Available online at www.sciencedirect.com
SCIENCE DIRECT®
Systems & Control Letters 55 (2006) 293–303
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Input-to-state stable finite horizon MPC for neutrally stable linear discrete-time systems with input constraints☆

Jung-Su Kim[a], Tae-Woong Yoon[b,*], Ali Jadbabaie[c], Claudio De Persis[d]

[a]ASRI, School of Electrical Engineering and Computer Science, Seoul National University, Kwanak P.O. Box 34, Seoul 151-600, Republic of Korea
[b]Department of Electrical Engineering, Korea University, Anam-dong 5-1, Sungbuk-ku, Seoul 136-701, Republic of Korea
[c]Department of Electrical and Systems Engineering, University of Pennsylvania, 365 GRW Moore Bldg. 200 South 33rd Street Philadelphia, PA 19104, United States
[d]Dipartimento di Informatica e Sistemistica "A. Ruberti", Università degli Studi di Roma "La Sapienza", Via Eudossiana, 18 00184 Roma, Italy

Received 19 March 2004; received in revised form 28 June 2005; accepted 15 August 2005
Available online 19 September 2005

Abstract

MPC or model predictive control is representative of control methods which are able to handle inequality constraints. Closed-loop stability can therefore be ensured only locally in the presence of constraints of this type. However, if the system is neutrally stable, and if the constraints are imposed only on the input, global asymptotic stability can be obtained; until recently, use of infinite horizons was thought to be inevitable in this case. A globally stabilizing finite-horizon MPC has lately been suggested for neutrally stable continuous-time systems using a non-quadratic terminal cost which consists of cubic as well as quadratic functions of the state. The idea originates from the so-called small gain control, where the global stability is proven using a non-quadratic Lyapunov function. The newly developed finite-horizon MPC employs the same form of Lyapunov function as the terminal cost, thereby leading to global asymptotic stability. A discrete-time version of this finite-horizon MPC is presented here. Furthermore, it is proved that the closed-loop system resulting from the proposed MPC is ISS (Input-to-State Stable), provided that the external disturbance is sufficiently small. The proposed MPC algorithm is also coded using an SQP (Sequential Quadratic Programming) algorithm, and simulation results are given to show the effectiveness of the method.

Keywords: Model predictive control (MPC); Input constraints; Global stability; Input-to-state stability (ISS); Non-quadratic Lyapunov function

1. Introduction

MPC or model predictive control is a receding horizon strategy, where the control is computed via an optimization procedure at every sampling instant. It is therefore possible to handle physical constraints on the input and/or state variables through the optimization [21]. Over the last decade, there have been many stability results on constrained MPC. Moreover, explicit solutions to constrained MPC are proposed recently [22,4]. These results reduce on-line computational burden regarded as a main drawback of MPC, and extend the applicability of MPC to faster plants as in electrical applications.

Particular attention is paid in this paper to input-constrained systems as all real processes are subject to actuator saturation. Generally, it is not possible to stabilize input-constrained plants globally. However, if the unconstrained part of the system is neutrally stable,[1] then global stabilization can be achieved. A typical example is the so-called small gain control [23,3,5]; it is noted that the

☆ This work was performed while the first author was a postgraduate student at Korea University.
* Corresponding author. Tel.: +82 2 3290 3240; fax: +82 2 928 8909.
E-mail address: twy@korea.ac.kr (T.-W. Yoon).

[1] All eigenvalues lie within the unit circle and those on the unit circle are simple.

0167-6911/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2005.08.005

Lyapunov functions used for stability analysis are non-quadratic functions containing cubic as well as quadratic terms.

Global stabilization of input-constrained neutrally stable systems is also possible via MPC; see e.g. [7]. As in [7], use of infinite horizons is generally thought to be inevitable. However, infinite horizon MPC can cause trouble in practice. For implementation, the optimization problem should be reformulated as a finite horizon MPC with a variable horizon, and it is not possible to predetermine a finite upper bound on the horizon in the presence of disturbances.

It is only fairly recently that globally stabilizing finite horizon MPC has been proposed for continuous-time neutrally stable systems [12]. This late achievement is based on two observations; firstly, the stability of an MPC system is mostly proved by showing that the terminal cost is a control Lyapunov function [21,14]. Secondly, the global stabilization of an input-constrained neutrally stable system can be achieved by using a non-quadratic Lyapunov function as mentioned above. By making use of these two facts, a new finite horizon MPC has been suggested in [12], where a non-quadratic Lyapunov function as in [23,3,5] is employed as the terminal cost, thereby guaranteeing the closed-loop stability. Here, we present a discrete-time version of this newly developed finite-horizon MPC in [12].

Recently, input-to-state stability (ISS) and its integral variant, integral-input-to-state stability (iISS) have become important concepts in nonlinear systems analysis and design [15,1,2]. ISS and iISS imply that the nominal system is globally stable, and that the closed-loop system is robust against a bounded disturbance and a disturbance with finite energy, respectively. There have been some reports on ISS properties of MPC [20,18,13]. However, these results are limited in that plants are assumed to be open-loop stable in [13], and only local properties are obtained in [20,18].

