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Automatica 37 (2001) 1971–1978

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Brief Paper

Comparison of nonlinear control design techniques on a model of the Caltech ducted fan☆

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Received 8 February 2000; revised 3 November 2000; received in final form 17 May 2001

Abstract

In this paper we compare different nonlinear control design methods by applying them to the planar model of a ducted fan engine. The methods used range from Jacobian linearization of the nonlinear plant and designing an LQR controller, to using model predictive control and linear parameter varying methods. The controller design can be divided into two steps. The first step requires the derivation of a control Lyapunov function (CLF), while the second involves using an existing CLF to generate a controller. The main premise of this paper is that by combining the best of these two phases, it is possible to find controllers that achieve superior performance when compared to those that apply each phase independently. All of the results are compared to the optimal solution which is approximated by solving a trajectory optimization problem with a sufficiently large time horizon. © 2001 Published by Elsevier Science Ltd.

Keywords: Nonlinear control design; Optimal control; LPV methods; Model predictive control; Control Lyapunov functions

1. Introduction

For more than three decades, nonlinear stabilization has been a major part of control engineering research. Despite the differences among nonlinear control design techniques, they can be divided into two major groups. The first group is comprised of those methods that generate a control Lyapunov function (CLF), while the second group takes an existing CLF and uses it to generate a stabilizing controller.

Motivated by the notion of combining the advantages of these two viewpoints into the design methodology, a workshop was organized at the American Control Conference in June 1997. Several nonlinear design techniques were applied to simple toy examples generated by the converse HJB method (Doyle, Primbs, Shapiro, & Nevisteć, 1996), and they were compared to the optimal solution which was known in advance. The purpose of this paper is to continue this comparison further, and add some other design methods to the existing ensemble. The methods are applied to a ducted fan, a research setup developed at Caltech as a testbed for nonlinear control design. In this paper, we report the simulation results performed using various nonlinear methods on a simplified model of the fan, and experimental results will be presented in our future papers.

In order to be able to focus only on the difficulties associated with *nonlinearity*, issues such as structured uncertainty and unmodeled dynamics are not considered at this point, and a quadratic performance index is selected. The studied methods include: Jacobian linearization, Frozen Riccati equations (FRE), linear parameter varying methods (LPV), control using global linearization, and finally model predictive (receding horizon) control (MPC), including hybrid approaches such as model predictive control combined with the CLF obtained using LPV.

This paper is organized as follows: In Section 2, we describe the state space model for the ducted fan engine. Section 3 deals with the methods that obtain a control Lyapunov function. In Section 4, we describe methods that utilize the CLF to generate a stabilizing controller. A comparison of different methods is performed in Section 5, and finally our conclusions are presented in Section 6.

☆A conference version of this paper was presented in IFAC world Congress, Beijing, China, 1999. This paper was recommended for publication in revised form by Associate editor Daizhan Cheng under the direction of Editor Hassan Khalil.

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0005-1098/01/$ - see front matter © 2001 Published by Elsevier Science Ltd.
PII: S0005-1098(01)00149-2

2. Caltech ducted fan model

The Caltech ducted fan is a small flight control experiment whose dynamics are representative of either a Harier in hover mode or a thrust vectored aircraft such as F18-HARV or X-31 in forward flight (Murray, 1998). This system has been used for a number of studies and papers. In particular, a comparison of several linear and nonlinear controllers was performed in Kantner, Bodenheimer, Bendotti, and Murray (1995), Bodenheimer, Bendotti, and Kantner (1996) and van Nieuwstadt and Murray (1996). In this section we describe the simple planar model of the fan shown in Fig. 1 which ignores the stand dynamics. This model is useful for initial controller design and serves as a good testbed for purposes of this paper.

Let (x, y, θ) denote the position and orientation of a point on the main axis of the fan that is distance l from the center of mass. We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r and a force f_2 parallel to the axis of the fan. Assuming m, J, and g to be the mass of the fan, the moment of inertia, and the gravitational constant respectively, the equations of motion can be written as follows:

$$\begin{aligned} m\ddot{x} &= -d\dot{x} + f_1 \cos\theta - f_2 \sin\theta, \\ m\ddot{y} &= -d\dot{y} + f_1 \sin\theta + f_2 \cos\theta - mg, \\ J\ddot{\theta} &= rf_1, \end{aligned} \tag{1}$$

where the drag terms are modeled as viscous friction with d being the viscous friction coefficient. It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let $u_1 = f_1$ and $u_2 = f_2 - mg$,

$$\begin{aligned} m\ddot{x} &= -mg\sin\theta - d\dot{x} + u_1 \cos\theta - u_2 \sin\theta, \\ m\ddot{y} &= mg(\cos\theta - 1) - d\dot{y} + u_1 \sin\theta + u_2 \cos\theta, \\ j\ddot{\theta} &= ru_1. \end{aligned} \tag{2}$$

