Moment-Based Analysis of Synchronization in Small-World Networks of Oscillators

Victor M. Preciado and Ali Jadbabaie

Abstract— **In this paper, we investigate synchronization in a small-world network of coupled nonlinear oscillators. This network is constructed by introducing random shortcuts in a nearest-neighbors ring. The local stability of the synchronous state is closely related with the support of the eigenvalue distribution of the Laplacian matrix of the network. We introduce, for the first time, analytical expressions for the first three moments of the eigenvalue distribution of the Laplacian matrix as a function of the probability of shortcuts and the connectivity of the underlying nearest-neighbor coupled ring. We apply these expressions to estimate the spectral support of the Laplacian matrix in order to predict synchronization in small-world networks. We verify the efficiency of our predictions with numerical simulations.**

I. Introduction

In recent years, systems of dynamical nodes interconnected through a complex network have attracted a good deal of attention [15]. Biological and chemical networks, neural networks, social and economic networks [7], the power grid, the Internet and the World Wide Web [6] are examples of the wide range of applications that motivate this interest (see also [11], [3] and references therein). Several modeling approaches can be found in the literature [6], [17], [1]. In this paper, we focus our attention on the so-called small-world phenomenon and a model proposed by Newman and Strogatz to replicate this phenomenon.

Once the network is modeled, one is usually interested in two types of problems. The first involves *structural properties* of the model. The second involves the performance of *dynamical processes* run on those networks. In the latter direction, the performance of random walks [9], Markov processes [4], consensus in a network of agents [12], [8], or synchronization of oscillators [16], [13], are very well reported in the literature.

The *eigenvalue spectrum* of an undirected graph contains a great deal of information about structural and dynamical properties [5]. In particular, we focus our attention on the spectrum of the *(combinatorial) Laplacian matrix* uniquely associated with an undirected graph [2]. This spectrum contains useful information about, for example, the number of spanning trees, or the stability of synchronization of a network of oscillators. We analyze the low-order moments of the Laplacian matrix spectrum corresponding to small-world networks.

This work was supported by ONR MURI N000140810747, and AFOR's complex networks program.

The authors are with the Department of Electrical and Systems Engineering, University of Pennsylvania, 3451 Walnut Street, {preciado,jadbabai}@seas.upenn.edu

The paper is organized as follows. In Section II, we review the master-stability-function approach. In Section III, we derive closed-form expressions for the low-order moments of the Laplacian eigenvalue distribution associated with a probabilistic small-world network. Our expressions are valid for networks of asymptotically large size. Section IV applies our results to the problem of synchronization of a probabilistic small-world network of oscillators. The numerical results in this section corroborate our predictions.

II. Synchronization of Nonlinear Oscillators

In this section we review the master-stability-function (MSF) approach, proposed by Pecora and Carrol in [13], to study local stability of synchronization in networks of nonlinear oscillators. Using this approach, we reduce the problem of studying local stability of synchronization to the algebraic problem of studying the spectral support of the Laplacian matrix of the network. First, we introduce some needed graph-theoretical background.

A. Spectral Graph Theory Background

In the case of a network with symmetrical connections, undirected graphs provide a proper description of the network topology. An undirected graph G consists of a set of N nodes or vertices, denoted by $V = \{v_1, ..., v_n\}$, and a set of edges E, where $E \in V \times V$. In our case, $(v_i, v_j) \in E$ implies $(v_j, v_i) \in E$, and this pair corresponds to a single edge with no direction; the vertices v_i and v_j are called *adjacent* vertices (denoted by $v_i \sim v_j$) and are *incident* to the edge (v_i, v_j). We only consider simple graphs (i.e., undirected graphs that have no self-loops, so $v_i \neq v_j$ for an edge (v_i, v_j), and no more than one edge between any two different vertices). A *walk* on G of length k from v_0 to v_k is an ordered set of vertices $(v_0, v_1, ..., v_k)$ such that $(v_i, v_{i+1}) \in E$, for $i = 0, 1, ..., k-1$; if $v_k = v_0$ the walk is said to be *closed*.

The *degree* d_i of a vertex v_i is the number of edges incident to it. The *degree sequence* of G is the list of degrees, usually given in non-increasing order. The *clustering coefficient*, introduced in [17], is a measure of the number of triangles in a given graph, where a triangle is defined by the set of edges $\{(i,j),(j,k),(k,i)\}$ such that $i \sim j \sim k \sim i$. Specifically, we define clustering as the total number of triangles in a graph, $T(G)$, divided by the number of triangles in a complete (all-to-all) graph with N vertices, i.e., the coefficient is equal to $T(G) \Big/ \binom{N}{3}$.