This paper presents a globally stabilizing MPC for input-constrained neutrally stable discrete-time plants, which is also (globally) ISS with a restriction on the external disturbance. The rest of the paper is organized as follows: Section 2 gives a brief summary on MPC. The stability and ISS properties of the proposed MPC are then obtained in Sections 3 and 4, respectively, by showing that the optimal cost with a non-quadratic terminal cost is an ISS Lyapunov function. The proposed MPC is coded using a sequential quadratic programming (SQP) algorithm, and simulation results are given to show the effectiveness of the method in Section 5. Finally, Section 6 concludes the paper.

2. An overview of MPC

Following [21], a brief summary on MPC is given in this section. Consider a discrete-time system described by

$$x^{+} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, x^+ the successor state (i.e. state at the next sampling instant), $u \in \mathbb{R}^m$ the control input, and (A, B) a controllable pair. Defining

$$\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}, \tag{2}$$

the MPC law is obtained by minimizing with respect to $\mathbf{u}$

$$J_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} l(x(i), u(i)) + V(x(N))$$

subject to

$$\begin{aligned} &x(i+1) = Ax(i) + Bu(i), \ \ x(0) = x, \\ &x(i) \in \mathscr{X}, \quad u(i) \in \mathscr{U}, \ \ i \in [0, N-1], \\ &x(N) \in \mathscr{X}_f \subset \mathscr{X}, \end{aligned}$$

where

$$l(x(i), u(i)) = x(i)^{\mathrm{T}} Q x(i) + u(i)^{\mathrm{T}} R u(i) \tag{3}$$

with Q and R being positive definite, $V(x(N))$ is the terminal cost, the sets $\mathscr{U}$, $\mathscr{X}$ represent the input and state constraints, and $x(N) \in \mathscr{X}_f$ is the artificial terminal constraint employed for stability guarantees. Note that $V(x)$ and $\mathscr{X}_f$ are chosen such that $V(x)$ is a control Lyapunov function in $\mathscr{X}_f$. This minimization problem, referred to as $\mathscr{P}_N(x)$, yields the optimal control sequence

$$\mathbf{u}^*(x) = \{u^*(0; x), u^*(1; x), \ldots, u^*(N-1; x)\}, \tag{4}$$

and the optimal cost

$$J_N^*(x) = J_N(x, \mathbf{u}^*(x)). \tag{5}$$

Then the MPC law, denoted by $k_N(\cdot)$, is written as

$$k_N(x) = u^*(0; x). \tag{6}$$

The entire procedure is repeated at every sampling instant. The stability properties of the resulting closed-loop are summarized below.

Theorem 1 (*Mayne et al. [21]*). *For some local controller* $k_f : \mathscr{X}_f \to \mathbb{R}^m$, *suppose the following*:

A1. $\mathscr{X}_f$ *is closed, and* $0 \in \mathscr{X}_f \subset \mathscr{X}$;
A2. $k_f(x) \in \mathscr{U}, \ \forall x \in \mathscr{X}_f$ (*feasibility*);
A3. $Ax + Bk_f(x) \in \mathscr{X}_f, \ \forall x \in \mathscr{X}_f$ (*invariance*);
A4. $V(Ax + Bk_f(x)) - V(x) + l(x, k_f(x)) \leqslant 0, \ \forall x \in \mathscr{X}_f$.

Then the optimization problem is guaranteed to be feasible at all times as long as the initial state can be steerable to $\mathscr{X}_f$ *in N steps or less while satisfying the control and state constraints* (*i.e. the problem is initially feasible*). *In addition, the optimal cost* $J_N^*(x)$ *is monotonically non-increasing such that*

$$\begin{aligned} J_N^*(x^+) &= J_N^*(Ax + Bk_N(x)) \\ &\leqslant J_N^*(x) - l(x, k_N(x)), \end{aligned} \tag{7}$$

thereby ensuring asymptotic convergence of the closed-loop state to zero.

Outline of proof. Suppose that the problem $\mathscr{P}_N(x)$ is feasible, and thus the optimal sequence in (4) exists. Consider for the problem $\mathscr{P}_N(x^+)$

$$\tilde{\mathbf{u}} = \{u^*(1;x),\ldots,u^*(N-1;x),k_f(x^*(N;x))\} \tag{8}$$

where $x^*(N;x)$ is given by the recursion

$$x^*(i+1;x) = Ax^*(i;x) + Bu^*(i;x),\quad i \in [0, N-1]. \tag{9}$$

Note that $x^*(1;x) = x^+$ as $u^*(0;x) = k_N(x)$, and that $x^*(N;x) \in \mathscr{X}_f$. It then follows from A1 and A2 that $\tilde{\mathbf{u}}$ is also feasible for the problem $\mathscr{P}_N(x^+)$. This implies that if the optimization problem is feasible at some time, then the feasibility is ensured from that time on. Also, from assumption A4, we have

$$\begin{aligned} &J_N^*(x^+)\\ &\quad = J_N^*(Ax+Bk_N(x)) \leqslant J_N(Ax+Bk_N(x),\tilde{\mathbf{u}})\\ &\quad = J_N^*(x) - l(x,k_N(x)) + l(x^*(N;x),k_f(x^*(N;x))\\ &\qquad + V(Ax^*(N;x)+Bk_f(x^*(N;x)))\\ &\qquad - V(x^*(N;x)) \leqslant J_N^*(x) - l(x,k_N(x)). \end{aligned}$$

This completes the proof. □

Remark 1. In addition to the conditions in Theorem 1, if there exist two $\mathscr{K}_\infty$ functions[2] $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(\|x\|) \leqslant J_N^*(x) \leqslant \alpha_2(\|x\|), \tag{10}$$

then asymptotic stability of the closed-loop also results from (7).