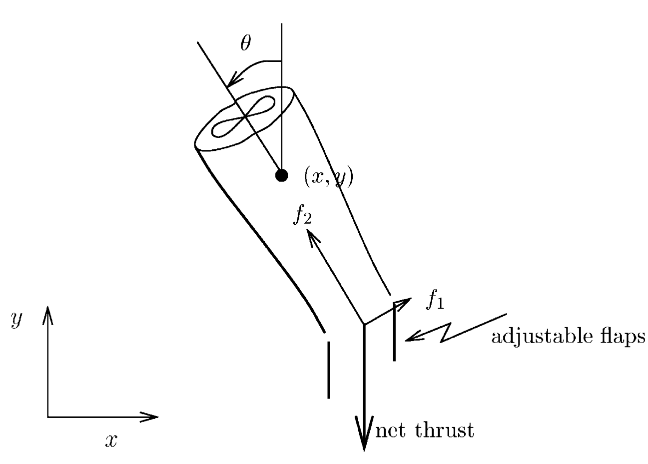

Fig. 1. Planar ducted fan model.

These equations are referred to as the *planar ducted fan equations*. The following quadratic cost function was used for comparison of different design techniques:

$$\mathscr{J} = \int_0^\infty (\bar{x}^{\mathrm{T}}(t) Q \bar{x}(t) + u^{\mathrm{T}}(t) R u(t))\, \mathrm{d}t,$$

where $\bar{x} = [x\ \dot{x}\ y\ \dot{y}\ \theta\ \dot{\theta}]^{\mathrm{T}}$, $R = I$, and Q is chosen to be a diagonal matrix with the following diagonal terms:

$$Q = diag[10\ \ 1\ \ 10\ \ 1\ \ 1\ \ 1]. \tag{3}$$

Associated with this optimal control problem is the corresponding *value function*, defined as

$$V^*(x) = \min_{u(t); x_0 = x} \mathscr{J} \tag{4}$$

which is also the solution to the Hamilton–Jacobi–Bellman (HJB) partial differential equation

$$\frac{\partial V^*}{\partial x} f - \frac{1}{4}\frac{\partial V^*}{\partial x} g R^{-1} g^{\mathrm{T}} \frac{\partial V^{*\mathrm{T}}}{\partial x} + x^{\mathrm{T}} Q x = 0, \quad V^*(0) = 0.$$

To make the problem more challenging, we chose the following initial condition:

$$[x\ \dot{x}\ y\ \dot{y}\ \theta\ \dot{\theta}] = [5\ \ 5\ \ 5\ \ 0\ \ -0.9\pi/2\ \ 0]. \tag{5}$$

3. Generation of CLFs

A concept that underlies many nonlinear design methodologies is that of a CLF. In simple terms, a CLF is the natural extension of the Lyapunov methodology to control systems. More formally, consider the following nonlinear system:

$$\dot{x} = f(x) + g(x)u, \tag{6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. A CLF is a $\mathbf{C}^1$, proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\inf_u\, [V_x(x) f(x) + V_x(x) g(x) u] < 0, \tag{7}$$

for all $x \neq 0$ (Sontag, 1989). This definition is motivated by the following consideration. Assume we are supplied with a positive-definite function V and asked whether we might be able to use this function as a Lyapunov function for a system we would like to stabilize. To determine if this is possible, we would calculate the time derivative of this function along trajectories of the system, i.e.

$$\dot{V}(x) = V_x(f(x) + g(x)u).$$

If it is possible to make the derivative negative at every point by an appropriate choice of u, then we have achieved our goal and can stabilize the system with V a Lyapunov function for the closed loop that we choose. This is exactly the condition given in (7). It can be shown that the existence of a CLF for system (6) is equivalent to the existence of a globally asymptotically stabilizing control law $u = k(x)$ which is smooth everywhere except possibly at $x = 0$ (Artstein, 1983).

