Receding Horizon Control of Spatially Distributed Systems over Arbitrary Graphs†

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Abstract— **In this paper, we study the problem of receding horizon control of spatially distributed systems with arbitrary interconnection topologies. The key idea is the introduction of *spatially decaying* operators (SD) which serve as the main ingredient in the cost function that couples the state and control of individual agents with those of others. It is shown that coupling between subsystems of many well-known spatially distributed systems such as some of the recently studied models of distributed motion coordination with nearest neighbor interactions as well as spatially invariant systems can be characterized using such operators. We prove that for spatially distributed systems with input and state constraints in which the coupling is through an SD operator in a finite horizon cost function, optimal receding horizon controllers are piece-wise affine (represented as a convolution sum plus an offset). More importantly, we prove that the kernel of each convolution sum decays exponentially in the spatial domain, thereby providing evidence that even centralized solutions to the receding horizon control problems for such systems has an inherent spatial locality. Our theoretical results are verified by numerical simulations.**

I. Introduction

Over the past few years, there has been a rapidly growing interest in systems and control community in the study of coordination and control algorithms for networked dynamic systems [1]–[3]. On the other hand with advances in real-time optimization-based control, there have been several attempts to develop distributed control algorithms that can handle constraints and can be implemented in real-time [4]–[10].

In parallel to such research efforts, there have been some recent developments in the study of infinite-dimensional spatially distributed systems with certain symmetries in their structure, such as linear spatially invariant systems. In [11], Bamieh *et al.* used spatial Fourier transforms and operator theory to study linear spatially invariant systems with quadratic performance criteria such as LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ problems. It was shown that such problems can be tackled by solving a parameterized family of finite-dimensional problems in Fourier domain. Furthermore, it was also shown that the optimal controller has a degree of spatial localization. In [12], the authors developed conditions for well-posedness, stability, and performance of spatially interconnected systems whose model is spatially discrete (e.g. over a one-dimensional or two-dimensional lattice) in terms of linear matrix inequalities (LMI). Another interesting work in this area is reported in [13] where the authors use operator theoretic tools, motivated by results of [14] to analyze time-varying systems, and design optimal controllers for heterogeneous systems which are not shift invariant with respect to spatial or temporal variables. Several other authors such as [13], [15]–[18], have dealt with analysis and design of optimal controllers for a larger class of interconnection topologies with symmetries, spatially distributed systems with boundary, and systems interconnected over an arbitrary graph. In all of the above results, the underlying system is assumed to be unconstrained.

The systems considered in this work are not spatially invariant and the corresponding operators are not translation invariant either. The spatial structures studied in [11] are Locally Compact Abelian (LCA) groups [19] such as $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,\oplus)$. As a result, the group operation naturally induces a translation operator for functions defined on the group. However, when the dynamics of individual subsystems are not identical and the spatial structure does not necessarily enjoy the symmetries of LCA groups, standard tools such as Fourier analysis cannot be used to analyze the system.

To this end, we consider the receding horizon control (RHC) problem of spatially distributed systems over arbitrary graphs. In Section V, it is shown that the optimal controller resulting from RHC which minimizes a quadratic performance criterion subject to discrete-time LTI systems with constraints on inputs and states, has a spatial localization property similar to [11] and [20]. Similar to [21], the resulting receding horizon controller is piece-wise affine (represented as a convolution sum) and continuous, and can be determined (at least in theory) in an explicit manner. More importantly, a new class of linear operators, called *spatially decaying* (SD), are introduced in Section IV that can be used to show that the kernel of the optimal controller decays exponentially in spatial domain, thereby extending our earlier results [20] to constrained receding horizon control over *arbitrary graphs*.

II. Preliminaries

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of non-negative real numbers, and $\mathbb{C}$ the set of complex numbers. Consider an undirected connected graph with a nonempty set $\mathbb{G}$ of nodes. We refer to $\mathbb{G}$ as the spatial domain. $\|\,.\,\|$ denotes the induced Euclidean 2-norm. The Banach space $\ell_p(\mathbb{G})$ for $1 \le p < \infty$ is defined to be the set of all sequences $x = (x_i)_{i\in\mathbb{G}}$ in which $x_i \in \mathbb{R}^{n_i}$ satisfying $\sum_{i\in\mathbb{G}} \|x_i\|^p < \infty$ endowed with the norm $\|x\|_p := \left(\sum_{i\in\mathbb{G}} \|x_i\|^p\right)^{\frac{1}{p}}$. The Banach space $\ell_\infty(\mathbb{G})$ denotes the set of all bounded