Theorem 1 shows that if $\mathscr{X}_f$ is a feasible and invariant set for $x^+ = Ax + Bk_f(x)$, MPC is stabilizing and its domain of attraction is the set of the initial state vectors which can be steerable to $\mathscr{X}_f$ in N steps or less while satisfying the control and state constraints. An interesting consequence is that the MPC can be globally stabilizing if $k_f(x)$ is found such that $\mathscr{X}_f = \mathbb{R}^n$. This is in fact possible if the unconstrained plant is neutrally stable and if constraints are imposed only on the input, i.e. $\mathscr{X} = \mathbb{R}^n$, as discussed on small gain control in the introduction.

3. Globally stabilizing MPC for input-constrained neutrally stable systems

3.1. A non-quadratic Lyapunov function for global stability

We first present a slight extension of the previous results [23,3], in which the case where all poles are simple on the unit circle is considered first and then the general case for neutrally stable systems is handled via coordinate transformations. On the other hand, neutrally stable systems are directly dealt with in this paper.

Consider the following neutrally stable plant

$$x^+ = Ax + B\,\mathrm{sat}(u), \tag{11}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and it is assumed as in the previous section that (A, B) is controllable and all the eigenvalues of A lie within and on the unit circle with those on the unit circle being simple. The saturation function, denoted by $\mathrm{sat}(\cdot)$, is defined as follows:

$$\mathrm{sat}(u) = [\mathrm{sat}(u_1)\ \mathrm{sat}(u_2) \cdots \mathrm{sat}(u_m)]^{\mathrm{T}},$$

where

$$\mathrm{sat}(u_i) = \begin{cases} u_{\max}, & u_i > u_{\max} \\ u_i, & |u_i| \leqslant u_{\max} \\ -u_{\max}, & u_i < -u_{\max} \end{cases}$$

and $u_{\max}$ is a positive constant. Then for any L_u satisfying $L_u u_{\max} > 1$, we have

$$\|\mathrm{sat}(u) - u\| \leqslant L_u u^{\mathrm{T}} \mathrm{sat}(u). \tag{12}$$

It also follows from the neutral stability that there exists a positive definite matrix M_c satisfying [6,3]

$$A^{\mathrm{T}} M_c A - M_c \leqslant 0. \tag{13}$$

Now globally stabilizing small gain control [23,6,3] is given by

$$u = -\kappa B^{\mathrm{T}} M_c A x =: Kx, \tag{14}$$

where $\kappa\,(>0)$ satisfies

$$\kappa B^{\mathrm{T}} M_c B < I. \tag{15}$$

This control law is similar to those in [23,3,6]. It can then be shown that there exists a positive definite matrix M_q such that

$$(A+BK)^{\mathrm{T}} M_q (A+BK) - M_q = -I. \tag{16}$$

The stability properties of the resulting closed-loop are given below.

Theorem 2. *For the closed-loop system* (11) *and* (14), *there exists a Lyapunov function* $W(x)$ *such that*

$$\begin{aligned} &W(x) = W_q(x) + \lambda W_c(x) = x^{\mathrm{T}} M_q x + \lambda (x^{\mathrm{T}} M_c x)^{3/2}\\ &W(Ax + B\,\mathrm{sat}(Kx)) - W(x) \leqslant -\|x\|^2, \end{aligned} \tag{17}$$

where K *is defined in* (14), *and*

$$\lambda = \frac{2\kappa L_u \sigma_{\max}(A_c^{\mathrm{T}} M_q B)}{\sqrt{\lambda_{\min}(M_c)}} \tag{18}$$

with $\sigma_{\max}$ *and* $\lambda_{\min}$ *denoting the maximum singular value and the minimum eigenvalue, respectively.*

[2] A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$ [19].

Proof. It follows from (17) that there exist $\mathscr{K}_\infty$ functions α_{W1} and α_{W2} such that

$$\alpha_{W1}(\|x\|) \leqslant W(x) \leqslant \alpha_{W2}(\|x\|). \tag{19}$$

Note that the Lyapunov function in (17) consists of a cubic term ($W_c(x)$) as well as a conventional quadratic term ($W_q(x)$). Now, we consider the difference of the quadratic term W_q along the trajectories, which is given by

$$\begin{aligned} &W_q(Ax + B\text{sat}(Kx)) - W_q(x) \\ &\quad \leqslant -\|x\|^2 + 2a_1 L_u \|x\| (Kx)^{\mathrm{T}} \text{sat}(Kx) \end{aligned}$$

where $a_1 = \sigma_{\max}(A^{\mathrm{T}} M_q B)$. Then, after some manipulations similar to those in [23], we obtain the difference of the cubic term W_c as follows:

$$\begin{aligned} &W_c(Ax + B\text{sat}(Kx)) - W_c(x) \\ &\quad \leqslant -\frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \text{sat}(Kx), \end{aligned}$$

where $a_2 = \sqrt{\lambda_{\min}(M_c)}$. From these differences of the quadratic and cubic components of the Lyapunov function, it is clear that if λ is chosen to satisfy (18), then we have

$$W(Ax + B\text{sat}(Kx)) - W(x) \leqslant -\|x\|^2.$$

This, together with the inequality in (19), completes the proof [24] (pp. 266–267). $\square$

3.2. A globally stabilizing MPC

On the basis of the discussions in Sections 2 and 3.1, we derive a globally stabilizing MPC for the plant in (11). The key idea is to use a non-quadratic function of the form (17) as the terminal cost; since (17) is a global Lyapunov function, the resulting MPC can be globally stabilizing in view of A4 of Theorem 1.