As was mentioned in the Introduction, nonlinear control design can be thought of as having two stages. The first, and perhaps the most challenging stage, is to find a CLF. In what follows, we present some of the widely used methods in nonlinear control design, and interpret each approach in the context of the search for a control Lyapunov function. The discussed methods are Jacobian linearization, global linearization, FRE, and LPV methods. Other methods such as feedback linearization (Khalil, 1996), backstepping (Kristić, Kanellakopoulus, & Kokotović, 1995), and the taylor series expansion (Lukes, 1969; Tsiotras, Corless, & Rotea, 1998) can also be used, but will not be covered in this paper.

3.1. Jacobian linearization

Perhaps the simplest method of deriving a CLF is to use the Jacobian linearization of the system and generate a CLF by solving an LQR problem. It is a well known result that the problem of minimizing the quadratic performance index

$$\mathscr{J} = \int_0^\infty (x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}Ru(t))\,\mathrm{d}t$$

subject to

$$\dot{x} = Ax + Bu, \quad u = -Kx$$

results in finding the positive-definite solution of the following Riccati equation:

$$A^{\mathrm{T}}P + \mathrm{PA} - \mathrm{PB}R^{-1}B^{\mathrm{T}}P + Q = 0. \tag{8}$$

The optimal control action is given by

$$u = -R^{-1}B^{\mathrm{T}}Px.$$

In the case of the nonlinear system

$$\dot{x} = f(x) + g(x)u,$$

A and B are assumed to be

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}, \quad B = \left.\frac{\partial g(x)}{\partial x}\right|_{x=0}.$$

Obviously, the obtained CLF $V(x) = x^{\mathrm{T}}Px$ will be valid only in a region around the equilibrium. Therefore, we should not expect good performance from initial conditions far from the origin. This is indeed the case as simulation results show that this method cannot stabilize the planar ducted fan model for large initial conditions.

3.2. Global linearization

The idea of global linearization has its roots in early works from the Soviet Union (Lur'e & Postnikov, 1944) on the problem of absolute stability. The basic idea behind this approach is to model a nonlinear system as a polytopic linear differential inclusion (PLDI) (Boyd, Ghaoui, Ferm, & Balakrishnan, 1994). The dynamics of the nonlinear system are approximated as a convex hull of a set of linear models. The problem of quadratic stability of the obtained PLDI, i.e., stability provable by a quadratic Lyapunov function, can be recast as an LMI feasibility problem which can be solved very efficiently using interior point convex optimization methods. The PLDI describing the planar ducted fan model can be written as

$$\dot{x} = \sum_{i=1}^{2} \alpha_i(t)(A_i x + B_i u), \quad u = -Kx. \tag{9}$$

Using the same cost function $\mathscr{J}$ as before, the problem of minimizing an upper bound on the cost $\mathscr{J}$ can be written as the following convex optimization problem:

minimize Z

subject to $Y > 0$,

$$\begin{bmatrix} YA_i^{\mathrm{T}} + A_iY - B_iX - X^{\mathrm{T}}B_i^{\mathrm{T}} & YQ^{1/2} & X^{\mathrm{T}}R^{1/2} \\ Q^{1/2}Y & -I & 0 \\ R^{1/2}X & 0 & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Z & x_0^{\mathrm{T}} \\ x_0 & Y \end{bmatrix} > 0, \quad i = 1, 2.$$

where $Y = P^{-1}$, and $X = KY$ are the change of variables made to recast the matrix inequalities as LMIs (Boyd et al., 1994). Q and R are chosen as before, and A_1, B_1 are obtained by linearization of the ducted fan model at the origin and A_2 and B_2 are chosen such that the dynamics lie in the convex hull described by (9). Despite the fact that this method is very powerful, it turns out to be conservative, since there are many trajectories that are a trajectory of the PLDI, but are not a trajectory of the nonlinear system. Using the LMI formulation of the LQR problem for PLDIs (Boyd et al., 1994), we can find a CLF (given by $V(x) = x^{\mathrm{T}}Px$) for the ducted fan model for positive values of θ. However, a global constant quadratic CLF does not exist. Simulation results for this method show that the closed-loop system is stable, but may suffer from poor performance.

