\epsilon$ for all i and for any $\epsilon > 0$. Consider for the moment only discontinuities h_k because of switching indexes i, j and switching times η, μ. Since $\dot{\mathcal{V}}$ decreases at most linearly in time, there exists a $\delta_1 > 0$ such that $\dot{\mathcal{V}}(t) \leq -\frac{\epsilon}{2}$ for all $t \in [\chi_i - \delta_1, \chi_i]$. This holds true even if $h_k \in [\chi_i - \delta_1, \chi_i]$ for any k because $\dot{\mathcal{V}}$ increases at these discontinuities. This behavior is the main difference to the proof in [32] where $\dot{\mathcal{V}}$ has to be uniformly continuous in $[\chi_i - \delta_1, \chi_i]$. Now, assume χ_i is right after a discontinuities $\overline{h}_k$. In these cases, there is a dwell time $\min\{h, \tilde{h}\}$ until the next discontinuity and we have $\dot{\mathcal{V}}(t) \leq -\frac{\epsilon}{2}$ for all t in some interval of length $\delta_2 > 0$ that contains χ_i. Integrating, we obtain

$$\int_0^\infty \dot{\mathcal{V}}(\eta) d\eta \leq \sum_{i=1}^{\infty} \frac{1}{2} \delta \dot{\mathcal{V}}(\chi_i), \leq -\sum_{i=1}^{\infty} \frac{\epsilon}{2} \delta,$$

where $\delta = \min\{\delta_1, \delta_2\}$. This is obviously a contradiction to $\mathcal{V}(t) \geq 0, \forall t$. We conclude that $\lim_{t\to\infty} \dot{\mathcal{V}}(t) = 0$. This implies $\lim_{t\to\infty} \mathcal{V}_i(t) =$ const., i.e. the agents with maximal and minimal states eventually keep fixed states. With the same arguments that we used in Theorem 2.5 to show that $\alpha = \beta$, we conclude that the consensus set $\mathcal{X}_\tau$ is asymptotically attracting even if topologies change. ∎

IV. Synchronization in Oscillator Networks

We now consider the related problem of oscillator synchronization [2]. Let N oscillators with phases $\theta_i \in [0, 2\pi)$ and natural frequencies ω_i be coupled on a network whose topology is given by an undirected graph $\mathcal{G}$. The properties of the so-called Kuramoto model [2],

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i} \sum_{j=1}^{N} A_{ij} \sin\left(\theta_j - \theta_i\right) \tag{15}$$

where A is the adjacency matrix of the graph and K is the coupling strength, was the subject of an earlier paper [33], for $M_i = N$, $i = 1, \ldots, N$. Here M_i is a scaling factor, which could be the number of neighbours of vertex v_i, or N, the total population. Considering a network of identical oscillators, i.e., $\omega_i = \omega$, and switching to a rotating frame $\theta_i = \phi_i + \omega t$, it was shown in [33] that the phase-locked equilibrium set $\phi_i = c =$ constant is asymptotically attracting for arbitrary connected topologies. See also [34].

A time-delayed version for the above system can be analyzed using the tools developed in Section II. The dynamics of the i-th oscillator are:

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i} \sum_{j=1}^{N} A_{ij} \sin\left(\theta_j(t - \tau_{ij}) - \theta_i(t)\right) \tag{16}$$

Most available results on the above system concern the linearization about the equilibrium set in the rotating frame given by $\phi_i(t) = c$, $i = 1, \ldots, N$, i.e., the system

$$\dot{\phi}_i(t) = \frac{K}{M_i} \sum_{j=1}^{N} G_{ij}(\phi_j(t - \tau_{ij}) - \phi_i(t)) \tag{17}$$

where $G_{ij} = A_{ij} \cos(\Omega \tau_{ij})$. In [35] and [36] the case of regular connected graphs (i.e. $M_i = d$, the (identical) degree of the vertices in the graph) with $\tau_{ij} = \tau$ was investigated, and synchronization criteria were established that required $G > 0$. In [37] the general connected graph case was considered with $M_i = d_i$ again for the case in which $\tau_{ij} = \tau$ yielding the condition $G > 0$. In both these cases the parameter Ω solves the ‘self-consistency’ relation

$$\Omega = \omega - K \sin(\Omega \tau). \tag{18}$$

In [25] the synchronization of oscillator networks for inhomogeneous delays τ_{ij} was investigated for scalings $M_i = N$ and $M_i = d_i$ for the linearization of system (16). In this case, (16) achieves uniform rotations $\theta(t) = \Omega t + \phi(t)$ under the self-consistency conditions

$$\Omega = \omega_i - \frac{K}{M_i} \sum_{j} A_{ij} \sin(\Omega \tau_{ij}) \tag{19}$$

for all i. If $G_{ij} > 0$, the phase-locked equilibrium set is asymptotically attracting (even for non-identical oscillators) [25].

We can derive conditions for phase-locking to be achieved for (16) based on the results of Section II, assuming that given the oscillator frequencies ω_i and the coupling strength K, one chooses the delays τ_{ij} and Ω judiciously so that there exists a compatible solution to Equations (19). Note that if $\cos(\Omega\tau_{ij}) > 0$ for all $i, j = 1, \dots, N$, then the functions

$$f_{ij}(y) = \sin(-\Omega\tau_{ij} + y) - \sin(\Omega\tau_{ij}) \tag{20}$$

have a positive derivative at $x = 0$, i.e. they are *locally passive*. Then the following is true:

Corollary 4.1: Consider (16) and assume $\cos(\Omega\tau_{ij}) > 0$ for all $i, j = 1, \dots, n$, where the graph $\mathcal{G}$ is connected. Define γ by

$$\gamma = \min\left(\left|\frac{2k\pi + \pi - 2\Omega\tau_{ij}}{2}\right|\right), \quad \forall\ k \text{ and } i, j = 1, \dots, N,$$

and consider initial conditions ψ that satisfy $|\psi_i(\theta)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \dots, N$. Then the phase-locked equilibrium is asymptotically attracting.

