Synchonization in Oscillator Networks with Heterogeneous Delays, Switching Topologies and Nonlinear Dynamics

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***Abstract*— This paper investigates the attractivity properties of the locked-in-phase equilibria set in oscillator networks, in the presence of multiple, non-commensurate communication delays. The dynamics that the oscillators are endowed with are in the form of nonlinear delay differential equations, with Kuramoto-type interactions. Using an appropriate LaSalle invariance principle we assess the attractivity properties of this set for arbitrary topology interconnections. We then show that this set is also asymptotically attracting even if the network topology is allowed to change.**

I. INTRODUCTION

In the past few years, there has been a lot of interest in the properties of systems that are formed by *arbitrary* interconnections of smaller systems, known as large-scale or networked systems. The fact that the size of these systems is not known *a priori* emphasizes the importance of *scalability* of the functional properties of the system, whether this is a design objective or a fact to be established. These properties could be robust stability, performance etc., but also some times permanence and invariance.

There are several systems in nature that have such properties. A particular example is *synchronization*, a phenomenon which can be used to explain how pacemaker cells generate and pace the heartbeat, how fireflies can flash in sync and crickets chirp in unison. Also, the phenomenon is observed in semiconductor laser arrays [1], [2].

Beyond these physical examples, are technological systems that have been designed to have such properties. One example is coordinated motion of multi-vehicle systems for arbitrary interconnection topologies, which can also be allowed to change, as long as connectivity is retained over time [3]–[11]. The related issues of self-ordered particle motion [12], [13] and distributed computation were also investigated in [14], [15].

A well known model for synchronization is the Kuramoto model, a system of structured ordinary differential equations [16]–[18]. In this paper we include time delays in the oscillator couplings, a feature that has been neglected in some of the earlier work on the subject. These can be used to model the effect of propagation of phase information in spatially large networks. Even though many times they are small, in general delays can deteriorate the system's performance or even destabilize it.

Previous work in this area has revealed that the locked-in-phase equilibrium set is asymptotically attracting for specific types of topologies, for homogeneous delays (i.e., identical for all interactions) and identical oscillators with linear dynamics, under some specific assumptions [19], [20]. Recently, it was shown that this set is asymptotically attracting for arbitrary topologies with heterogeneous delays and non-identical oscillators with linear dynamics, under similar assumptions as before [21]. In this paper we will show that the locked-in-phase equilibrium set is asymptotically attracting for arbitrarily connected topologies for the *nonlinear* Kuramoto model, even in the presence of heterogeneous delays with non-identical oscillators under some assumptions. The methods we use are different from [21], and provide flexibility for the analysis of other large-scale nonlinear delay differential equation models.

Another aspect of the problem, which is also a subject of this paper, is whether attractivity can be ensured for *switching* topologies even if delays are present in the system. We use ideas from [22], [23] and [24] to formulate our result, which, because of the structure of the Kuramoto dynamics, is restricted to oscillators with homogeneous delays and regular, connected graphs.

The paper is organized as follows. In Section II we present the tools that will be used in the sequel; in particular, we state an invariance principle for Lyapunov-Razumikhin functions for delay-differential equations. In Section III we analyze the attractivity properties of locked-in-phase equilibria sets in oscillator networks with nonlinear, heterogeneously delayed Kuramoto dynamics and fixed underlying topologies. In Section IV we show that the same conclusion can be reached if the topology is allowed to change, as long as the switching signal is non-chattering with a dwell time and we are switching between connected graphs. Section V concludes the paper.

II. PRELIMINARIES

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the standard norm $|\cdot|$. Given $\tau > 0$ let $C([-\tau,0],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-\tau,0] \subset \mathbb{R}$ into $\mathbb{R}^n$ with the topology of uniform convergence. The norm on C is defined as $\|\varphi\| = \sup_{-\tau\leq s\leq 0}|\varphi(s)|$. Let $\rho \geq 0$ and $x \in C([-\tau,\rho],\mathbb{R}^n)$; then for any $t \in [0,\rho]$, define $x_t \in C$ by $x_t(s) = x(t+s), s \in [-\tau, 0]$.