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sequences endowed with the norm $\|x\|_\infty := \sup_{i\in\mathbb{G}} \|x_i\|$. Throughout the paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. The space ℓ_2 is a Hilbert space with inner product $\langle x, y\rangle := \sum_{i\in\mathbb{G}} \langle x_i, y_i\rangle$ for all $x, y \in \ell_2$. An operator $\mathcal{Q} : \ell_p \to \ell_q$ for $1 \le p, q \le \infty$ is bounded if it has a finite induced norm, i.e., the following quantity

$$\|\mathcal{Q}\|_{\ell_p\to\ell_q} := \sup_{\|x\|_p=1} \|\mathcal{Q}x\|_q \tag{1}$$

is bounded. The identity operator is denoted by $\mathcal{I}$. The set of all bounded linear operators of ℓ_p into itself is denoted by $\mathcal{L}(\ell_p)$. An operator $\mathcal{Q} \in \mathcal{L}(\ell_p)$ has an *algebraic* inverse if it has an inverse $\mathcal{Q}^{-1}$ in $\mathcal{L}(\ell_p)$ [22]. The adjoint operator of $\mathcal{Q} \in \mathcal{L}(\ell_2)$ is the operator $\mathcal{Q}^*$ in $\mathcal{L}(\ell_2)$ such that $\langle \mathcal{Q}x, y\rangle = \langle x, \mathcal{Q}^* y\rangle$ for all $x, y \in \ell_2$. An operator $\mathcal{Q}$ is self-adjoint if $\mathcal{Q} = \mathcal{Q}^*$. An operator $\mathcal{Q} \in \mathcal{L}(\ell_2)$ is *positive definite*, shown as $\mathcal{Q} \succ 0$, if there exists a number $\alpha > 0$ such that $\langle x, \mathcal{Q}x\rangle > \alpha \|x\|_2^2$ for all nonzero $x \in \ell_2$.

The concatenation of a sequence of vectors $x_i \in \mathbb{R}^{n_i}$ is defined as $\underset{i\in\mathbb{G}}{\mathrm{cat}}\ x_i$. For a given sequence of matrices $A_i \in \mathbb{R}^{n_i\times n_i}$ with $i \in \mathbb{G}$, the diagonal operator $\underset{i\in\mathbb{G}}{\mathrm{diag}} A_i$ is defined to be an operator that maps $x = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ x_i$ to $y = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ y_i$ such that $y_i = A_i x_i$ for all $i \in \mathbb{G}$. Symbol $\otimes$ stands for the Kronecker product. In this paper, we are interested in linear operators which have matrix representations.

III. Formulation of Receding Horizon Control Problem

Let $\mathbb{G}$ be the set of nodes of an undirected connected graph in which each node represents an individual dynamical system (see Fig.1). Assume that each subsystem can be modeled as a discrete-time linear system

$$\begin{bmatrix} x_i(t+1) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix} \tag{2}$$

with initial condition $x_i(0) = x_0^i$. Index $i \in \mathbb{G}$ is the spatial variable, $x_i \in \mathbb{R}^{n_i}$ is the state variables, x_0^i is the initial condition, $u_i \in \mathbb{R}^{m_i}$ is the control input , and $y_i \in \mathbb{R}^{p_i}$ the sensor output of agent i, with all matrices having the appropriate dimension. The state evolution of the system started at x_0^i is denoted by $x_i(t; x_0^i)$ for $t > 0$, and similarly for the output of the system $y_i(t; x_0^i)$. We assume that (A_i, B_i) is stabilizable for all $i \in \mathbb{G}$.

Furthermore, we assume that the collective objective of the entire system can be posed as a constrained finite horizon problem with a quadratic cost as follows

$$\begin{aligned} \min_{\mathbf{u}^N}\ & \mathfrak{J}(x_0, \mathbf{u}^N; N) && \qquad (3)\\ \text{s.t.}\ & \text{Eq. (2)} && , 0 \le t \le N \\ & u_{\min}^i \preceq u_i(t) \preceq u_{\max}^i && , 0 \le t \le N_c \\ & y_{\min}^i \preceq y_i(t; x_0^i) \preceq y_{\max}^i && , 0 \le t \le N_c \\ & u_i(t) = \sum_{j\in\mathbb{G}} [\mathcal{K}_N]_{ij}\, x_j(t; x_0^j) && , N_u \le t \le N-1 \\ & i \in \mathbb{G} \end{aligned}$$

where the objective function is given by:

$$\begin{aligned} \mathfrak{J}(x_0, \mathbf{u}^N; N) = {} & \langle x(N; x_0), \mathcal{P}_N x(N; x_0)\rangle + \\ & \sum_{t=0}^{N-1} \langle x(t; x_0), \mathcal{Q}x(t; x_0)\rangle + \langle u(t), \mathcal{R}u(t)\rangle \end{aligned} \tag{4}$$