It is often convenient to represent graphs via matrices. There are several choices for such a representation. For example, the *adjacency matrix* of an undirected graph G,

denoted by $A(G) = [a_{ij}]$, is defined entry-wise by $a_{ij} = 1$ if nodes i and j are adjacent, and $a_{ij} = 0$ otherwise. (Note that $a_{ii} = 0$ for simple graphs.) Notice also that the degree d_i can be written as $d_i = \sum_{j=1}^{N} a_{ij}$. We can arrange the degrees on the diagonal of a diagonal matrix to yield the *degree matrix*, $D = diag(d_i)$. The *Laplacian matrix* (also called Kirchhoff matrix, or combinatorial Laplacian matrix) is defined in terms of the degree and adjacency matrices as $L(G) = D(G) - A(G)$. For undirected graphs, $L(G)$ is a symmetric positive semidefinite matrix [2]. Consequently, it has a full set of N real and orthogonal eigenvectors with real non-negative eigenvalues. Since all rows of L sum to zero, it always admits a trivial eigenvalue $\lambda_1 = 0$, with corresponding eigenvector $\mathbf{v}_1 = (1, 1, ..., 1)^T$.

The moments of the Laplacian eigenvalue spectrum are central to our paper. Denote the eigenvalues of our $N \times N$ symmetric Laplacian matrix $L(G)$ by $0 = \lambda_1(G) \leq ... \leq \lambda_N(G)$. The k-th order moment of the eigenvalue spectrum of $L(G)$ is defined as:

$$q_k(G) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i(G)^k$$

(which is also called the k-th *order spectral moment*[1]).

In the following subsection, we illustrate how a network of identical nonlinear oscillators synchronizes whenever the Laplacian spectrum is contained in a certain region on the real line. This *region of synchronization* is exclusively defined by the dynamics of each isolated oscillator and the type of coupling [13]. This simplifies the problem of synchronization to the problem of locating the Laplacian eigenvalue spectrum.

B. Synchronization as a Spectral Graph Problem

Several techniques have been proposed to analyze the synchronization of coupled identical oscillators. We pay special attention to the master-stability-function (MSF) approach, [13]. This approach provides us with a criterion for local stability of synchronization based on the numerical computation of Lyapunov exponents. Even though quite different in nature, the mentioned techniques emphasize the key role played by the graph eigenvalue spectrum.

In this paper we consider a time-invariant network of N identical oscillators, one located at each node, linked with 'diffusive' coupling. The state equations modeling the dynamics of the network are

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \gamma \sum_{j=1}^{N} a_{ij} \Gamma (\mathbf{x}_j - \mathbf{x}_i), \; i = 1, ..., N \qquad (1)$$

where $\mathbf{x}_i$ represents an n-dimensional state vector corresponding to the i-th oscillator. The nonlinear function $\mathbf{f}(\cdot)$ describes the (identical) dynamics of the isolated nodes. The positive scalar γ can be interpreted as a global coupling strength parameter. The $n \times n$ matrix Γ represents how states in neighboring oscillators couple linearly, and a_{ij} are the entries of the adjacency matrix. By simple algebraic manipulations, one can write down Eq. (1) in terms of the Laplacian entries, $L(G) = [l_{ij}]$, as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \gamma \sum_{j=1}^{N} l_{ij} \Gamma \mathbf{x}_j, \text{ for } i = 1, ..., N. \qquad (2)$$

We say that the network of oscillators is at a synchronous equilibrium if $\mathbf{x}_1(t) = \mathbf{x}_2(t) = ... = \mathbf{x}_N(t) = \phi(t)$, where $\phi(t)$ represents a solution for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. In [13], the authors studied the local stability of the synchronous equilibrium. Specifically, they considered a sufficiently small perturbation, denoted by $\varepsilon_i(t)$, from the synchronous equilibrium, i.e.,

$$\mathbf{x}_i(t) = \phi(t) + \varepsilon_i(t).$$

After appropriate linearization, one can derive the following equations to approximately describe the evolution of the perturbations:

$$\dot{\varepsilon}_i = \mathbf{Df}(t)\, \varepsilon_i(t) - \gamma \sum_{j=1}^{n} l_{i,j} \Gamma \varepsilon_j(t), \text{ for } i = 1, ..., N. \qquad (3)$$