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Let Ω be a subset of C, $f : \Omega \to \mathbb{R}^n$ a given function, and '$\dot{}$' represent the right-hand derivative. Then we call

$$\dot{x}(t) = f(x_t) \tag{1}$$

an autonomous Retarded Functional Differential Equation (RFDE) on Ω. Given $\varphi \in C$ and $\rho > 0$, a function $x(\varphi)$ is said to be a solution to Equation (1) on $[-\tau, \rho)$ with initial condition φ, if $x \in C([-\tau, \rho), \mathbb{R}^n)$, $x_t \in \Omega$, $x(t)$ satisfies (1) for $t \in [0, \rho)$ and $x(\varphi)(0) = \varphi$. Such a solution exists and is unique if f is continuous and $f(\varphi)$ is Lipschitzian in each compact set in Ω; see [25] for more details.

In this paper we will be concerned with invariance. For this, we need to define ω-limit sets of solutions, and provide LaSalle-type theorems for functional differential equations. In [25] the definitions to follow are based on what constitutes a *process*.

Definition 1: Let $\varphi \in \Omega$. An element ψ of Ω is in $\omega(\varphi)$, the *ω-limit set of* φ, if $x(\varphi)(t)$ is defined on $[-\tau, \infty)$ and there is a sequence of non-negative real numbers $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$. A set $M \subset \Omega$ is said to be *positively invariant* for (1) if for any φ in M there is a solution $x(\varphi)(t)$ of (1) that is defined on $[-\tau, \infty)$ such that $x_t \in M$ for all $t \geq 0$ and $x_0 = \varphi$.

If $x(\varphi)(t)$ is a solution of (1) that is defined and bounded on $[-\tau, \infty)$ then the orbit through φ, i.e., the set $\{x_t(\varphi) : t \geq 0\}$ is precompact, $\omega(\varphi)$ is non-empty, compact, connected and invariant, and $x_t(\varphi) \to \omega(\varphi)$ as $t \to \infty$.

There are two types of Lyapunov certificates for stability of equilibria time-delay systems, namely Lyapunov-Krasovskii and Lyapunov-Razumikhin, with Lyapunov-Krasovskii being the natural extension of Lyapunov's theorem for ODEs. A Lyapunov-Krasovskii invariance theorem can be found in [25]. Here we are interested in a Lyapunov-Razumikhin-type theorem, which is more difficult to state because these functions are not non-decreasing along the trajectories of the system [25]. Such an invariance principle has been developed in [26].

Let $D \subseteq \mathbb{R}^n$. By a *Lyapunov-Razumikhin Function* $V = V(x)$ we mean a continuous function $V : D \to \mathbb{R}$; the upper right-hand derivative of V with respect to (1) is defined by:

$$\dot{V}(\phi) = \lim \sup_{h \to 0^+} \frac{1}{h}(V(\phi(0) + hf(\phi)) - V(\phi(0)))$$

For a given set $\Omega \subset C$, define:

$$E = \{\varphi \in \Omega : \max_{s \in [-\tau, 0]} V(x_t(\varphi)(s)) = \max_{s \in [-\tau, 0]} V(\varphi(s)),\ \forall t \geq 0\} \tag{2}$$

$$L = \text{Largest set in } E \text{ that is invariant with respect to Equation (1).} \tag{3}$$

Here L is the set of functions $\varphi \in \Omega$ which can serve as initial conditions for (1) so that $x_t(\varphi)$ satisfies

$$\max_{-\tau \leq s \leq 0} V(\varphi(s)) = \max_{-\tau \leq s \leq 0} V(x_t(\varphi)(s))$$

for all $t \in (-\infty, \infty)$. Note that the above condition for $\overline{V}$ defined as

$$\overline{V}(\varphi) = \max_{-\tau \leq s \leq 0} V(\varphi(s)) \tag{4}$$

is indeed a condition that $\dot{\overline{V}}(\varphi) = 0$. In particular, for a Lyapunov-Razumikhin function V and for any $\varphi \in E$, we have $\dot{V}(x_t(\varphi)) = 0$ for any $t > 0$ such that $\max_{-\tau \leq s \leq 0} V(x_t(\varphi)(s)) = V(x_t(\varphi)(0))$.