in which $x(t; x_0) = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ x_i(t; x_0^i)$, $x_0 = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ x_0^i$, and $u(t) = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ u_i(t)$ for $t \ge 0$. Also, N_u is the control prediction horizon, N_c the constraint horizon, and N the state prediction horizon. Furthermore, we assume that $N_u \le N$ and $N_c \le N-1$ (cf. [21]). Assume that the linear operators $\mathcal{Q} \succeq 0$, $\mathcal{R} \succ 0$, $\mathcal{P}_N \succ 0$ are bounded and self-adjoint. The state feedback kernel $\mathcal{K}_N$ and the terminal weighting cost $\mathcal{P}_N$ are determined by solving the corresponding LQR problem and the corresponding Riccati equation. It is assumed that $u_{\min},\ u_{\max},\ y_{\min},\ y_{\max} \in \ell_2$ where $u_{\min} = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ u_{\min}^i$, etc. Given a horizon length, there is a polyhedral set of initial conditions for which feasible trajectories exists, over which the receding horizon controller is stabilizing (cf. [21], [23]–[26] and the references therein). The set of all initial conditions for which an optimal solution of (3) exist is a polyhedral set $\mathbb{X}$ which can be characterized as $\mathbb{X} = \{x \in \ell_2 : \mathcal{E}x \preceq d\}$ where $\mathcal{E}$ is a bounded linear operator. For simplicity, we will assume that $N_u = N-1$. For $N_u < N-1$ optimization is performed only over N_u control variables, and for the rest of the horizon we may use the pre-computed feedback gain as in (3) to find control inputs, which in turn will reduce the complexity of the problem. The prediction model for the entire system is given by

$$\begin{aligned} \mathbf{x}^N &= \underset{i\in\mathbb{G}}{\mathrm{diag}}\mathbf{A}_i\ x_0\ +\ \underset{i\in\mathbb{G}}{\mathrm{diag}}\mathbf{B}_i\ \mathbf{u}^N \\ &= \mathcal{A}\ x_0\ +\ \mathcal{B}\ \mathbf{u}^N \end{aligned} \tag{5}$$

in which $\mathbf{x}^N = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ \mathbf{x}_i^N$ and $\mathbf{u}^N = \underset{i\in\mathbb{G}}{\mathrm{cat}}\ \mathbf{u}_i^N$ where

$$\begin{aligned} \mathbf{x}_i^N &= [x_i(1; x_0^i), \ldots, x_i(N; x_0^i)]^* \\ \mathbf{u}_i^N &= [u_i(0), \ldots, u_i(N-1)]^* \end{aligned}$$

where $\mathbf{A}_i$ and $\mathbf{B}_i$ are completely determined from A_i and B_i. Substituting (5) into (4) gives us

$$\begin{aligned} \mathfrak{J}(x_0, \mathbf{u}^N; N) = {} & \langle x_0, (\mathcal{Q} + \mathcal{A}^*\mathcal{L}_x\mathcal{A})\, x_0\rangle + \\ & 2\left\langle x_0, \mathcal{A}^*\mathcal{L}_x\mathcal{B}\ \mathbf{u}^N\right\rangle + \left\langle \mathbf{u}^N, (\mathcal{L}_u + \mathcal{B}^*\mathcal{L}_x\mathcal{B})\, \mathbf{u}^N\right\rangle \end{aligned}$$

in which operators $\mathcal{L}_x$ and $\mathcal{L}_u$ are defined as

$$\begin{aligned} [\mathcal{L}_x]_{ij} &:= \mathrm{diag}\left(I_{N-1} \otimes [\mathcal{Q}]_{ij}, [\mathcal{P}_N]_{ij}\right) \\ [\mathcal{L}_u]_{ij} &:= I_N \otimes [\mathcal{R}]_{ij}. \end{aligned}$$

The input and output constraints in (3) can be written as

$$G_i \mathbf{u}_i^N \preceq E_i + F_i x_0^i$$

where

$$E_i = \begin{bmatrix} \mathbf{1}_N \otimes y_{\max}^i \\ -\mathbf{1}_N \otimes y_{\min}^i \\ \mathbf{1}_N \otimes u_{\max}^i \\ -\mathbf{1}_N \otimes u_{\min}^i \end{bmatrix}, G_i = \begin{bmatrix} \mathbf{D}_i \\ -\mathbf{D}_i \\ I_{Nm_i} \\ -I_{Nm_i} \end{bmatrix}, F_i = \begin{bmatrix} -\mathbf{C}_i \\ \mathbf{C}_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

for all $i \in \mathbb{G}$, $\mathbf{1}_N = [1\ 1\ \dots\ 1]^*$, $\mathbf{C}_i$ and $\mathbf{D}_i$ are completely determined from C_i and D_i. Therefore, the following optimization problem is equivalent to (3)

$$\begin{aligned} \min_{\mathbf{u}^N} \ & \frac{1}{2}\left\langle \mathbf{u}^N, \mathcal{P}\mathbf{u}^N \right\rangle + \left\langle \mathbf{u}^N, \mathcal{H}x_0 \right\rangle \\ \text{s.t.} \ & \mathcal{G}\ \mathbf{u}^N \preceq E + \mathcal{F}x_0 \end{aligned} \tag{6}$$