For the neutrally stable plant in (11), consider the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad J_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} l(x(i), u(i)) + V(x(N)) \\ &\text{subject to} \\ &\qquad x(i+1) = Ax(i) + Bu(i), \;\; x(0) = x \\ &\qquad u(i) = \text{sat}(u(i)), \;\; i \in [0, N-1], \end{aligned} \tag{20}$$

where

$$V(x(N)) = \Theta W(x(N)),$$

with $W(x)$ as in Eq. (17), $\mathbf{u}$ and $l(x, u)$ are defined in Eqs. (2) and (3), N is a positive integer, and Θ is a positive constant to be specified below. For the optimal control sequence, the optimal cost, and the MPC law resulting from this minimization problem, we use the same symbols as in (4), (5), and (6), respectively, for the purpose of presentation.

Theorem 3. *Consider the neutrally stable plant in* (11) *and the MPC law $k_N(x)$ resulting from the optimization problem* (20). *Then, given any positive integer N, the closed-loop system $x^+ = Ax + Bk_N(x)$ is globally asymptotically stable for some positive Θ.*

Proof. We first show that all the assumptions in A.1–A.4 of Theorem 1 are satisfied for $\mathscr{X}_f = \mathscr{X} = \mathbb{R}^n$. Note that assumptions A.1 and A.3 trivially hold; we thus find Θ such that assumption A.2 and A.4 are satisfied. To this end, choose

$$k_f(x) = \text{sat}(Kx), \tag{21}$$

where K is defined in (14)–(15). It is now clear that A.2 holds. Consider also

$$\begin{aligned} l(x, k_f(x)) &\leqslant x^{\mathrm{T}} Q x + \text{sat}(Kx)^{\mathrm{T}} R \text{sat}(Kx) \\ &\leqslant x^{\mathrm{T}} Q x + \kappa^2 x^{\mathrm{T}} A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A x \\ &\leqslant \lambda_{\max}(Q + \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A) \|x\|^2. \end{aligned}$$

Hence, in view of the inequality in (7), if Θ is chosen such that

$$\Theta \geqslant \lambda_{\max}(Q + \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A),$$

we have

$$\begin{aligned} &V(Ax + Bk_f(x)) - V(x) \leqslant -l(x, k_f(x)), \\ &\forall x \in \mathscr{X}_f = \mathbb{R}^n. \end{aligned}$$

This leads to

$$J_N^*(Ax + Bk_N(x)) - J_N^*(x) \leqslant -l(x, k_N(x)) \tag{22}$$

for all $x \in \mathbb{R}^n$.

In addition, the optimal cost $J_N^*(x)$ can be shown to satisfy such an inequality as in (10), with an upper bound resulting from a simple substitution, say $u(i) = 0$, into the cost, and an obvious lower bound of $x^{\mathrm{T}} Q x$. This, together with the inequality in (22), completes the proof (see Remark 2 of Section 2). $\square$

Remark 2. The proposed MPC is no longer a quadratic programming (QP) optimization problem. However, it is still convex, and can thus be dealt with effectively via various convex optimization solvers. For example, we employ a sequential quadratic programming (SQP) algorithm for simulations in Section 5.

Remark 3. In [7], infinite horizon MPC is needed in order to achieve global asymptotic stability. However, *any fixed finite* horizon can be used in the proposed MPC.

Remark 4. Recently a number of results have been presented on stability of nonlinear systems in the sense of ISS or iISS. Among these is an interesting report proving that global stability is equivalent to iISS in discrete-time [1]. Hence the proposed MPC is actually integral-input-to-state-stabilizing [1,16], i.e. the state of the resulting closed-loop is guaranteed to be bounded even in the presence of an external disturbance provided that the disturbance has finite energy. A consequence of this iISS property is that it may be possible to design a switching-based adaptive MPC for uncertain input-constrained neutrally stable systems. See e.g. [10] and [8] for switching-based adaptive control of nonlinear systems with iISS properties.

4. ISS of the proposed MPC

It is well known that ISS is impossible to achieve for input-constrained neutrally stable systems in general. In this section, we prove ISS with a restriction on the external disturbance, i.e. ISS under the assumption that the ∞-norm of the disturbance is sufficiently small, by finding an ISS Lyapunov function for the proposed MPC scheme. To this end, firstly ISS of the small gain control is obtained, which is then used to derive an ISS characterization of the proposed MPC. To begin with, the definition of ISS and a theorem regarding the Lyapunov characterization of ISS are stated below, before presenting our main results.

Consider the following discrete-time nonlinear system with an external disturbance

$$x^+ = f(x, w). \tag{23}$$

Definition 1 (*Jiang and Wang [15]*). The system in (23) is ISS if there exist $\gamma \in \mathscr{K}_\infty$ and $\beta \in \mathscr{KL}$[3] such that for each input $w(\cdot) \in l_\infty$ and each x_0, it holds that

$$\|x(k, x_0, w(\cdot))\| \leqslant \beta(\|x_0\|, k) + \gamma(\|w(\cdot)\|), \tag{24}$$

where $x(k, x_0, w(\cdot))$ is the solution of (23) at time k, with x_0 being the initial state at time 0.