We then have the following theorem [26]:

Theorem 2: Suppose there exists a Lyapunov-Razumikhin function $V = V(x)$ and a closed set Ω that is positively invariant with respect to (1) such that:

$$\dot{V}(\varphi) \leq 0 \text{ for all } \varphi \in \Omega \text{ s.t. } V(\varphi(0)) = \max_{-\tau \leq s \leq 0} V(\varphi(s)).$$

Then for any $\varphi \in \Omega$ such that $x(\varphi)(\cdot)$ is defined and bounded on $[-\tau, \infty)$, $\omega(\varphi) \subseteq L \subseteq E$. Hence,

$$x_t(\varphi) \to L \text{ as } t \to \infty.$$

In the next two sections, which constitute the core of this paper, we will apply the ideas developed in this section to oscillator networks with nonlinear, heterogeneously delayed Kuramoto Dynamics. Before that, some notation on Algebraic Graph Theory.

A. Algebraic Graph Theory

Throughout the paper we will be using the following notation to capture the topology of the network interactions. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E}$. We denote each vertex by $v_i \in \mathcal{V}$ for $i = 1, \ldots, N$, and each edge by $e = (v_i, v_j) \in \mathcal{E}$. All graphs in this paper are undirected. If $v_i, v_j \in \mathcal{V}$ and $(v_i, v_j) \in \mathcal{E}$, then v_i and v_j are neighbors. The valence of each vertex v_i, i.e., the number of its neighbors, is denoted by d_i. The valency matrix $\Delta = \Delta(\mathbb{G})$ is an $N \times N$ diagonal matrix in which the (i, i) element is the valence of vertex i. If a graph is regular, then $d_i = d$ and $\Delta(\mathcal{G}) = dI$ where $I \in \mathbb{R}^{N \times N}$ denotes the identity matrix. A path of length r from vertex v_i to v_j is a sequence of $r + 1$ distinct vertices starting from v_i and ending at v_j so that consecutive vertices are neighbors. A graph $\mathcal{G}$ is said to be connected if there is a path between any two vertices in it. The adjacency matrix $A = A(\mathcal{G}) = [a_{ij}]$ of an (undirected) graph is an $N \times N$ symmetric matrix such that $a_{ij} = 1$ if v_i and v_j are neighbors, and $a_{ij} = 0$ otherwise. More on graph theory can be found in [27].

III. KURAMOTO OSCILLATOR NETWORKS: FIXED TOPOLOGY

As mentioned in the Introduction, this paper is concerned with synchronization phenomena in oscillator networks [28] with heterogeneously delayed Kuramoto dynamics.

Consider a set of N coupled oscillators with phases $\theta_i \in [0, 2\pi]$ and natural frequencies ω_i. The phase of each oscillator θ_i is associated to a vertex $v_i \in \mathcal{V}$ of an underlying undirected graph $\mathcal{G}$ with no loops and adjacency matrix A.

The original Kuramoto model, proposed by Kuramoto [17] in 1975 was for a complete graph (the 'all-to-all' case), and took the form:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

where K is the coupling strength between the oscillators, assumed to be the same for all interactions. When the natural frequencies ω_i are the same, i.e., when $\omega = \omega_i$, one can employ a transformation to the rotating frame $\theta_i(t) = \omega t + \phi_i(t)$ and investigate the behavior of the $\phi_i(t)$ dynamics. In this case, we have:

$$\dot{\phi}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i)$$

Equilibrium points for this system form *sets*; of particular importance is the set which satisfies

$$\phi_i^* = c, \quad \forall i = 1, \ldots, N.$$

Other equilibria sets may appear depending on the topology [29]. When this equilibrium set is asymptotically attracting, we say that the network achieves phase locking. When the ω_i are not the same, then depending on K and the distribution of the ω_i, the ϕ_i (defined now by the transformation $\theta_i(t) = \overline{\omega}t + \phi_i(t)$ where $\overline{\omega}$ is the mean natural frequency) may tend to constant values, which however will not be the same, i.e., this is not a phase locked equilibrium set. See [30] and [31] and the references therein.