3.3. Frozen Riccati equation (FRE) method

This method was first introduced by Cloutier, D'Souza, and Mracek (1996). The basic idea behind this method, sometimes called *state-dependent Riccati equations*, is to solve the Riccati equations online, at each time step. Although results are often promising, there are no rigorous justifications for even maintaining mere stability. Nevertheless, the simplicity of the implementation makes the FRE approach a plausible alternative in some applications. To apply this method, the planar ducted fan model is written as

$$\dot{x} = A(x)x + g(x)u. \tag{10}$$

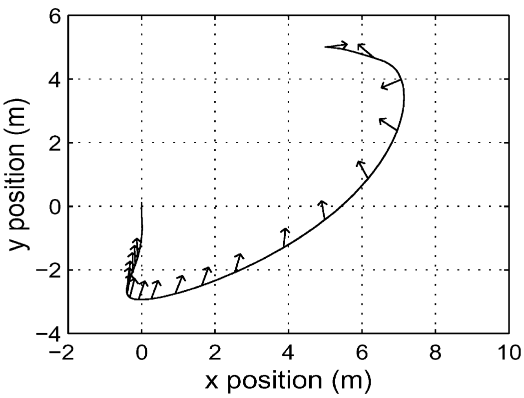

Fig. 2. The FRE controller.

At each *frozen state* the Riccati equation is solved, and then the resulting state fedback controller is feedback to the system, that is, the state feedback nonlinear control law is obtained by solving the following:

$$0 = A(x)^{\mathrm{T}}P(x) + P(x)A(x) - P(x)g(x)g^{\mathrm{T}}(x)P(x) + Q,$$

$$u = -g^{\mathrm{T}}(x)P(x)x. \tag{11}$$

The quantity $V(x) = x^{\mathrm{T}}P(x)x$ generated by this technique is in general only a local CLF. Furthermore, one of the major drawbacks of this method is the lack of a systematic procedure for selecting, among the infinite possibilities, a single parameterization for $f(x)$ (in the form of Eq. (10)) which achieves stability and acceptable performance. However, in the case of the ducted fan model, the *obvious parameterization* of $f(x)$ appears to work in simulation studies. The dynamics of the fan are written as follows:

3.4. Linear parameter varying (LPV) methods

In this technique, the following so-called quasi-LPV representation of a nonlinear input-affine system is used to design a state feedback controller

$$\dot{x} = A(\rho(x))x + B(\rho(x))u. \tag{13}$$

Assume the underlying parameter ρ varies in the allowable set

$$\mathscr{F}_{\mathscr{P}}^{v} := \{\rho \in \mathscr{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho \in \mathscr{P}, \underline{v}_i(\rho) \leqslant \dot{\rho}_i \leqslant \bar{v}_i(\rho),\ i = 1, \ldots, m\}, \tag{14}$$

where $\mathscr{P} \subset \mathbf{R}^m$ is a compact set. If there exists a positive definite $X(\rho)$ such that the following inequality is satisfied:

$$\begin{bmatrix} -\sum_{i=1}^{m} \underline{\bar{v}}_i(\rho)\dfrac{\partial X}{\partial \rho_i} + A(\rho)X(\rho) + X(\rho)A^{\mathrm{T}}(\rho) - B(\rho)B^{\mathrm{T}}(\rho) & X(\rho)C^{\mathrm{T}}(\rho) \\ C(\rho)X(\rho) & -I \end{bmatrix} < 0 \tag{15}$$

for all $\rho \in \mathscr{P}$ where $C(\rho) = Q^{1/2}(\rho(x))$, then the closed-loop system is stable with the state feedback

$$u(x) = -B^{\mathrm{T}}(\rho(x))X^{-1}(\rho(x))x.$$

Moreover, an upper bound on the optimal value function $V^*(x)$ (which also serves as a CLF) is given by

$$V(x) = x^{\mathrm{T}}X^{-1}(\rho(x))x \geqslant V^*(x).$$

The notation $\sum_{i=1}^{m} \underline{\bar{v}}_i(\rho)$ in (15) means that every combination of $\bar{v}_i(\rho)$ and $\underline{v}_i(\rho)$ should be included in the inequality. For instance, when $m = 2$, $\bar{v}_1(\rho) + \bar{v}_2(\rho)$, $\bar{v}_1(\rho) + \underline{v}_2(\rho)$, $\underline{v}_1(\rho) + \bar{v}_2(\rho)$ and $\underline{v}_1(\rho) + \underline{v}_2(\rho)$ should be checked individually. In other words, (15) actually represents 2^m inequalities. Additionally, solving (15) involves gridding the parameter space $\mathscr{P}$ and choosing a finite set of basis for $X(\rho)$ (see Wu, Hua Xin, Yang, Packard, & Becker (1996) for details).