in which

$$\mathcal{P} := \mathcal{L}_u + \mathcal{B}^*\mathcal{L}_x\mathcal{B} \quad , \quad \mathcal{H} := \mathcal{B}^*\mathcal{L}_x\mathcal{A}$$
$$\mathcal{G} = \operatorname*{diag}_{i\in\mathbb{G}} G_i \quad , \quad E = \operatorname*{cat}_{i\in\mathbb{G}} E_i \quad , \quad \mathcal{F} = \operatorname*{diag}_{i\in\mathbb{G}} F_i.$$

Formulation in (6) gives a clear picture of the relationship between the control input variables and initial conditions x_0^i. Problem (6) is a multi-parametric optimization problem on $\mathbb{G}$ in which x_0 is treated as vector of parameters. Problem (6) depending on the size of the graph could be a finite or infinite dimensional optimization problem. The key results developed in the coming sections are applicable to any problem that can be posed in the convex quadratic form of (6).

In what follows, we introduce an important class of linear operators which are useful in analysis of spatially distributed systems. Later in section V, we will apply results of section IV to problem (6) and prove that the kernel of the optimal solution decays exponentially in space, and as a result it can be localized around each agent.

IV. Spatially Decaying Operators

While translation invariant operators can be defined on LCA groups and analyzed by Fourier methods, the analysis does not extend to arbitrary topologies. Simply put, if we replace "space" with "time", we get a more familiar analogue of this problem: Fourier methods can not be used directly for analysis of linear time-varying systems. In order to address this problem, we extend the notion of spatial decay in a natural way from linear translation invariant operators to a larger class of linear operators, which we call SD operators for short. Without loss of generality, we assume that operators in the following definitions are self-adjoint.

A. Definitions and Examples

Definition 1: The distance function between nodes of a given graph is defined as a single-valued, nonnegative, real function $\operatorname{dis}(k,i)$ defined for all $i, j, k \in \mathbb{G}$ which has the following properties:

(1) $\operatorname{dis}(k,i) = 0 \quad \text{iff} \quad k = i$.
(2) $\operatorname{dis}(k,i) = \operatorname{dis}(i,k)$.
(3) $\operatorname{dis}(k,i) \leq \operatorname{dis}(k,j) + \operatorname{dis}(j,i)$.

Definition 2: Suppose that a distance function $\operatorname{dis}(.,.)$ is given. The linear operator $\mathcal{Q} \in \mathcal{L}(\ell_p)$ is SD if there exists a number $b > 1$ such that operator $\hat{\mathcal{Q}}(\zeta)$ defined by $[\hat{\mathcal{Q}}(\zeta)]_{ki} = [\mathcal{Q}]_{ki}\ \zeta^{\operatorname{dis}(k,i)}$ is bounded on ℓ_p for all $\zeta \in [1, b)$. The number b is referred to as the *decay margin*.

In general, determining the boundedness of the auxiliary operator depends on the choice of p. The following result gives us a simple sufficient condition for an operator to be SD in all ℓ_p-norms.

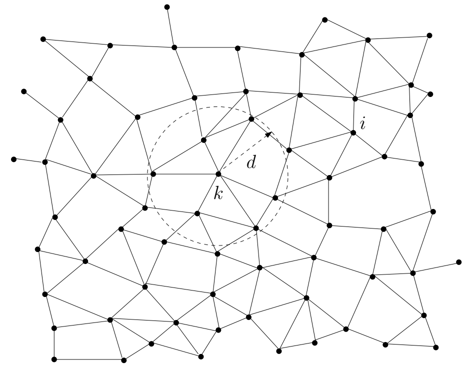

Fig. 1. Topology of the system on an arbitrary connected graph. Coupling between two agents through cost function is shown by an undirected edge between them.

Proposition 1: A linear operator $\mathcal{Q}$ is SD in all ℓ_p-norms if there exists a number $b > 1$ such that

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\|\ \zeta^{\operatorname{dis}(k,i)} < \infty \tag{7}$$

for all $\zeta \in [1, b)$.