In order to allow for interactions that are not all-to-all, we introduce an underlying graph structure on the topology using the graph's adjacency matrix A:

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i) \tag{5}$$

This system was the subject of an earlier paper [30], for $M_i = N$, $i = 1, \ldots, N$. Here M_i is a scaling factor, which could be the number of neighbors of node i, i.e., the degree of node i, or N, the total population. It was found there, that a network of identical oscillators ($\omega_i = \omega$) converges to the phase-locked equilibrium set for arbitrary connected topologies.

In this paper we consider a time-delayed version of the above system. We introduce a time delay τ_{ij} in the coupling between two vertices v_i and v_j if these are neighbors. This models the time required for oscillator j to pass the phase information to oscillator i and vice-versa, i.e., $\tau_{ij} = \tau_{ji}$. We assume that these delays are inhomogeneous (i.e., unequal and non-commensurate). In particular we consider the following delayed Kuramoto dynamics at the i-th oscillator:

$$\dot{\theta}_i = \omega_i + \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j(t - \tau_{ij}) - \theta_i(t)) \tag{6}$$

and we are interested in uniform rotations, i.e., $\theta(t) = \Omega t + \phi(t)$. For this, self-consistency relations of the following form are imposed:

$$\Omega = \omega_i - \frac{K}{M_i}\sum_{j} A_{ij}\sin(\Omega\tau_{ij}) \tag{7}$$

for all i. We assume that a compatible solution Ω exists; the τ_{ij} can be chosen judiciously for this purpose.

Most available results on the above system can be found in the physics literature, and concern the linearization of the above system about the equilibrium set given by $\phi_i^*(t) = c$, $i = 1, \ldots, N$. In particular, if we linearize (6) about the set of equilibria satisfying $\phi_i(t) = c$ we have:

$$\dot{\phi}_i(t) = \frac{K}{M_i}\sum_{j=1}^{N} G_{ij}(\phi_j(t - \tau_{ij}) - \phi_i(t)) \tag{8}$$

where $G_{ij} = A_{ij}\cos(\Omega\tau_{ij})$. In [19] and [20] the case of regular connected graphs (i.e., $M_i = d$, the (identical) degree of the nodes in the graph) with $\tau_{ij} = \tau$ was investigated, and stability criteria were established that required $G > 0$. In [32] the general connected graph case was considered with $M_i = d_i$ again for the case in which $\tau_{ij} = \tau$ yielding the condition $G > 0$. In both these cases the parameter Ω solves the 'self-consistency' relation

$$\Omega = \omega - K\sin(\Omega\tau). \tag{9}$$

In [21] the stability of locked-in-phase equilibria sets in oscillator networks for inhomogeneous delays τ_{ij} was investigated for scalings $M_i = N$ and $M_i = d_i$ for the linearization of system (6). Under the condition $G_{ij} > 0$, it was shown that phase locking is achievable (even for non-identical oscillators).

Here we consider the nonlinear system given by (6) and derive attractivity conditions for the locked-in-phase equilibria when inhomogeneous delays are present. Let us consider the dynamics of oscillator i in the rotating frame. To do this, we write $\theta_i(t) = \Omega t + \phi_i(t)$ to get:

$$\begin{aligned}\dot{\phi}_i(t) = {} & \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin(-\Omega\tau_{ij} + \phi_j(t - \tau_{ij}) - \phi_i(t)) \\ & + \frac{K}{M_i}\sum_{j=1}^{N} A_{ij}\sin(\Omega\tau_{ij}).\end{aligned} \tag{10}$$