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{d}{m} & 0 & 0 & -\dfrac{g\sin\theta}{\theta} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{d}{m} & \dfrac{g(\cos\theta - 1)}{\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ \dfrac{\cos\theta}{m} & -\dfrac{\sin\theta}{m} \\ 0 & 0 \\ \dfrac{\sin\theta}{m} & \dfrac{\cos\theta}{m} \\ 0 & 0 \\ \dfrac{r}{j} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{12}$$

Results of this approach are shown in Fig. 2.

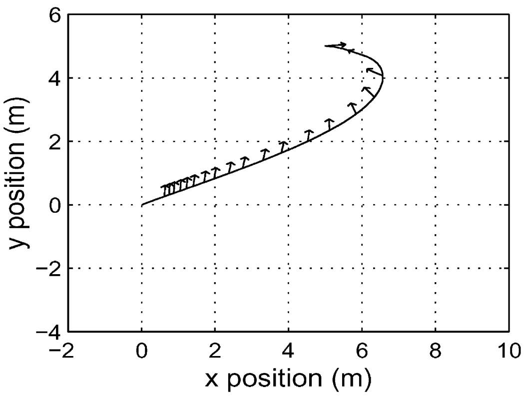

Fig. 3. The LPV controller.

For the ducted fan, $\rho = \theta$ is chosen as the varying parameter, and the operation range $\mathscr{P} = [-\pi/2, \pi/2]$. The bound on the rate variation on θ is set to 10, i.e., $|\dot{\theta}| \leqslant 10$. Both $A(\rho)$ and $B(\rho)$ are the same as in the model used for the FRE method. A set of basis functions is chosen to compute $X(\rho)$, i.e., $X(\rho) = \sum_{i=1}^{5} f_i(\rho) X_i$, where X_i's are symmetric coefficient matrices and $\{f_i(\rho)\}$ are fifth order Legendre polynomials on $\mathscr{P}$:

$$\{f_i(\rho)\} = \left\{1, \frac{2}{\pi}\theta, \left(3\left(\frac{2}{\pi}\theta\right)^2 - 1\right)\Big/2, \left(5\left(\frac{2}{\pi}\theta\right)^3 - 3\left(\frac{2}{\pi}\theta\right)\right)\Big/2, \left(35\left(\frac{2}{\pi}\theta\right)^4 - 30\left(\frac{2}{\pi}\theta\right)^2 + 3\right)\Big/2\right\}.$$

Simulation of the closed-loop system is shown in Fig. 3.

4. CLF-based control schemes

So far, we have discussed several methods for generating a CLF. Each of the above mentioned methods have their own technique for generating a controller. However, once a CLF is obtained, there are number of alternative methods that can be used to implement a controller purely from the knowledge of the CLF. In what follows, some of the important ones are described.

4.1. Sontag's formula

Perhaps the most important formula for producing a stabilizing controller based on the existence of a CLF was introduced in by Sontag (1989) and has come to be known as Sontag's formula. Below, we present a slight variation of Sontag's original formula.

$$u_s = \begin{cases} -\left[\dfrac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}}{V_x g g^T V_x^T}\right] g^T V_x^T, & V_x g \neq 0, \\ 0, & V_x g = 0, \end{cases} \tag{16}$$

where $V_x = \partial V / \partial x$. While this formula has obtained recognition for possessing many desirable properties (particularly related to the continuity of the control law Sontag, 1989), it also has an interpretation in terms of optimal control. Sontag's formula, in essence, uses the directional information given by the CLF, V, and scales it properly to solve the Hamilton–Jacobi–Bellman (HJB) equation. That is, Sontag's formula can be "derived" by assuming the control action to be of the form

$$u = -\frac{\lambda(x)}{2} g^T V_x^T$$

and determining λ by solving the HJB equation pointwise with $\lambda(x) V_x$ substituting for the gradient of the value function. In particular, if V has level curves that agree with those of the value function, then Sontag's formula produces the optimal controller. On the other hand, when a CLF does not closely resemble the value function, poor performance may result (Freeman & Primbs, 1996). Results from Sontag's formula with the CLF from LPV can be found in the comparisons section.