where $\mathbf{Df}(t)$ is the Jacobian of $\mathbf{f}(\cdot)$ evaluated along the trajectory $\phi(t)$. This Jacobian is an $n \times n$ matrix with time-variant entries. Following the methodology introduced in [13], Eq. (3) can be similarity transformed into a set of linear time-variant (LTV) ODEs of the form:

$$\dot{\xi}_i = [\mathbf{Df}(t) + (\gamma \lambda_i(G))\, \Gamma]\, \xi_i, \text{ for } i = 1, ..., N, \qquad (4)$$

where $\{\lambda_i(G)\}_{1 \leq i \leq N}$ is the set of eigenvalues of $L(G)$. Based on the stability analysis presented in [13], the network of oscillators in (1) presents a locally stable synchronous equilibrium if the corresponding maximal nontrivial Lyapunov exponents of (4) is negative for $i = 2, ..., N$.

Inspired in Eq. (4), Pecora and Carroll studied in [13] the stability of the following parametric LTV-ODE in the parameter σ:

$$\dot{\xi} = [\mathbf{Df}(t) + \sigma \Gamma]\, \xi, \qquad (5)$$

where $\mathbf{Df}(t)$ is the linear time-variant Jacobian in Eq. (3). The master stability function (MSF), denoted by $F(\sigma)$, is defined as the value of the maximal nontrivial Lyapunov exponent of (5) as a function of σ. Note that $F(\sigma)$ depends exclusively on $\mathbf{f}(\cdot)$ and Γ, and is independent of the coupling topology, i.e., independent of $L(G)$. The region of synchronization is, therefore, defined by the range of $\sigma > 0$ for which $F(\sigma) < 0$. For a broad class of systems, the MSF is negative in the interval $\sigma \in [0, \sigma_{\max}] \equiv S$ (although more generic stability sets are also possible, we assume, for simplicity, this is the case in subsequent derivations). In order to achieve synchronization, the set of scaled nontrivial Laplacian eigenvalues, $\{\gamma \lambda_i\}_{2 \leq i \leq N}$, must be located inside the region of synchronization, S. This condition is equivalent to: $\gamma \lambda_2 > 0$ and $\gamma \lambda_N < \sigma_{\max}$.

We illustrate how to use of the above methodology in the following example:

[1]Given that our interest is in networks of growing size (i.e., number of nodes N), a more explicit notation for μ and q_k would perhaps have been $\mu^{(N)}$ and $q_k^{(k)}$. However, for notational simplicity, we shall omit reference to N in there and other quantities in this paper.

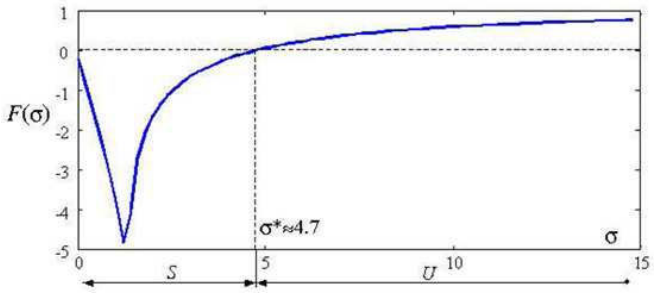

Fig. 1. Numerical values of the maximum Floquet exponent of Eqn. (7) for $\sigma \in [0,15]$, discretizing at intervals of length 0.2.

Example 1: Study the stability of synchronization of a ring of 6 coupled Rössler oscillators [10]. The dynamics of each oscillator is described by the following system of three nonlinear differential equations:

$$\begin{aligned} \dot{x}_i &= -(y_i + z_i), \\ \dot{y}_i &= x_i + a\,y_i, \\ \dot{z}_i &= b + z_i(x_i - c). \end{aligned}$$

The adjacency entries, a_{ij}, of a ring graph of six nodes are $a_{i,j} = 1$ if $j \in \{(i+1) \bmod 6, (i-1) \bmod 6\}$, for $i = 1,2,...,6$, and $a_{ij} = 0$ otherwise. The dynamics of this ring of oscillators are defined by:

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -(y_i + z_i) \\ x_i + a\,y_i \\ b + z_i(x_i - c) \end{bmatrix} + \gamma \sum_{j \in R(i)} \begin{bmatrix} x_j - x_i \\ 0 \\ 0 \end{bmatrix} \qquad (6)$$

where we have chosen to connect the oscillators through their x_i states exclusively. Our choice is reflected in the structure of the 3×3 matrix, Γ, inside the summation in Eqn. (6).