Theorem 4 (*Jiang and Wang [15]*). *The system* (23) *is ISS if and only if there exists a continuous ISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that for $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathscr{K}_\infty$, V satisfies*

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|)$$

and

$$V(x^+) - V(x) \leqslant -\alpha_3(\|x\|) + \sigma(\|w\|). \tag{25}$$

4.1. ISS of the small gain control

In this subsection, an ISS characterization of the small gain control is derived. Consider the following discrete-time neutrally stable plant with input saturation

$$x^+ = Ax + B\mathrm{sat}(u) + w, \tag{26}$$

where w is an external disturbance as in (23). The next theorem shows that the small gain control (14) stabilizes (26) in the sense of ISS with a restriction on w.

[3] A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathscr{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathscr{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$ [19].

Theorem 5. *For the closed-loop system* (26) *and* (14), *the function $W(x)$ given in* (17) *and* (18) *is an ISS Lyapunov function such that, for all $x \in \mathbb{R}^n$, and for all w satisfying $\|w\| \leqslant \delta$ with some positive δ*

$$\begin{aligned} &W(Ax + B\mathrm{sat}(x) + w) - W(x) \\ &\quad \leqslant -\varepsilon_W\|x\|^2 + \beta_{w1}\|w\| + \beta_{w2}\|w\|^2, \end{aligned} \tag{27}$$

where ε_W, β_{w1} and β_{w2} are positive.

Proof. Proceeding as in [23,3], we obtain the differences of the quadratic and cubic terms as follows:

$$\begin{aligned} &W_q(Ax + B\mathrm{sat}(Kx) + w) - W_q(x) \\ &\quad \leqslant -\|x\|^2 + a_1 2\|x\| L_u(Kx)^{\mathrm{T}}\mathrm{sat}(Kx) + \varepsilon_{xq}\|x\|^2 \\ &\qquad + \beta_{wq2}\|w\|^2 + \beta_{wq1}\|w\| \end{aligned} \tag{28}$$

$$\begin{aligned} &W_c(Ax + B\mathrm{sat}(Kx) + w) - W_c(x) \\ &\quad \leqslant 2c_1c_2\|w\|\|x\|^2 + \varepsilon_{xc}\|x\|^2 + \beta_{wc1}\|w\| \\ &\qquad + \beta_{wc2}\|w\|^2 - \frac{a_2}{\kappa}\|x\|(Kx)^{\mathrm{T}}\mathrm{sat}(Kx), \end{aligned} \tag{29}$$

where ε_{xq} and ε_{xc} are arbitrarily small positive constants, β_{wq1}, β_{wq2}, β_{wc1} and β_{wc2} are positive, a_1 and a_2 are defined as in Theorem 2, $c_1 = 2\sigma_{\max}(A^{\mathrm{T}}M_c)$, and $c_2 = \sqrt{\lambda_{\max}(M_c)}$. See Appendix A for the details of deriving (28) and (29). With λ in (18), we obtain

$$\begin{aligned} &W(Ax + B\mathrm{sat}(Kx) + w) - W(x) \\ &\quad \leqslant (-1 + \lambda 2c_1c_2\|w\|)\|x\|^2 + \varepsilon_x\|x\|^2 \\ &\qquad + \beta_{w2}\|w\|^2 + \beta_{w1}\|w\| \\ &\quad \leqslant (-1 + \lambda 2c_1c_2\delta)\|x\|^2 + \varepsilon_x\|x\|^2 + \beta_{w2}\|w\|^2 \\ &\qquad + \beta_{w1}\|w\|, \end{aligned}$$

where $\varepsilon_x = \varepsilon_{xq} + \lambda\varepsilon_{xc}$, $\beta_{w1} = \beta_{wq1} + \lambda\beta_{wc1}$, and $\beta_{w2} = \beta_{wq2} + \lambda\beta_{wc2}$. Note that ε_x can be made arbitrarily small as ε_{xq} and ε_{xc} are arbitrarily small. Now for δ such that

$$-1 + \lambda 2c_1c_2\delta \leqslant -\varepsilon_W - \varepsilon_x \tag{30}$$

with $0 < \varepsilon_W < 1$, we have

$$\begin{aligned} &W(Ax + B\mathrm{sat}(Kx) + w) - W(x) \\ &\quad \leqslant -\varepsilon_W\|x\|^2 + \beta_{w2}\|w\|^2 + \beta_{w1}\|w\|. \end{aligned} \tag{31}$$

Together with the inequality in (19) and the continuity of $W(x)$, the inequality in (31) implies that $W(x)$ is an ISS Lyapunov function. □

Remark 5. Note that the inequality in (31) is obtained by making use of the assumption that $\|w\|$ is sufficiently small. This is inevitable, as an input-constrained neutrally stable system cannot be stabilized when there is a large disturbance.

Remark 6. As the right hand side of (30) can be made arbitrarily small, the inequality in (31) is valid for any δ strictly less than $1/(2c_1c_2\lambda)$.

Remark 7. The ISS characterization in Theorem 5 is similar to, but is slightly different from those in [23,3]. In [3], the ISS characterization is obtained for the case where the external disturbance enters into the system in the following manner

$$x^+ = Ax + B\mathrm{sat}(u + w).$$

In [23], the ISS characterization is derived for the matched case, i.e.

$$x^+ = Ax + B\mathrm{sat}(u) + Bw.$$

Note also that Theorem 5 can be extended to a discrete time version of Proposition 14.1.5 of [11].