Let $\tau = \max_{i,j}\tau_{ij}$, and assume throughout that $\cos(\Omega\tau_{ij}) > 0$ for all $i, j = 1, \ldots, N$. This condition means that the functions

$$f_{ij}(y) = \sin(-\Omega\tau_{ij} + y) - \sin(-\Omega\tau_{ij}) \tag{11}$$

have a positive derivative at $y = 0$. In fact these functions are *locally passive*: $yf_{ij}(y) > 0$ for all $y \in [-\sigma_{ij}^-, \sigma_{ij}^+]$ for all $i, j = 1, \ldots, N$, where $\sigma_{ij}^- > 0$ and $\sigma_{ij}^+ > 0$. The function $f_{ij}(y)$ defined by (11) crosses the $y-$axis at $y = 2k\pi$ and $y = 2k\pi + \pi - 2\Omega\tau_{ij}$ for all integer k. We define γ as

$$\gamma = \min\left(|2k\pi + \pi - 2\Omega\tau_{ij}|\right), \quad \forall\ k \text{ integer}, \tag{12}$$

for $i, j = 1, \ldots, N$.

We then have the following lemma:

Lemma 3: Let $\tau = \max_{i,j=1,\ldots,N}\tau_{ij}$ and consider (10). Define γ as in (12), and suppose the initial conditions ψ satisfy

$$|\psi_i(s)| \leq \frac{\gamma}{2}, \quad \forall i = 1, \ldots, N, \quad s \in [-\tau, 0]. \tag{13}$$

Then

$$-\frac{\gamma}{2} \leq \phi_i(t) \leq \frac{\gamma}{2}$$

for all $t \geq -\tau$.

Proof: From the bounds on the initial condition, we have:

$$-\frac{\gamma}{2} \leq \psi_i(s) \leq \frac{\gamma}{2}$$

for $s \in [-\tau, 0]$. Suppose this condition is first violated at time t^*. When this happens, the following hold:

1) $-\frac{\gamma}{2} \leq \phi_i(t) \leq \frac{\gamma}{2}$ for $t \in [-\tau, t^*)$ for all $i = 1, \ldots, N$;
2) At t^* there is an $i \in \{1, \ldots, N\}$ such that we have either:

$$\left\{\phi_i(t^*) = \tfrac{\gamma}{2} \text{ and } \dot{\phi}_i(t^*) > 0\right\}$$
$$\text{or}$$
$$\left\{\phi_i(t^*) = -\tfrac{\gamma}{2} \text{ and } \dot{\phi}_i(t^*) < 0\right\}$$

Suppose the first case holds. Recall the structure of the dynamics:

$$\dot{\phi}_i = k_i \sum_{j=1}^{N} A_{ij} \sin(\phi_j(t - \tau_{ij}) - \phi_i(t))$$

Now since $\phi_i(t^*) = \frac{\gamma}{2} \geq \phi_j(t^* - \tau_{ij})$ from the observations above, and $|\phi_j(t)| \leq \frac{\gamma}{2}$ for $t \in [-\tau, t^*)$, we can see that each term on the right hand side (the vector field) is non-positive and so

$$\dot{\phi}_i(t^*) \leq 0$$

which leads to a contradiction. The same is true for the second case. In conclusion we have that

$$-\frac{\gamma}{2} \leq \phi_i(t) \leq \frac{\gamma}{2}$$

for all $t \geq -\tau$. ■

Therefore we have identified an invariant set, and we will now proceed to show that the equilibrium set is asymptotically attracting using Theorem 2.

Theorem 4: Consider (10) and assume $\cos(\Omega\tau_{ij}) > 0$ for all $i, j = 1, \ldots, N$, where the graph $\mathcal{G}$ is connected and Ω solves (7). Define γ as in (12), and consider initial conditions ψ that satisfy (13). Then the locked-in-phase equilibrium set is asymptotically attracting.