Numerical simulations of an isolated Rössler oscillator unveil the existence of a periodic trajectory with period $T = 5.749$ when the parameters in Eqn. (6) take the values $a = 0.2, b = 0.2$, and $c = 2.5$. We denote this periodic trajectory by $\phi(t) = [\phi_x(t), \phi_y(t), \phi_z(t)]$. In our specific case, the LTP differential equation (5) takes the following form:

$$\dot{\xi} = \left(\begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \phi_z(t) & 0 & c \end{bmatrix} + \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \xi, \qquad (7)$$

where the leftmost matrix in the above equation represents the Jacobian of the isolated Rössler evaluated along the periodic trajectory $\phi(t)$, and the rightmost matrix represents Γ.

In Fig. 1, we plot the numerical values of the maximum Floquet exponent of Eqn. (7) for $\sigma \in [0,15]$, discretizing at intervals of length 0.2. This plot shows the range in which the maximal Floquet exponent is negative. This range of stability is $S = (0, \sigma^*)$, for $\sigma^* \approx 4.7$. The MSF criterion introduced in [13] states that the synchronous equilibrium is locally stable if the set of values $\{\gamma\ \lambda_i(G)\}_{i=2,...,n}$ lies inside the stability range, S. For the case of a 6-ring configuration, the eigenvalues of $L(G)$ are $\{0,1,1,3,3,4\}$, so the set $\{\gamma\lambda_i\}_{i=2,...,n}$ is $\{\gamma, \gamma, 3\gamma, 3\gamma, 4\gamma\}$. Therefore, we achieve stability for $\gamma \in (0, \sigma^*/\lambda_n(G))$, where in our case $\sigma^*/\lambda_n(G) \approx 1.175$.

In the next subsection, we propose an approach to estimating the support of the eigenvalue distribution of large-scale probabilistic networks from low-order spectral moments. This allows us to predict synchronization in a large-scale Chung-Lu network.

III. Spectral Analysis of Small-World Networks

In this section we study the Laplacian eigenvalue spectrum of a variant of Watts-Strogatz small-world network [17]. After describing the model, we use algebraic graph theory to compute explicit expressions for the Laplacian moments of a small-world network as a function of its parameters. Our derivations are based on a probabilistic analysis of the expected spectral moments of the Laplacian for asymptotically large small-world networks.

A. Small-World Probabilistic Model

We consider a one-dimensional lattice of N vertices, $\{v_1, ..., v_N\}$, with periodic boundary conditions, i.e., on a ring, and connect each vertex v_i to its $2k$ closest neighbors, i.e., v_i is connected to the set of nodes $\{v_j : j \in [(i-k) \bmod N, (i+k) \bmod N]\}$. Then, instead of rewiring a fraction of the edges in the regular lattice as proposed by Watts and Strogatz [17], we add some random 'shortcuts' to the one-dimensional lattice. These shortcuts are added by independently assigning edges between each pair of nodes (i,j), $1 \le i < j \le N$ with probability p. The resulting small-world graph is intermediate between a regular lattice (achieved for $p = 0$) and a classical random graph (achieved for $p = 1$).

An interesting property observed in this model was the following: for small probability of rewiring, $p \ll 1$, the number of triangles in the network is nearly the same as that of the regular lattice, but the average shortest-path length is close to that of classical random graphs. In the rest of the paper we assume we are in the range of p in which this property holds, in particular, we will prescribe p to be r/N, for a given parameter r.

In the coming sections, we shall study spectral properties of the Laplacian matrix associated to the above small-world model. In our derivations we will need the probabilistic distribution for the degrees. It is well known that, for asymptotically large graphs, the degree distribution of a classical random graph with average degree r is a Poisson distribution with rate r. Hence, the degree distribution of the above small-world network is

$$\Pr(d_i = d) = \begin{cases} 0, & \text{for } d < 2k, \\ \frac{r^{d-2k} e^{-r}}{(d-2k)!}, & \text{for } d \ge 2k, \end{cases} \qquad (8)$$

which corresponds to a Poisson with parameter r 'shifted' $2k$ units. The Poisson distribution is shifted to take into account the degree of the regular $2k$-neighbors ring superposed to the random shortcuts.

Furthermore, it is well known that the clustering coefficient (or, equivalently, the number of triangles) of the regular $2k$-neighbors rings is very lightly perturbed by the addition of random shortcuts for $p = r/N$. In particular, one can prove the following expression for the expected number of triangles:

$$\mathbb{E}[T] = (1+o(1))\,\frac{1}{3}N\binom{2k}{2}, \tag{9}$$

where the dominant term, $\frac{1}{3}N\binom{2k}{2}$, corresponds to the exact number of triangles in a $2k$-neighbors ring with N nodes.