Proof: First, we will show that the auxiliary operator $\hat{\mathcal{Q}}(\zeta)$ is simultaneously in $\mathcal{L}(\ell_1)$ and $\mathcal{L}(\ell_\infty)$. For a fixed $\zeta \in [1, b)$, $\hat{\mathcal{Q}}(\zeta) \in \mathcal{L}(\ell_1) \cap \mathcal{L}(\ell_\infty)$ if the following quantities

$$\|\hat{\mathcal{Q}}(\zeta)\|_{\ell_\infty\to\ell_\infty} \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\|\ \zeta^{\operatorname{dis}(k,i)}$$

and

$$\|\hat{\mathcal{Q}}(\zeta)\|_{\ell_1\to\ell_1} \leq \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\|\ \zeta^{\operatorname{dis}(k,i)}$$

are bounded. Finally, using the Riesz-Thorin theorem, we can show that $\hat{\mathcal{Q}}(\zeta)$ is also bounded on all intermediate spaces ℓ_p where $1 \leq p \leq \infty$. ■

Intuitively, we may interpret the norm of each block element $[\mathcal{Q}]_{ki}$ as the coupling strength between subsystems k and i. Let us fix a value for $\zeta \in (1, b)$. For an infinite graph, if we fix a node k, and move on the graph away from node k, the coupling strength is going to decay exponentially so that relation (7) holds. The following examples are of special interest in cooperative control of multi-agent systems and distributed control of networked systems.

1) Systems with Nearest Neighbor Coupling: Suppose that operator $\mathcal{Q}$ represents the nearest neighbor coupling given by a connected distance dependent *proximity* graph (e.g. Fig.1) with the node set $\mathbb{G}$ defined as follows

$$[\mathcal{Q}]_{ki} = \begin{cases} Q_{ki} & \text{if } \operatorname{dis}(k,i) \leq d \\ 0 & \text{if } \operatorname{dis}(k,i) > d \end{cases} \tag{8}$$

where $Q_{ki} \in \mathbb{R}^{n_k \times n_i}$. For this case, some common choices for the distance function are Euclidean distance with $d > 0$ and geodesic or minimum hop count distance (i.e., hop count on the shortest path) with $d = 1$. Examples of such operators arise in motion coordination of autonomous agents. For such examples, It can be shown that $\mathcal{Q}$ as defined above is SD and the decay margin is any real number $b > 1$.

2) Spatially Invariant Systems: Translation invariant operators defined on ℓ_2 are of special importance in studying of the spatially invariant systems defined on discrete groups (see [11], [12], [20] and references in there). We recall that operator $\mathcal{Q}$ is translation invariant if $\mathcal{Q}\mathbf{T} = \mathbf{T}\mathcal{Q}$ where $\mathbf{T}$ is the unit shift operator to the left. Suppose that $\mathbb{G} = \mathbb{Z}$ and operator $\mathcal{Q}$ is translation invariant with representation

$$\mathcal{Q}(\mathbf{T}) = \sum_{k\in\mathbb{G}} Q_k \mathbf{T}^k. \tag{9}$$

The equivalent representation is $[\mathcal{Q}]_{ki} = Q_{k-i}$. The Fourier transform of $\mathcal{Q}$ is denoted by $\hat{\mathcal{Q}}(z)$ evaluated on the unit circle in $\mathbb{C}$. We recall the following result from [20] which gives us a simple sufficient condition for a translation invariant operator to be SD in ℓ_2-norm.

Theorem 1: Let $\mathcal{Q}$ be defined by (9) on $\mathbb{Z}$ with discrete Fourier transform $\hat{\mathcal{Q}}(z)$. If $\mathcal{Q} \in \mathcal{L}(\ell_2)$, then the coefficients of operator $\mathcal{Q}$ decay exponentially in the spatial domain, i.e., for all $k \in \mathbb{Z}$

$$\|Q_k\| \leq \ \alpha \ e^{-\beta|k|} \tag{10}$$

for some $\alpha > 0$ and $0 < \beta < \ln(1+r)$ where r is the distance of the nearest pole of $\hat{\mathcal{Q}}(z)$ to the unit circle in $\mathbb{C}$.

In this case, since the interconnection topology is assumed to be a lattice, by defining the distance function as $\operatorname{dis}(k,i) = |k-i|$, it can be proved that the operator $\mathcal{Q}$ with the above mentioned properties is SD in ℓ_2-norm. Indeed, it can be shown that a bounded translation invariant operator is SD in ℓ_p-norm for all $1 \leq p \leq \infty$. The decay margin of $\mathcal{Q}$ is $1+r$. According to theorem 1, for a translation invariant operator $\mathcal{Q}$, if $\det(\hat{\mathcal{Q}}(z)) \neq 0$ on the unit circle, then operator $\mathcal{Q}$ is invertible and the inverse operator $\mathcal{P}$ can be represented as

$$\mathcal{P}(\mathbf{T}) = \sum_{k\in\mathbb{G}} P_k \mathbf{T}^k.$$

Furthermore,

$$\lim_{|k-i|\to\infty} \|[\mathcal{P}]_{ki}\| \ e^{m|k-i|} = 0 \tag{11}$$

for $0 < m < \ln(1+\rho)$ and

$$\rho = \sup\{r : \det(\mathrm{Q}(z)) \neq 0 \text{ for all } 1-r \leq |z| \leq 1+r\}.$$

B. Properties

As seen so far, the class of SD operators play an essential role in important applications such as control of multi-agent systems and networked systems. In the following, some important properties of this class of linear operators are studied. In the rest of the paper, we say $\mathcal{Q}$ is SD if it satisfies (7) for some $b > 1$.