Proof: Consider the function

$$V = \frac{1}{2} \max_i \phi_i^2$$

as a candidate Lyapunov-Razumikhin function. In order to satisfy the first condition in Theorem 2, we are interested in

$$V(\varphi(0)) = \max_{-\tau \leq s \leq 0} V(\varphi(s))$$

i.e., for the cases for which

$$\varphi_I^2(0) \geq \varphi_j^2(s), \quad s \in [-\tau, 0]. \tag{14}$$

where I is the index for which $|\phi_I| = \max_i |\phi_i|$. Now

$$\dot{V}(\varphi) = \varphi_I(0)\dot{\varphi}_I(0)$$

We want to ensure that while condition (14) holds, $\dot{V} \leq 0$. First suppose $\varphi_I(0) = c \geq 0$. Then from Equation (14), we have that

$$-c \leq \varphi_j(s) \leq c$$

for all $j = 1, \ldots, N$, $s \in [-\tau, 0]$. This implies that $\sin(\varphi_j(-\tau_{Ij}) - \varphi_I(0)) \leq 0$ for all j that are neighbor to I, for admissible initial conditions. Therefore $\dot{\varphi}_I(0) \leq 0$. In a similar fashion, we can show that for $\varphi_I(0) \leq 0$ we have $\dot{\varphi}_I(0) \geq 0$. In conclusion, while (14) holds, we have $\dot{V} \leq 0$.

We now proceed to compute the sets E and L defined by Equations (2-3). First of all, since $\varphi = 0$ is in L, this set is non-empty. Suppose $\varphi \in E$, i.e., let $\varphi \in \Omega$ be such that

$$\max_i \max_{-\tau \leq \theta \leq 0} \phi_i^2(\varphi)(t + \theta) = \max_i \max_{-\tau \leq \theta \leq 0} \varphi_i^2(\theta) \tag{15}$$

for all $t \geq 0$ and $\theta \in [-\tau, 0]$. For $\varphi \in E$ satisfying (14), there exists a t^* for which we have $\dot{V}(x_{t^*}(\varphi)) = 0$, as V attains a relative maximum for such t^*. For such a t^*, we have:

$$\dot{V} = \phi_I(t^*)\dot{\phi}_I(t^*) = 0$$

Let us first concentrate in the case $\dot{\phi}_I(t^*) = 0$ and $\phi_I > 0$. From (14) we have that $\phi_j(t^* + \theta) - \phi_I(t^*) \leq 0$ for all $\theta \in [-\tau, 0]$, and if $\dot{\phi}_I(t^*) = 0$ is to hold, this means that $\phi_I(t^*) = \phi_j(t^* - \tau_{Ij})$ for all j neighbour to I – this is the set E. An invariant set belonging to E is the locked-in-phase set and so we conclude, using Theorem 2, that the locked-in-phase set between node I and its neighbors is asymptotically attracting. The same conclusion can be drawn when $\phi_I < 0$ or when $\phi_I = 0$. Since the graph is connected, all nodes (not just ones that are neighbor to I) have to agree asymptotically because otherwise they would influence some neighbor of node I and hence that would contradict the previous conclusion. Therefore the locked-in-phase set is asymptotically attracting for connected topologies. ■

We remark that the above holds independent of the network topology, so long that this is fixed and connected. Moreover, the scaling $M_i > 0$ in (5) does affect the attractivity properties. We want to investigate whether the same properties hold for the system when the topology changes, which is a more complicated issue.

IV. SWITCHING TOPOLOGIES

The issue of coordination under changing topologies has been investigated in [3] [6] [7] [33] [14], for the case in which the system does not have any delays that make the state-space infinite-dimensional. But even if the time-delays are ignored, the problem of ensuring stability in this case is difficult. Switching arbitrarily among a set of possible topologies can be regarded as a problem of establishing stability for a switching system with an unknown switching rule. The (conservative) condition of *quadratic stability* has been used to ensure that a system comprised of M subsystems of the form $\dot{x} = A_p x$, $p = 1, \ldots, M$ is stable under arbitrary switching; the conditions in this case require the existence of a *common* Lyapunov function $V = x^T P x$, $P > 0$ so that $A_p^T P + P A_p < 0$ for all $p = 1, \ldots, M$. This argument is many times inconclusive, as this criterion is conservative. This conservativeness was observed in [3], where a different approach had to be taken to conclude coordination. Another problem of using such an argument in the class of systems we are interested is that an invariance principle would have to be invoked to conclude stability, and the switching signal itself should satisfy certain conditions, as outlined in [23]. Therefore there are many challenges that one faces when trying to prove stability for arbitrary switching when delays are taken into account.