In the following section, we shall derive explicit expressions for the first low-order spectral moments of the Laplacian matrix associated with the small-world model herein described. Even though our analysis is far from complete, in that only low-order moments are provided, valuable information regarding spectral properties can be retrieved from our results.

B. Algebraic Analysis of Spectral Moments

In this section we deduce closed-form expressions for the first three moments of the Laplacian spectrum of any simple graph G. First, we express the spectral moments as a trace using the following identity:

$$q_k(G) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i(G)^k = \frac{1}{N}\mathrm{tr}\,[D-A]^k. \tag{10}$$

This identity is derived from the fact that trace is conserved under diagonalization (in general, under any similarity transformation). In the case of the first spectral moment, we obtain

$$q_1 = \frac{1}{N}\mathrm{tr}\,(D-A) = \frac{1}{N}\sum_{i=1}^{N} d_i.$$

where $\overline{d}$ is the average degree of the graph.

The fact that D and A do not commute forecloses the possibility of using Newton's binomial expansion on $(D-A)^k$. On the other hand, the trace operator allows us to cyclically permute multiplicative chains of matrices. For example, $\mathrm{tr}\,(AAD) = \mathrm{tr}\,(ADA) = \mathrm{tr}\,(DAA)$. Thus, for words of length $k \leq 3$, one can cyclically arrange all binary words in the expansion of (10) into the standard binomial expression:

$$q_k = \sum_{\alpha=0}^{k}\binom{k}{\alpha}\frac{(-1)^\alpha}{N}\mathrm{tr}\left(A^\alpha D^{k-\alpha}\right), \qquad \text{for } k \leq 3. \tag{11}$$

Also, we can make use of the identity $\mathrm{tr}\left(A^\alpha D^{k-\alpha}\right) = \sum_{i=1}^{N}(A^\alpha)_{ii}\,d_i^{k-\alpha}$ to write

$$q_k = \sum_{\alpha=0}^{k}\sum_{i=1}^{N}\binom{k}{\alpha}\frac{(-1)^\alpha}{N}d_i^{k-\alpha}\,(A^\alpha)_{ii}, \qquad \text{for } k \leq 3. \tag{12}$$

Note that this expression is not valid for $k \geq 4$. For example, for $k = 4$, we have that $\mathrm{tr}(AADD) \neq \mathrm{tr}(DADA)$.

We now analyze each summand in expression (12) from a graph-theoretical point of view. Specifically, we find a closed-form solution for each term $\mathrm{tr}\left(A^iD^j\right)$, for all pairs $1 \leq i+j \leq 3$, as a function of the degree sequence and the number of triangles in the network. In our analysis, we make use of the following result from [2]:

Lemma 2: Let G be a simple graph. Denote by t_i the number of triangles touching node i. Then,

$$(A)_{ii} = 0, \;\; \left(A^2\right)_{ii} = d_i, \text{ and } \left(A^3\right)_{ii} = 2t_i. \tag{13}$$

After substituting (13) into (12), and straightforward algebraic simplifications, we obtain the following exact expression for the low-order normalized spectral moments of a given Kirchhoff matrix L:

$$q_k = \begin{cases} 1, & \text{for } k=1, \\ \frac{1}{N}\left(\sum_{i=1}^{N} d_i^2 + \sum_{i=1}^{N} d_i\right), & \text{for } k=2, \\ \frac{1}{N}\left[\left(\sum_{i=1}^{N} d_i^3 + 3\sum_{i=1}^{N} d_i^2\right) - 6T\right], & \text{for } k=3, \end{cases} \tag{14}$$

where $T = \frac{1}{3}\sum_{i=1}^{N} t_i$ is the total number of triangles in the network.

C. Probabilistic Analysis of Spectral Moments

In this section, we use Eq. (14) to compute the first three expected Laplacian moments of the small-world model under consideration. The expected moments can be computed if we had explicit expressions for the moments of the degrees, $\mathbb{E}[d_i]$, $\mathbb{E}[d_i^2]$, and $\mathbb{E}[d_i^3]$, and the expected number of triangles, $\mathbb{E}[T]$. Since we know the degree distribution (8) for this model, the moments of the degrees can be computed to be:

$$\begin{aligned} \mathbb{E}[d_i] &= r+2k, \\ \mathbb{E}[d_i^2] &= r^2 + (1+4k)\,r + 4k^2, \\ \mathbb{E}[d_i^3] &= r^3 + 15\,(3+6k)\,r^2 + \left(1+6k+12k^2\right)\,r - 8k^3. \end{aligned} \tag{15}$$