V. Conclusion

We have seen that for multi-agent consensus (5) and oscillator synchronization (16) networks, functionality is retained irrespective of the delay size, as long as certain connectivity conditions are satisfied, even in the case of switching topologies. This delay-independent condition is not conservative, and it is interesting that robust functionality can be ensured for this problem instance for arbitrary sizes and heterogeneous delays: recall that for independent, incommensurate delays $\tau_i, i = 1, \dots, M$ the problem of deciding whether the system

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{M} A_k x(t - \tau_k)$$

is stable independent of delay, is NP hard [38]. Here $x \in \mathbb{R}^n$ and $A_i \in \mathbb{R}^{n \times n}$ are real matrices.

The Lyapunov functions used to analyze the delayed version of the systems under study are simple extensions of the Lyapunov functions used to analyze the undelayed systems. In particular the Lyapunov-Razumikhin function used for the delayed case (8) is the same as the Lyapunov function (4) used for the undelayed system. Note that we used Lyapunov-Razumikhin functions to conclude delay-independent stability rather than Lyapunov-Krasovskii functionals, as Lyapunov-Razumikhin functions are easier to work with for nonlinear systems; however, they tend to give more conservative delay-dependent conditions. For delay-dependent analysis it may be preferable to use Lyapunov-Krasovskii functionals.

APPENDIX

Here we review briefly invariance principles for systems with delays, see [13], [39] for more details. For Ω a subset of C, $f : \Omega \to \mathbb{R}^n$ a given function, and '$\dot{}$' representing the right-hand derivative, we call

$$\dot{x}(t) = f(x_t) \tag{21}$$

a Retarded Functional Differential Equation (RFDE) on Ω. Given $\phi \in C$ and $\rho > 0$, a function $x(\phi)$ is said to be a solution to Equation (21) on $[-\tau, \rho)$ with initial condition ϕ, if $x \in C([-\tau, \rho), \mathbb{R}^n)$, $x_t \in \Omega$, $x(t)$ satisfies (21) for $t \in [0, \rho)$ and $x(\phi)(0) = \phi$. Such a solution exists and is unique under certain conditions.

Let $D \subseteq \mathbb{R}^n$. A *Lyapunov-Razumikhin Function* $V = V(x)$ is a continuous function $V : D \to \mathbb{R}$ whose upper right-hand derivative with respect to (21) is defined by:

$$\dot{V}(\phi) = \lim\sup_{h \to 0^+} \frac{1}{h}(V(\phi(0) + hf(\phi)) - V(\phi(0))).$$

Definition A-1: Let $\phi \in \Omega$. An element ψ of Ω is in $\omega(\phi)$, the *ω-limit set of* ϕ, if $x(\phi)(t)$ is defined on $[-\tau, \infty)$ and there is a sequence of non-negative real numbers $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$. A set $M \subset \Omega$ is said to be *positively invariant* for (21) if for any ϕ in M there is a solution $x(\phi)(t)$ of (21) that is defined on $[-\tau, \infty)$ such that $x_t \in M$ for all $t \geq 0$ and $x_0 = \phi$.

If $x(\phi)(t)$ is a solution of (21) that is defined and bounded on $[-\tau, \infty)$ then the orbit through ϕ, i.e., the set $\{x_t(\phi) : t \geq 0\}$ is precompact, $\omega(\phi)$ is non-empty, compact, connected and invariant, and $x_t(\phi) \to \omega(\phi)$ as $t \to \infty$.

Let $V = V(x)$ be a Lyapunov-Razumikhin function. For a given set $\Omega \subset C$, define:

$$E = \{\phi \in \Omega : \max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) \text{ for all } t \geq 0\} \tag{22}$$

$$L = \text{largest set in } E \text{ that is invariant with respect to Equation (21).} \tag{23}$$

Again, L is the set of functions $\phi \in \Omega$ which can serve as initial conditions for (21) so that $x_t(\phi)$ satisfies

$$\max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) = \max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta))$$

for all $t \in (-\infty, \infty)$. For a Lyapunov-Razumikhin function V and for any $\phi \in E$, we have $\dot{V}(x_t(\phi)) = 0$ for any $t > 0$ such that $\max_{-\tau \leq \theta \leq 0} V(x_t(\phi)(\theta)) = V(x_t(\phi)(0))$. The following is known [39]:

Theorem A-2: Suppose there exists a Lyapunov-Razumikhin function $V = V(x)$ and a closed set Ω that is positively invariant with respect to (21) such that:

$$\dot{V}(\phi) \leq 0 \text{ for all } \phi \in \Omega \text{ such that } V(\phi(0)) = \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)).$$

Then for any $\phi \in \Omega$ such that $x(\phi)(\cdot)$ is defined and bounded on $[-\tau, \infty)$, $\omega(\phi) \subseteq L \subseteq E$. Hence, $x_t(\phi) \to L$ as $t \to \infty$.

It is important to note that since V is bounded from below along $x_t(\phi)$, i.e.,

$$\lim_{t\to\infty} \left\{ \max_{-\tau\leq\theta\leq 0} V(x_t(\phi)(\theta)) \right\} = c \text{ exists.}$$

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