Consider a network of N oscillators with phases θ_i, $i = 1, \dots, N$ with an interaction topology chosen from a collection of graphs indexed by $p \in \mathcal{P} = \{1, \dots, M\}$. Each graph $\mathcal{G}^{(p)} = (\mathcal{V}, \mathcal{E}^{(p)})$ has an adjacency matrix $A^{(p)}$. Consider a piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ that is continuous from the right, which is non-chattering and with a dwell time $h > 0$, i.e., consecutive discontinuities of σ are separated by h. For more details, see [22].

We are interested in systems with uniform rotations; for the fixed topology case, we imposed conditions in the form (7) to enforce this. When the topology is allowed to change, these compatibility relations have to change to

$$\Omega^{(p)} = \omega_i - \frac{K}{M_i} \sum_{j=1}^{N} A_{ij}^{(p)} \sin(\Omega^{(p)} \tau_{ij}), \quad p = 1, \dots, m$$

We are interested in uniform rotations, i.e., $\Omega^{(p)} = \Omega$. It is easy to see that in that case, unless further restrictions are imposed on the structure of the networks and the dynamics of the oscillators, the above cannot have a compatible solution. We therefore impose that $\tau_{ij} = \tau$ for all $i, j = 1, \dots, N$, and we further assume that the graphs $\mathcal{G}^{(p)}$ we are switching amongst are connected and regular, so that $M_i = d$. Moreover, the oscillators are identical, i.e., $\omega_i = \omega$. This guarantees that there is a single Ω that solves the equation

$$\Omega = \omega - K \sin(\Omega \tau). \tag{16}$$

Under this assumption and using the new compatibility relation (16) we arrive to the following set of nonlinear Delay Differential Equations for each graph $\mathcal{G}^{(p)}$, on a rotating frame:

$$\begin{aligned} \dot{\phi}_i(t) = {} & \frac{K}{d} \sum_{j=1}^{N} A_{ij}^{(p)} \sin(-\Omega\tau + \phi_j(t-\tau) - \phi_i(t)) \\ & + \frac{K}{d} \sum_{j=1}^{N} A_{ij}^{(p)} \sin(\Omega \tau) \end{aligned} \tag{17}$$

Assume throughout that $\cos(\Omega\tau) > 0$. This condition means that the function

$$f(y) = \sin(-\Omega\tau + y) - \sin(-\Omega\tau) \tag{18}$$

is *locally passive*, as explained in the previous section. We define γ as:

$$\gamma = \min\left(|2k\pi + \pi - 2\Omega\tau|\right), \quad \forall\ k \text{ integer}. \tag{19}$$

It is easy to construct an invariant region even if the topologies change in a similar way as it was done for the fixed topology case, as long as there is no chattering. This is achieved by restricting the allowable switching signals to the ones that possess a dwell time. In particular, for γ as in (19) if the initial conditions ψ satisfy

$$|\psi_i(s)| \le \frac{\gamma}{2}, \quad \forall i = 1, \dots, N, \quad s \in [-\tau, 0]. \tag{20}$$

then

$$-\frac{\gamma}{2} \le \phi_i(t) \le \frac{\gamma}{2}$$

for all $t \ge -\tau$. Having identified an invariant region, we now proceed to establish the following result:

Theorem 5: Consider a piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ with a dwell time $h > 0$ where $\mathcal{P} = \{1, \dots, M\}$ and let switching happen between a collection of regular, connected graphs $\mathcal{G}^{(p)}$ in N vertices of valency d. At time t when $\sigma(t) = p$, assign to vertex v_i the dynamics given by Equation (17) with $\cos(\Omega\tau) > 0$. Define γ as in (19), and suppose the initial conditions ψ satisfy (20). Then the locked-in-phase equilibrium set is asymptotically attracting even if the topology changes.