We can therefore substitute the expressions (9) and (15) in Eq. (14) in order to derive the following expressions for the (non-normalized) expected Laplacian moments for $N \to \infty$:

$$\begin{aligned} \mathbb{E}[q_1] &= r+2k, \\ \mathbb{E}[q_2] &= r^2 + (4k+2)\,r + 4k^2 + 2k, \\ \mathbb{E}[q_3] &= r^3 + (6k+6)\,r^2 + \left(12k^2+18k+4\right)\,r \\ &\quad + 8k^2 + 8k^3 + 2k. \end{aligned} \tag{16}$$

In the following table we compare the numerical values of the Laplacian moments corresponding to one random realization of the model under consideration with the analytical predictions in (16). In particular, we compute the moments for a network of $N = 512$ nodes with parameters $p = r/N = 4/N$ and $k = 3$. It is important to point out that the indicated numerical values are obtained for one realization only, with no benefit from averaging.

Moment order	1^{st}	2^{nd}	3^{rd}
Numerical realization	10.14	116.96	1,467.6
Analytical expectations	10	114	1,431
Relative error	1.38%	2.53%	2.49%

In the next subsection, we use an approach introduced in [14] to estimate the support of the eigenvalue distribution using the first three spectral moments. In coming sections, we shall use this technique to predict whether the Laplacian spectrum lies in the region of synchronization.

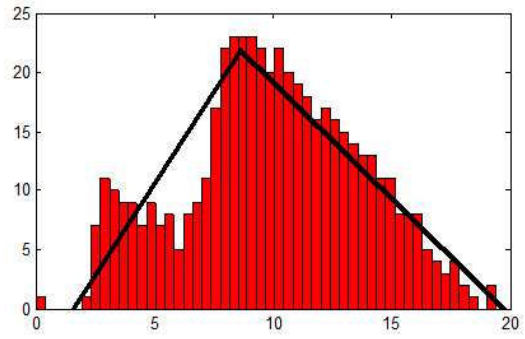

Fig. 2. Comparison between the histogram of the eigenvalues of one random realization of the Laplacian matrix of a small-world model with parameters $N = 512$, $p = 4/N$ and $k = 3$, and the triangular function that fits the expected spectral moments.

D. Piecewise-Linear Reconstruction of the Laplacian Spectrum

Our approach, described more fully in [14], approximates the spectral distribution with a triangular function that exactly preserves the first three moments. We define a triangular distribution $t(\lambda)$ based on a set of abscissae $x_1 \leq x_2 \leq x_3$ as

$$t(\lambda) := \begin{cases} \frac{h}{x_2 - x_1}(\lambda - x_1), & \text{for } \lambda \in [x_1, x_2), \\ \frac{h}{(x_2 - x_3)}(\lambda - x_3), & \text{for } \lambda \in [x_2, x_3], \\ 0, & \text{otherwise.} \end{cases}$$

where $h = 2/(x_3 - x_1)$. Our task is to find the set of values $\{x_1, x_2, x_3\}$ in order to fit a given set of moments $\{M_1, M_2, M_3\}$. The resulting system of equations is amenable to analysis. Following the methodology in [14], we can find the abscissae $\{x_1, x_2, x_3\}$ as roots of an explicitly defined polynomial.

The following example illustrates how this technique provides a reasonable estimation of the Laplacian spectrum for small-world Networks.

Example 3: Estimate the spectral support of the small-world model described in Subsection III-A for parameters $N = 512$, $p = 4/N$ and $k = 3$. In subsection III-C we computed the expected spectral moments of this particular network to be $\{M_1 = 10, M_2 = 114, M_3 = 1{,}431\}$. Thus, we apply the above technique with these particular values of the moments to compute the following set of abscissae for the triangular reconstruction $\{x_1 = 1.577, x_2 = 8.662, x_3 = 19.76\}$. In Fig. 2 we compare the triangular function that fits the expected spectral moments with the histogram of the eigenvalues of one random realization of the Laplacian matrix. We also observe that any random realization of the eigenvalue histograms of the Laplacian is remarkably close to each other. Although a complete proof of this phenomenon is beyond the scope of this paper, one can easily proof using the law of large numbers that the distribution of spectral moments in (14) concentrate around their mean values.

In the next section, we propose a methodology which uses results presented in previous sections to predict the local stability of the synchronous state in a small-world network of oscillators.