4.2. ISS of the proposed MPC

On the basis of the ISS property of the small-gain control given in Section 4.1, we prove here that the finite-horizon MPC proposed in Section 3.2 is ISS. The following theorem gives a global ISS characterization[4] with a restriction on the disturbance.

Theorem 6. *Consider the neutrally stable plant in* (26) *and the MPC law $k_N(x)$ resulting from the optimization problem* (20). *Then, given any positive integer N, there exists a $\Theta > 0$ such that the closed-loop $x^+ = Ax + Bk_N(x) + w$ is ISS for all w satisfying $\|w\| < \delta_1$ with some positive δ_1.*

Proof. First, note that there exist two $\mathscr{K}_\infty$ functions satisfying (10), as discussed in the proof of Theorem 3. Note also that the optimal cost $J_N^*(x)$ is continuous; as shown in [9], the optimal cost is continuous if the cost function is continuous, and if the constraints are placed only on the input with the constraining set $\mathscr{U}$ being compact. The proof is thus completed by showing that an equality like (25) holds for $J_N^*(x)$.

Let $\mathbf{u}^*(x)$ and $x^*(i; x)$ be the optimal control sequence in (4) and the resulting trajectories in (9), respectively. As in the proof of Theorem 1, consider the feasible (suboptimal) sequence $\tilde{\mathbf{u}}$ defined in (8) at the next sampling instant; $J_N(x^+, \tilde{\mathbf{u}})$ is then given by

$$\begin{aligned} J_N&(x^+, \tilde{\mathbf{u}}) \\ &= \sum_{i=1}^{N-1} l(x_i, u^*(i; x)) + l(x_N, k_f(x^*(N; x))) \\ &\quad + \Theta W(Ax_N + Bk_f(x^*(N; x))), \end{aligned}$$

where x_i's are the sequence resulting from the state x^+ and the control $\tilde{\mathbf{u}}$, i.e.

$$\begin{aligned} &x_i = x^*(i; x) + A^{i-1}w, \ \ i \in [1, N] \\ &x^+ = x_1. \end{aligned}$$

Hence, we have

$$\begin{aligned} J_N&(x^+, \tilde{\mathbf{u}}) \\ &= \sum_{i=1}^{N-1} l(x^*(i; x) + A^{i-1}w, u^*(i; x)) \\ &\quad + l(x^*(N; x) + A^{N-1}w, k_f(x^*(N; x))) \\ &\quad + \Theta W(Ax^*(N; x) + Bk_f(x^*(N; x)) + A^N w) \\ &= J_N^*(x, \mathbf{u}) - l(x, k_N(x)) + M_1 + M_2 + M_3, \end{aligned} \tag{32}$$

where

$$\begin{aligned} M_1 &= l(x^*(N; x) + A^{N-1}w, k_f(x^*(N; x))) \\ M_2 &= \Theta W(Ax^*(N; x) + Bk_f(x^*(N; x)) + A^N w) \\ &\quad - \Theta W(x^*(N; x)) \\ M_3 &= \sum_{i=1}^{N-1} \{l(x^*(i; x) + A^{i-1}w, u^*(i; x)) \\ &\quad - l(x^*(i; x), u^*(i; x))\}. \end{aligned}$$

We first consider M_1 as follows:

$$M_1 \leqslant \varepsilon_1 \|x^*(N; x)\|^2 + \beta_1 \|w\|^2, \tag{33}$$

where

$$\begin{aligned} \varepsilon_1 &= \lambda_{\max}(Q + \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A) + \varepsilon_y \\ \beta_1 &= \frac{1}{\varepsilon_y} \|QA^{N-1}\| + \lambda_{\max}(A^{N-1^{\mathrm{T}}} Q A^{N-1}), \end{aligned} \tag{34}$$

and ε_y is an arbitrary positive constant. See Appendix B for the details of deriving (33). Therefore, ε_1 can be any constant larger than $\lambda_{\max}(Q + \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A)$. In view of Theorem 5, an upper bound on M_2 for $\|w\| < \delta_1$ can be obtained as

$$\begin{aligned} M_2 &= \Theta W(Ax^*(N; x) + Bk_f(x^*(N; x)) + A^N w) \\ &\quad - \Theta W(x^*(N; x)) \\ &\leqslant -\Theta\varepsilon_2 \|x^*(N; x)\|^2 + \Theta\beta_2 \|w\| + \Theta\beta_3 \|w\|^2, \end{aligned} \tag{35}$$

where ε_2 is any constant satisfying $0 < \varepsilon_2 < 1$, β_2 and β_3 are positive, and δ_1 is such that

$$-1 + \lambda 2 c_1 c_2 A^N \delta_1 \leqslant -\varepsilon_2 - \varepsilon_x$$

with c_1, c_2, λ, and ε_x as in (30). In order to give an upper bound on M_3, we consider

$$x^*(i; x) = A^i x + W_{ci} U_i,$$

where

$$\begin{aligned} U_i &= [u^*(0; x) u^*(1; x) \cdots u^*(i-1; x)]^{\mathrm{T}}, \\ W_{ci} &= [A^{i-1}B A^{i-2}B \cdots B]. \end{aligned}$$

[4] The previous results reported in [20,18] provide local ISS characterizations; for the notion of local ISS, see [19, p. 192].