4.2. MPC + CLF:

This section deals with combining the CLF obtained using methods described in the previous section with a model predictive control (MPC) strategy. First, however we review some results on MPC.

4.2.1. Model predictive control (MPC)

In model predictive control (cf. García, Prett, & Morari, 1989; Mayne & Michalska, 1990), the current control at state x and time t is obtained by determining on-line the optimal control $\hat{u}$ over the interval $[t, t+T]$ respecting the following objective:

$$\inf_{u(\cdot)} \int_t^{t+T_h} (q(x(\tau)) + u^T(\tau) u(\tau))\, d\tau + \Phi(x(t+T))$$

s.t. $\dot{x} = f(x) + g(x)u$.

The resulting control trajectory is implemented until a new state update occurs, usually at pre-specified sampling intervals of time T_s. Repeating these calculations yields a feedback control law. This type of implementation is commonly referred to as receding or moving horizon. As is evident from this sort of control scheme, obtaining a reduced value of the performance index is of utmost importance.

Model predictive control has traditionally been plagued by stability questions. For nonlinear systems, these have been addressed in various ways, including the use of end constraints (which require that the state be identically zero at the end of the horizon T), and by various other methods (Mayne & Michalska, 1990; Michalska & Mayne, 1993). Unfortunately, these do not always result in satisfactory solutions when considered in the framework of practical implementation. Often such additional constraints introduce a number of problems related to the feasibility of the on-line optimization. Hence, while receding horizon control exploits the ability to compute on-line, this also introduces many difficulties ranging from stability to feasibility of the on-line optimization. Nevertheless, as computer power increases, receding horizon control and on-line computation become increasingly attractive.

4.2.2. *MPC extensions of CLF formulas*

Freeman and Kokotović (1995), proposed to use a CLF in conjunction with the following optimization to determine a control law.

Pointwise Min-Norm:

$$\text{minimize} \quad u^{\mathrm{T}}u \tag{17}$$

$$\text{subject to} \quad V_x(f+gu) \leqslant -\sigma(x), \tag{18}$$

where σ is some positive definite function and the optimization is solved pointwise (i.e. for each x). This formula pointwise minimizes the control energy used while requiring that V be a Lyapunov function for the closed loop system and decrease by at least σ at every point. Sontag's formula (16) results by choosing

$$\sigma_{\mathrm{s}} = \sqrt{(V_x f)^2 + q(x)(V_x g g^{\mathrm{T}} V_x^{\mathrm{T}})}.$$

Primbs, Nevistić, and Doyle (2000) proposed the following MPC extension of pointwise min-norm controllers:

MPC + CLF:

$$\text{minimize} \quad \int_t^{t+T_{\mathrm{h}}} (q(x) + u^{\mathrm{T}}u)\,\mathrm{d}t \tag{19}$$

$$\text{subject to} \quad V_x(f+gu(t)) \leqslant -\varepsilon\sigma(x), \tag{20}$$

$$V(x(t+T_{\mathrm{h}})) \leqslant V(x_\sigma(t+T_{\mathrm{h}})), \tag{21}$$

where $1 \geqslant \varepsilon > 0$ and x_σ represents the state trajectory from the pointwise min-norm controller with parameter $\sigma(x)$. Constraint (20) requires that V is a Lyapunov function for the closed-loop system and is applied over the sampling interval $[t, t+T_{\mathrm{s}}]$ in which the optimizing control trajectory is implemented. The second constraint (21) obviates the need for a terminal weight, using the information from the CLF instead.

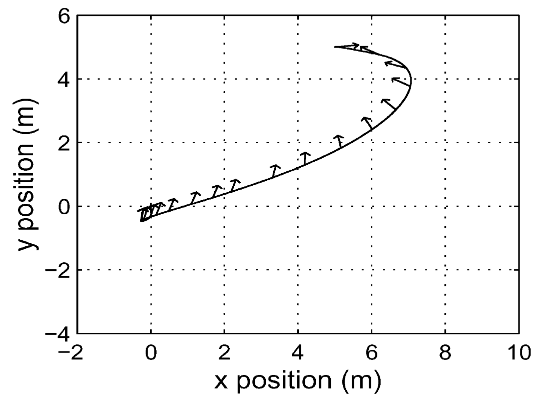

Fig. 4. Optimal.