IV. Analytical Estimation of Synchronization

In this section we use the expressions in (16) and the triangular reconstruction in the above subsection to predict synchronization in a large small-world network of coupled nonlinear oscillators. Specifically, we study a network of coupled Rössler oscillators, as those in Example 1. We build our prediction based on the following steps:

1) Determine the *region of synchronization* following the technique presented in Subsection II-B.
2) Compute the *expected spectral moments* of the Laplacian eigenvalue spectrum for a given set of parameters using the set of Eqns. in (16).
3) Estimate the *support of the Laplacian eigenvalue spectrum,* $\{\lambda_i^{(K)}\}_{i=2,\ldots,N}$, using the methodology presented in Subsection III-D.
4) Compare the region of stability in **Step 1** with the estimation of the spectral support in **Step 3**.

Following the above steps, one can easily verify that our estimated spectral support, $(1.57\ \gamma, 19.76\ \gamma)$, lies inside the region of stability, $(0, \sigma^* \approx 4.7)$, for $0 < \gamma < 4.7/19.76 \approx 0.238$. Therefore, the small-world network of 512 coupled Rössler oscillators is predicted to synchronize whenever the global coupling strength satisfies $\gamma \in (0, 0.238)$.

A. Numerical Results

In this section we present numerical simulations supporting our conclusions. We consider a set of identical 512 Rössler oscillators (as the one described in Example 1) interconnected through the Small–World network defined in Example ($p = 4/N$ and $k = 3$). Using the methodology proposed above, we have predicted that the synchronous state of this system is locally stable if the coupling parameter γ lies in the interval $(0, 0.238)$. We run several simulations with the dynamics of the oscillators presenting different values of the global coupling strength γ. For each coupling strength, we present two plots: *(i)* the evolution of the 512 x-states of the Rössler oscillators in the time interval $0 \leq t \leq 40$, and *(ii)* the evolution of $x_i(t) - \bar{x}(t)$ for all i, where $\bar{x}(t) = \frac{1}{N}\sum_i x_i(t)$.

In our first simulation, we use a coupling strength $\gamma = 0.1 \in (0, 0.238)$; thus, we predict the synchronous state to be locally stable. Fig. 3 (a) and (b) represents the dynamics x-states for the 512 oscillators in the small-world network. In this case, we observe a clear exponential convergence of the errors to zero. In the second simulation, we choose $\gamma = 0.3 \notin (0, 0.238)$; thus, we predict the synchronous state to be unstable. In fact, we observe in Figs. 4.a and 4.b how synchronization is clearly not achieved.

V. Conclusions and Future Research

In this paper, we have studied the eigenvalue distribution of the Laplacian matrix of a large-scale small-world networks. We have focused our attention on the low-order moments of the spectral distribution. We have derived explicit expressions of these moments as functions of the parameters in the small-world model. We have then applied our results to the problem of synchronization of a network

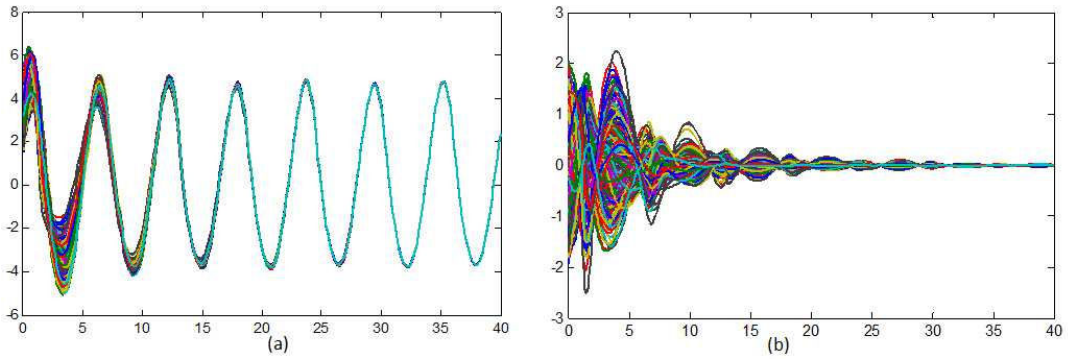

Fig. 3. We plot the dynamics of the x-states for 512 Rössler oscillators (as the one described in Example 1) interconnected through the Small–World network with $p = 4/N$ and $k = 3$, in Fig.a. In Fig.b, we observe a clear exponential convergence of the errors towards zero.