Proposition 2: If linear operators $\mathcal{Q}$ and $\mathcal{R}$ are SD, then so are $\mathcal{Q}+\mathcal{R}$, $\mathcal{Q}\mathcal{R}$.

Proof: We refer to [27] for a proof. ■

The main result of this section is the next theorem which can be thought of as the extension of result (11) to SD operators.

Theorem 2: Assume that linear operator $\mathcal{Q}$ is SD with decay margin $b > 1$ and has an algebraic inverse on $\mathcal{L}(\ell_2)$. Then the inverse operator $\mathcal{Q}^{-1}$ is also SD with decay margin $\tilde{b} > 1$ where $1 < \tilde{b} < b$. Furthermore, each nonzero block element of the inverse operator satisfies

$$\|[\mathcal{Q}^{-1}]_{ki}\| \ \leq \ c \ \xi^{\operatorname{dis}(k,i)} \tag{12}$$

for all $\xi \in \left(\frac{1}{\hat{b}}, 1\right]$ and some $c > 0$, where

$$\hat{b} = \sup\left\{\zeta : \varphi(\zeta) - \varphi(1) < \|\mathcal{Q}^{-1}\|^{-1}_{\ell_\infty\to\ell_\infty}\right\} \tag{13}$$

and $\varphi(\zeta) = \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\| \ \zeta^{\operatorname{dis}(k,i)}$.

Proof: We refer to [28] for a proof. ■

The following theorem shows that the unique solution of discrete algebraic Riccati equation (DARE), which can be used as terminal weighting cost operator in (3)-(4), is SD.

Theorem 3: Let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{R}$ be SD with decay margin $b > 1$ and $\mathcal{Q} \succeq 0$, $\mathcal{R} \succ 0$. Whenever the unique positive definite solution of the following (DARE)

$$\mathcal{P} = \mathcal{Q} + \mathcal{A}^*\mathcal{P}\mathcal{A} - \mathcal{A}^*\mathcal{P}\mathcal{B}\left(\mathcal{R} + \mathcal{B}^*\mathcal{P}\mathcal{B}\right)^{-1}\mathcal{B}^*\mathcal{P}\mathcal{A}$$

exists, it is SD with decay margin b.

Proof: See [27] for a similar proof. ■

This is the extension of results from spatially invariant systems [11] to spatially distributed systems over arbitrary graphs. By using the analogous version of theorem 1 for continuous groups $\mathbb{G} = \mathbb{R}$, it is shown that the kernel of the optimal LQR controller decays exponentially in the spatial domain. Theorem 3 extends the decay property of the ARE solution to the class of SD operators.

V. Analysis of Receding Horizon Control for Spatially Distributed Systems over Arbitrary Graph

As mentioned before, (6) can be solved as a multi-parametric optimization problem, in which parameters are the initial states x_0^i. It can be shown that the set of admissible states $\mathbb{X}$ can be partitioned into countably many partitions [21] over each of which the optimal control law is an affine function of the initial states. We now show that a similar explicit representation exists for the optimal receding horizon controller in the case of spatially distributed systems over arbitrary graphs. In this scenario, however, the affine representation is in the form of a convolution sum with the controller gains appearing as the kernel. We will show that convolution kernel corresponding to the optimal controllers on each partition, have decay exponentially over the spatial dimension. Before stating the theorem, we recall that if $\mathcal{P} \succ 0$, then for every $x_0 \in \mathbb{X}$, there exist a unique optimal solution for (6) [29]. The following theorem is the main result of our paper and extends the main result of [20] to arbitrary graphs.

Theorem 4: Let $\mathcal{Q} \succeq 0$ and $\mathcal{R} \succ 0$ in (4) be SD with decay margins $b > 1$. For any $i \in \mathbb{G}$, assume that some combination of constraints in (6) are active and the corresponding rows to these active constraints from matrices G_i, E_i, and F_i form the full-row rank matrix $\bar{G}_i$, $\bar{F}_i$, and $\bar{E}_i$. Let $\Omega \subseteq \mathbb{X}$ be the set of all x_0 so that such combinations are

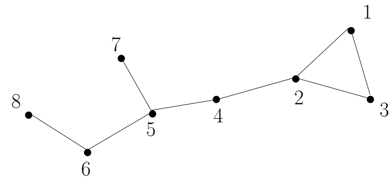

Fig. 2. Interconnection topology of 8 agents with nearest neighbor interactions.

active at the optimal solution. Then the optimal solution $\mathbf{u}_k^N$ of (6), as well as the corresponding Lagrange multipliers to index k, are

(a) *affine* maps of x_0 over Ω, especially

$$\mathbf{u}_k^N = \sum_{i \in \mathbb{G}} [\mathcal{K}_f]_{ki}\, x_0^i + c_k. \tag{14}$$

(b) *spatially distributed*, in the sense that the coupling decays exponentially in the spatial domain, i.e.,

$$\|[\mathcal{K}_f]_{ki}\| \;\leq\; c\; \xi^{\mathrm{dis}(k,i)}$$

for $\xi \in (\frac{1}{\tilde{b}}, 1]$ ($\tilde{b}$ is determined explicitly in the proof).