Using this expression, we obtain an upper bound on M_3 as follows:

$$M_3 \leqslant \varepsilon_3\|x\|^2 + \beta_4\|w\|^2 + \beta_5\|w\|, \tag{36}$$

where ε_3 is an arbitrary positive constant, and β_4 and β_5 are positive. See Appendix B for the details of deriving (36). Note that we use $2U_i^{\mathrm{T}} W_{ci}^{\mathrm{T}} Q A^{i-1} w \leqslant \beta_5\|w\|$ above, which ensues from the fact that the inputs are bounded due to saturation. To sum up, (33), (35) and (36) result in an upper bound on the difference of the optimal cost as follows:

$$\begin{aligned} J_N^*(x^+) - J_N^*(x) &\leqslant J_N(x^+, \tilde{\mathbf{u}}) - J_N(x, \mathbf{u}^*) \\ &\leqslant -l(x, k_N(x)) + \varepsilon_3\|x\|^2 + (\varepsilon_1 - \Theta\varepsilon_2)\|x^*(N; x)\|^2 \\ &\quad + \beta_6\|w\|^2 + \beta_7\|w\|, \end{aligned}$$

where $\beta_6 = \beta_1 + \Theta\beta_3 + \beta_4$ and $\beta_7 = \Theta\beta_2 + \beta_5$. Therefore, if Θ is set to a value larger than or equal to $\varepsilon_1/\varepsilon_2$, we have

$$\begin{aligned} J_N^*(x^+) - J_N^*(x) \leqslant &-(\lambda_{\min}(Q) - \varepsilon_3)\|x\|^2 \\ &+ \beta_6\|w\|^2 + \beta_7\|w\|. \end{aligned} \tag{37}$$

The proof is then completed by observing that ε_3 in (37) can be made arbitrarily small; note that this ISS characterization holds with a restriction on w as $\|w\|$ is assumed to be sufficiently small when deriving (35) using Theorem 5.

Remark 8. In view of (34) with ε_y being arbitrary and $0 < \varepsilon_2 < 1$, if

$$\Theta > \Theta_0 := \lambda_{\max}(Q + \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A), \tag{38}$$

the proposed MPC is ISS. Note that for global asymptotic stability, $\Theta \geqslant \Theta_0$ is shown to be sufficient in the proof of Theorem 3. Therefore, selecting Θ satisfying the inequality in (38) leads to both asymptotic and input-to-state stability.

5. Simulation

To demonstrate the effectiveness of the proposed MPC scheme, we consider the following plant

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w, \quad -1 \leqslant u \leqslant 1,$$

where $w(k)$ is an external disturbance. Note that the unconstrained part of the system is neutrally stable with one integrator. For implementation, we employ an SQP algorithm. The MPC parameters used in the simulation are summarized below.

$$N = 3, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.8,$$

$$M_c = \begin{bmatrix} 0.06 & 0.3 \\ 0.3 & 2 \end{bmatrix}, \quad \kappa = \frac{0.95}{\lambda_{\max}(B^{\mathrm{T}} M_c B)},$$

$$\Theta = 1.05 \cdot \Theta_0.$$

As shown in Fig. 1, the proposed MPC successfully stabilizes the neutrally stable plant with $x_0 = [11, 8]$ while satisfying the saturation constraint.

Fig. 2 concerns the case where the initial state changes to $x_0 = [-1000, -1000]$; once again, stabilization is achieved. Note that the finite horizon MPC with no cubic term in the terminal cost fails to stabilize the plant in this case because of the large initial state, which causes infeasibility.

Fig. 3 shows the results for the case where the disturbance is a random bounded sequence ranging from 0 to 0.5. Despite the presence of the persistent disturbance, the output remains bounded as suggested by Theorem 6.

Fig. 4 concerns the case where the disturbance has finite energy ($w(k) = 5 \cdot 0.9^k$). As the iISS property guaranteed by global stability implies, the output is bounded and tends to zero.

As seen in the figures, the closed-loop is stable, and is robust against disturbances. The stability properties resulting from the proposed MPC are global, and consequently the optimization problem involved is guaranteed to be feasible at all times. This is in sharp contrast with finite-horizon quadratic MPC, where the domain of attraction does not cover the whole space and thus infeasibility can always occur

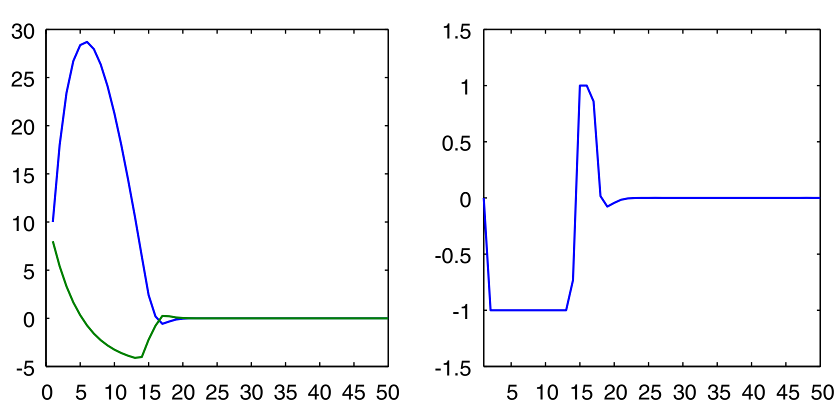

Fig. 1. State and input with $x_0 = [10, 8]$ and $w(k) = 0$.