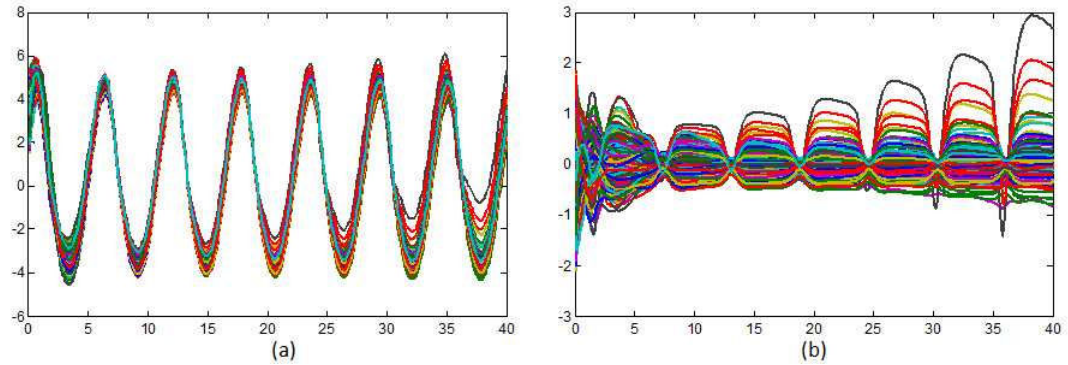

Fig. 4. We plot the dynamics of the x-states for the 512 Rössler oscillators for $\gamma = 0.3 \notin (0, 0.238)$ in Fig.a. We observe in Fig.b. how the errors do not converge to zero.

of nonlinear oscillators. Using our expressions, we have studied the local stability of the synchronous state in a large-scale small-world network of oscillators. Our approach is based on performing a triangular reconstruction matching the first three moments of the unknown spectral measure. Our numerical results match our predictions with high accuracy. Several questions remain open. The most obvious extension would be to derive expressions for higher-order moments of the Kirchhoff spectrum. A more detailed reconstruction of the spectral measure can be done based on more moments.

VI. ACKNOWLEDGMENTS

The first author gratefully acknowledges George C. Verghese and Vincent Blondel for their comments and suggestions on this work.

REFERENCES

[1] A. L. Barabási, and R. Albert, "Emergence of Scaling in Random Networks," *Science*, vol. 285, pp. 509-512, 1999.

[2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, second edition, 1993.

[3] S. Boccaletti S., V. Latora, Y. Moreno, M. Chavez, and D.-H. Hwang, "Complex Networks: Structure and Dynamics," *Physics Reports*, vol. 424, no. 4-5, pp. 175-308, 2006.

[4] P. Bremaud, *Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues*, Springer, 2001.

[5] F.R.K. Chung, *Spectral Graph Theory*, AMS: CBMS series, vol. 92, 1997.

[6] S.N. Dorogovtsev, and J.F.F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW*, Oxford University Press, 2003.

[7] M.O. Jackson, *Social and Economic Networks*, Princeton University Press, 2008.

[8] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor," *IEEE Trans. Autom. Control*, vol. 50, no.1, 2003.

[9] L. Lovász, "Random Walks on Graphs: A Survey," *Combinatorics*, Paul Erdös is Eighty (vol. 2), pp. 1-46, 2003.

[10] S.C. Manrubia, A.S. Mikhailov, and D. Zanette, *Emergence of Dynamical Order*, World Scientific, 2004.

[11] M.E.J. Newman, "The Structure and Function of Complex Networks," *SIAM Review* vol. 45, pp. 167-256, 2003.

[12] R. Olfati-Saber, J.A. Fax, and R.M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proc. IEEE*, vol. 95, pp. 215-233, 2007.

[13] L.M. Pecora, and T.L. Carroll, "Master Stability Functions for Synchronized Coupled Systems," *Phys. Rev. Lett.*, vol. 80, no. 10, pp. 2109-2112, 1998.

[14] V.M. Preciado, *Spectral Analysis for Stochastic Models of Large-Scale Complex Dynamical Networks*, Ph.D. dissertation, Dept. Elect. Eng. Comput. Sci., MIT, Cambridge, MA, 2008.

[15] S.H. Strogatz, "Exploring Complex Networks," *Nature*, vol. 410, pp. 268-276, 2001.

[16] S.H. Strogatz, *Sync: The Emerging Science of Spontaneous Order*, New York: Hyperion, 2003.

[17] D.J. Watts, and S. Strogatz, "Collective Dynamics of Small World Networks," *Nature,* vol 393, pp. 440-42, 1998.