This formulation of MPC avoids the stability problems of standard MPC by incorporating the stability properties of the CLF. This allows the horizon to be varied on-line and the optimization to be preempted before the optimizing solution is found, all without loss of stability. When no horizon is present, it reduces to the pointwise min-norm problem, which also serves as a feasible trajectory for the MPC problem when a horizon is present. In order to test this hybrid strategy on the planar model of the ducted fan, we used the CLF obtained by the Quasi-LPV method described in Section 3.4. Simulations were performed for two different time horizons, $T_h = 0.1$ and 0.3 are resulted in trajectories quite similar to the optimal trajectory (Fig. 4). Note that as we increase the time horizon, the response improves as expected.

5. Comparisons

As was mentioned earlier, by choosing a large time horizon we found the optimal cost for the quadratic cost $\mathscr{J}$ from the chosen initial condition by solving a single trajectory optimization. Values of the cost function for all of the methods described in this paper are given in Table 1. A review of Table 1 leads to some interesting observations. At the top is MPC without the use of a terminal weight or a CLF. Despite the fact that it exploits on-line computation to directly address the performance aspect of the optimal control problem, it lacks a stability guarantee, and in this case is actually unstable. Not so surprising a result is that JL + LQR is found to be unstable. This illustrates the true nonlinear nature of the problem and indicates that nonlinear techniques are needed. Only slightly more sophisticated than JL + LQR is the FRE technique. Even though it also lacks global stability guarantees, it is stabilizing from the chosen

Table 1
Values of the cost function $\mathscr{J}$ using different methods for the ducted fan example

Method	Cost
MPC w/o CLF ($T_h = 0.3$, $T_s = 0.05$)	Unstable
Jacobian linearization + LQR	Unstable
FRE	2801
Global linearization + LQR using LMIs	2423
Quasi-LPV	1795
CLF from LPV + Sontag	1761
CLF from LPV + MPC ($T_h = 0.1$ s, $T_s = 0.05$)	1572
CLF from LPV + MPC ($T_h = 0.3$ s, $T_s = 0.05$)	1451
'Optimal'	1362

initial condition, although with a rather poor cost. Simulation results are supplied in Fig. 2.

Next, we find that the Global Linearization techniques provide a guarantee of stability and reduce the cost even further. When even more off-line computation is thrown at the problem as in LPV techniques, the cost reduces even further. The result of the LPV simulation is given in Fig. 3. Note that in LPV and FRE techniques, a design choice is involved in the selection of a state dependent representation. Although no systematic procedure was used (nor does one exist), the results obtained here were the best of the state dependent representations that were tested.

The results from LPV on down were all qualitatively similar to the optimal trajectory. Applying Sontag's formula with the aid of the CLF from LPV results in a cost and trajectory very similar to that obtained from the standard LPV implementation. When this approach was extended to an on-line MPC implementation, a significant drop in the cost was observed. As the horizon was increased from $T_h = 0.1$ to 0.3, the cost steadily decreased, providing the lowest cost observed in any of the simulations. Due to the similarity in results, only the optimal trajectory is supplied in Fig. 4 for reference.

The simulation results indicate that the hybrid techniques outperform the "CLF-only"" techniques. While this is just a numerical observation, theoretical justification can also be given; see Primbs (1998) and Jadbabaie (2000) for more details.

To summarize, in general the following trends were observed. While not uniformly true, the more detailed and sophisticated techniques, which generally involve extensive off-line analysis, tended to outperform the simpler, less theoretically sound techniques. Extensive computation was also found to be extremely beneficial, especially when employed in an on-line manner, but only when used under the guidance of a solid theoretical framework. This accounts for the difference between the near optimal MPC + CLF controllers and the unstable standard MPC approach. A question of increasing importance will be whether on-line techniques such as MPC + CLF can realistically be implemented in a real time environment such as the Caltech ducted fan experiment. This will be a topic of ongoing interest as this research continues.

6. Conclusions

The purpose of this paper was to perform a comparison between several existing nonlinear design methods on the planar model of a ducted fan. The design schemes were divided into two groups. The first dealt with different methods of generating control Lyapunov functions (CLFs). Despite the fact that the CLF generating methods have their standard method of generating a controller, it was shown via simulation that if the CLF methodology is combined with an MPC framework, the hybrid method works better than any of the individual ones. Simulation results were performed for a specific initial condition which incorporated the true nonlinear nature of the problem. Further research in this direction is under progress, and our goal is to do a similar comparison with more sophisticated models and eventually with the Caltech ducted fan experiment.

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