Proof: We will show that the Lyapunov-Razumikhin function we used in the proof of Theorem 4 is a common Lyapunov-Razumikhin function for all the systems indexed by $p \in P$. Indeed, consider

$$V = \frac{1}{2} \max_i \phi_i^2$$

Suppose that at time t, subsystem $p \in P$ is active and node I achieves the above maximum. Then we have:

$$\dot{V} = \varphi_I(0) \frac{K}{d} \sum_{j=1}^{N} A_{ij}^{(p)} f(\varphi_j(-\tau) - \varphi_I(0))$$

where f is given by (18). We are interested, furthermore, in the set for which

$$\varphi_I^2(0) \ge \varphi_j^2(s), \quad s \in [-\tau, 0]. \tag{21}$$

as explained in the proof of Theorem 4. Irrespective of p, $\dot{V} \le 0$ as while $\varphi_I(0) \le 0$, the function f is non-negative, and when $\varphi_I(0) \ge 0$, the function f is non-positive. Therefore V is a common Lyapunov-Razumikhin function.

Denote the time instances at which switching occurs by t_m, $m = 1, 2, \dots$. Note that since the switching signal has a dwell time $h > 0$, we have $t_{m+1} - t_m > h$; during the time interval $[t_m, t_{m+1})$ the topology index is $\sigma(t_m) = p_m \in \{1, \dots, M\}$. Recall that V is not non-decreasing for all time, but $\overline{V} = \max_{\theta \in [-\tau, 0]} V(\theta)$ is. Denote by $\theta_0(t)$ the value of θ for which the maximum is achieved at time t. We then have:

$$\begin{aligned} & \overline{V}(\phi(t_{i+L} + \theta)) - \overline{V}(\phi(t_i + \theta)) \\ & = \sum_{m=0}^{L} \int_{t_{i+m}}^{t_{i+m+1}} \frac{K}{d} \phi_I(t + \theta_0(t)) \sum_{j=1}^{N} A_{Ij}^{(p_{i+m})} \\ & \qquad f\left(\phi_j(t + \theta_0(t) - \tau_{Ij}) - \phi_I(t + \theta_0(t))\right) dt. \end{aligned}$$

Since $\overline{V}$ is bounded from below and is non-increasing, the left hand side of this equation goes to a constant as

$L \to \infty$. In turn this means that the series on the right hand side of this equation has to converge. A Barbalat argument cannot, however, be immediately applied to conclude that the integrand goes to zero asymptotically. Nonetheless, since switching happens with a dwell time, a Barbalat-like argument [22] can be used to conclude this. An invariant set that satisfies this condition is the locked-in-phase equilibrium set, and we therefore conclude that the locked-in-phase set between vertex I and its neighbors as the topology switches is asymptotically attracting. But this would have to be the case for all the nodes (see argument in proof of Theorem 4) and so the consensus set is asymptotically attracting even if the topology is allowed to change, as long as switching occurs between connected topologies and with a dwell time. ■

We should emphasize that the restriction imposed on the structure of the network is due to the compatibility conditions that had to be satisfied. This is not the case for other, multi-agent systems with similar dynamics. Indeed, for a system with dynamics

$$\dot{x}_i(t) = \frac{K}{M_i} \sum_{j=1}^{N} A_{ij}^{(p)} f_{ij}(x_j(t - \tau_{ij}) - x_i(t))$$

with f_{ij} locally passive, we would expect to conclude asymptotic attractivity of the consensus state as long as the graphs we are switching amongst are jointly connected over time. This is a topic of future research.

V. CONCLUSIONS

In this paper we have investigated the attractivity properties of the phase-locked equilibria sets in nonlinear, heterogeneously delayed oscillator networks under fixed and switching topologies endowed with Kuramoto dynamics.

The techniques we have used can be readily applied to other such systems of interest, that are related to multi-agent coordination, a subject of future research.

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