Proof: The Lagrange function corresponding to (6) is

$$\mathfrak{L}(\mathbf{u}^N, \lambda) = \frac{1}{2}\left\langle \mathbf{u}^N, \mathcal{P}\mathbf{u}^N \right\rangle \;+\; \left\langle \mathbf{u}^N, \mathcal{H}x_0 \right\rangle \;+\; \left\langle \lambda\;,\; \mathcal{G}\mathbf{u}^N - E - \mathcal{F}x_0 \right\rangle.$$

KKT conditions are given by

$$\mathcal{P}\mathbf{u}^N + \mathcal{H}x_0 + \mathcal{G}^*\lambda = 0 \tag{15}$$

$$\lambda_i^j \;\left(G_i \mathbf{u}_i^N - E_i - F_i\; x_0^i\right)^j = 0 \tag{16}$$

$$j = 1, ..., 2N(p_i + m_i)$$

$$\lambda_i \succeq 0 \tag{17}$$

$$\mathcal{G}\mathbf{u}^N \preceq E + \mathcal{F}x_0 \tag{18}$$

where $i \in \mathbb{G}$, $\lambda_i \in \mathbb{R}^{2N(p_i+m_i)}$, and λ_i^j or $(\;.\;)^j$ represents the j^{th} row. Since $\mathcal{P} \succ 0$ is invertible, (15) gives us

$$\mathbf{u}^N = -\mathcal{P}^{-1}\left(\mathcal{G}^*\lambda + \mathcal{H}x_0\right). \tag{19}$$

According to equation (16) and (17), all Lagrange multipliers corresponding to inactive constraints are zero, and the Lagrange multipliers corresponding to active constraints, stacked in a column vector $\bar{\lambda}_i$, are non-negative numbers. From assumptions, active constraints result in the following set of equations which allow us to solve them along with (19) for $\bar{\lambda}_i$. By assuming $x_0 \in \Omega$ and using (19), we have

$$\mathbf{u}^N = -\mathcal{P}^{-1}\bar{\mathcal{G}}^*\bar{\lambda} - \mathcal{P}^{-1}\mathcal{H}x_0 \tag{20}$$

where $\bar{\mathcal{G}} = \underset{i \in \mathbb{G}}{\mathrm{diag}}\; \bar{G}_i$ and $\bar{\lambda} = \underset{i \in \mathbb{G}}{\mathrm{cat}}\; \bar{\lambda}_i$. Substituting (20) into (16), results in

$$\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\;\bar{\lambda} = -\bar{E} - \left(\bar{\mathcal{G}}\mathcal{P}^{-1}\mathcal{H} + \bar{\mathcal{F}}\right)x_0$$

where $\bar{E} = \underset{i \in \mathbb{G}}{\mathrm{cat}}\; \bar{E}_i$ and $\bar{\mathcal{F}} = \underset{i \in \mathbb{G}}{\mathrm{diag}}\bar{F}_i$. Since $\bar{\mathcal{G}}$ is surjective and $\bar{\mathcal{G}}^*$ injective, $\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*$ is invertible on ℓ_2, and that we

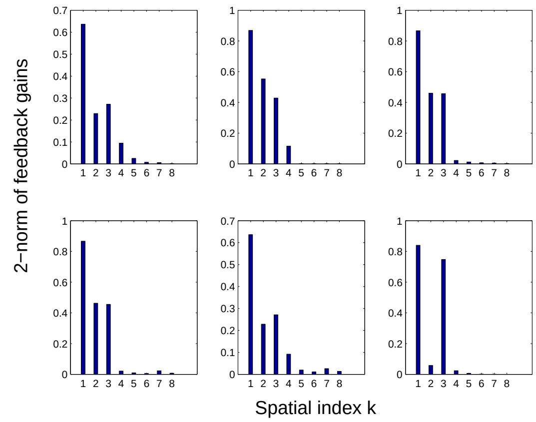

Fig. 3. Exponential decay of the kernel of the optimal controller of agent 1 with respect to other agents for six different regions. Note that agents 2 and 3 are in the same hop distance from agent 1.

have

$$\bar{\lambda} = -\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\right)^{-1}\bar{E} - \left(\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\right)^{-1}\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\mathcal{H} + \bar{\mathcal{F}}\right)x_0$$

and

$$\mathbf{u}^N = \mathcal{K}_f\; x_0 \;+\; \mathcal{K}_o\; \bar{E} \tag{21}$$

where

$$\mathcal{K}_f \;:=\; \mathcal{P}^{-1}\mathcal{G}^*\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\right)^{-1}\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\mathcal{H} + \bar{\mathcal{F}}\right) + \mathcal{P}^{-1}\mathcal{H}$$

$$\mathcal{K}_o \;:=\; \mathcal{P}^{-1}\mathcal{G}^*\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\right)^{-1}.$$

Since $\mathcal{Q}, \mathcal{R}$ and $\mathcal{P}_N$ are SD with decay margin b, so are $\mathcal{L}_x$ and $\mathcal{L}_u$ with decay margin b. Thus, operators $\mathcal{P}$ and $\mathcal{H}$ are SD with decay margin b. Applying theorem 2 to $\mathcal{P}$, we get

$$\|[\mathcal{P}^{-1}]_{ki}\| \leq c_0\; \xi^{\mathrm{dis}(k,i)}$$

for $\xi \in (\frac{1}{b_1}, 1]$ and some $b_1 > 1$. Similarly, it follows that

$$\|[\left(\bar{\mathcal{G}}\mathcal{P}^{-1}\bar{\mathcal{G}}^*\right)^{-1}]_{ki}\| \leq c_1\; \xi^{\mathrm{dis}(k,i)}$$

for $\xi \in (\frac{1}{b_2}, 1]$ and some $b_2 > 1$. Using proposition 2 and the above results, we have

$$\|[\mathcal{K}_f]_{ki}\| \leq c\; \xi^{\mathrm{dis}(k,i)}$$

for some $c > 0$ and $\xi \in (\frac{1}{\tilde{b}}, 1]$ where $\tilde{b} = \min(b, b_1, b_2)$. A similar result holds for $\mathcal{K}_0$. This completes the proof. ∎

VI. Simulation Results

Consider a group of autonomous agents (e.g., points or particles), labeled 1 through 8 as in Fig. 2, all moving in the plane with the same speed but with different headings. Each agent is modeled by a simple kinematic model

$$\theta_i(t+1) \;=\; \theta_i(t) + u_i(t)$$

$$y_i(t) \;=\; \theta_i(t).$$

The objective is to align the velocity vectors of all agents and have them "flock" in the same direction. Each agents heading is updated using a receding horizon control scheme similar to that of (3)-(4) with weight matrices

$$\mathcal{Q} = \mathcal{L}^2 \quad , \quad \mathcal{R} = \mathrm{I} \quad , \quad \mathcal{P}_N \succ 0$$

where $\mathcal{L}$ is the graph Laplacian. Therefore the cost to be minimized represents the control effort as well as the misalignment energy between the velocity vectors. We select the terminal weighting cost such that $\mathcal{P}_N \succeq \mathcal{L}$ (see [30] for details). We choose $\mathcal{P}_N = \mathcal{L} + \varepsilon \mathrm{I}$ with $\varepsilon = 0.001$. The prediction horizon is $N = 4$ and the following constraints are imposed on the inputs

$$-2 \leq u_i(t) \leq 2 \quad , \quad 0 \leq t \leq N-1.$$

The Hybrid Toolbox [31] is used to run simulations for this problem. In simulations, range of states is defined to be $x_{\min} = -4$ and $x_{\max} = 4$. Software returned 14527 partitions for the admissible region and corresponding affine controllers. After careful evaluations, it turns out that number of different optimal control laws are 3048. Figure 3 shows the dependency of the optimal control law of agent number 1 to the other agents. Note that the distance function in this example is the hop-distance (shortest path between agents). Six different affine controllers are chosen randomly and norm of the feedback gain is depicted versus the spatial index in Fig. 3. As we can see in Fig. 3, the coupling in the optimal controller gains decays exponentially with the spatial index. The dependency of the optimal control of agent 1 to agent number 5, 6, 7 and 8 is negligible, and that agent 1 needs only to communicate with agents 2, 3 and 4.

VII. CONCLUSION

In this paper, we developed a formal study of receding horizon control (RHC) of spatially distributed systems that are defined over an arbitrary graph. The systems are assumed to be physically decoupled and heterogeneous. However the coupling between agents occurs through a finite horizon quadratic performance index which is defined through a spatially decaying operator. We proved that for spatially distributed systems with constraints, optimal receding horizon controllers are piece-wise affine functions of the initial condition (convolution sum). Furthermore, using tools from operator theory, we showed that the kernel of each convolution sum decays exponentially in the spatial domain. Thereby extending the previous analysis for infinite horizon optimal controllers defined over symmetric graphs to finite horizon constrained receding horizon control defined over arbitrary graphs. Future work will be focused on synthesis of localized receding horizon control of spatially distributed systems over arbitrary graphs.

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