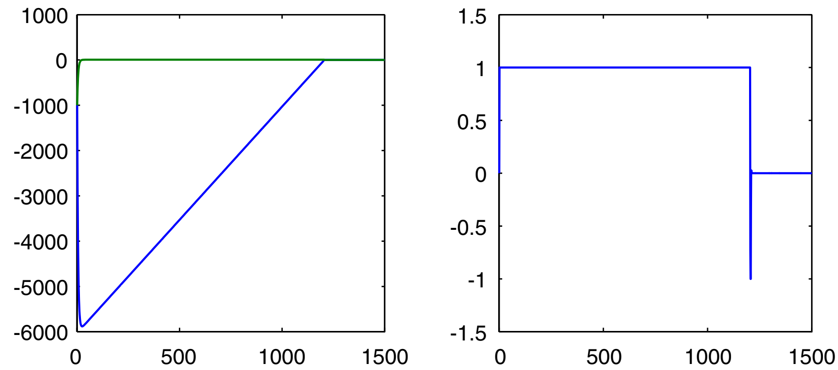

Fig. 2. State and input with $x_0 = [-1000, -1000]$ and $w(k) = 0$.

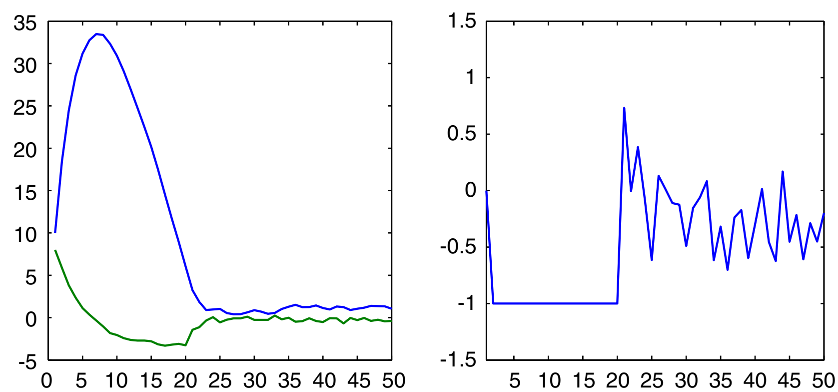

Fig. 3. State and input with $x_0 = [10, 8]$ and $w(k)$ random.

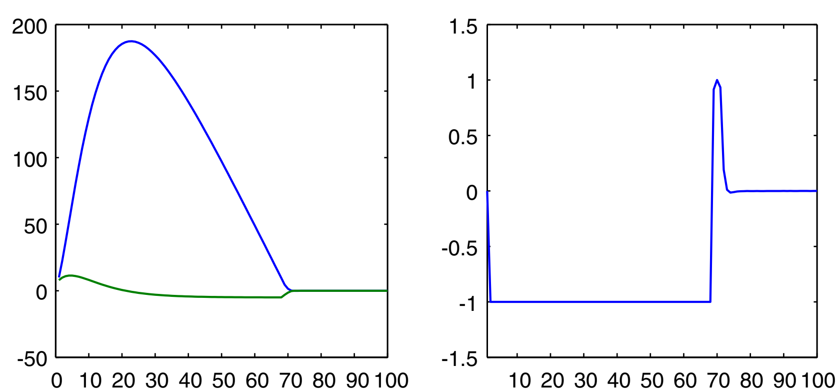

Fig. 4. State and input with $x_0 = [10, 8]$ and $w = 5 \cdot 0.9^k$.

due to the disturbance even when the initial state belongs to the domain of attraction.

6. Conclusion

In this paper, a finite horizon MPC is proposed, which globally stabilizes discrete-time neutrally stable linear systems subject to input constraints. The global stabilization is achieved by employing a non-quadratic function as the terminal cost, which consists of cubic as well as quadratic functions of the state. This is a discrete-time version of a recent work for continuous-time systems. Furthermore, an ISS characterization for the resulting closed-loop is derived under the assumption that the external disturbance is sufficiently small. Simulations using an SQP algorithm show the effectiveness of the proposed MPC. Possible directions for future research include switching-based adaptive MPC [17] for uncertain input-constrained neutrally stable systems; it is expected that the ISS property obtained here may be used to devise a switching algorithm.

Acknowledgement

The second author is grateful to financial support by Korea University.

Appendix A. Deriving the differences of the quadratic and cubic functions in Theorem 5

Along the trajectories of the system (26) and (14), the difference of the quadratic term in (17) can be evaluated as follows:

$$\begin{aligned}
&W_q(Ax+B\mathrm{sat}(Kx)+w)-W_q(x)\\
&\quad=(Ax+B\mathrm{sat}(Kx)+w)^{\mathrm{T}}M_q\\
&\qquad\times(Ax+B\mathrm{sat}(Kx)+w)-x^{\mathrm{T}}M_qx\\
&\quad=(A_cx-BKx+B\mathrm{sat}(Kx)+w)^{\mathrm{T}}M_q\\
&\qquad\times(A_cx-BKx+B\mathrm{sat}(Kx)+w)-x^{\mathrm{T}}M_qx\\
&\quad=x^{\mathrm{T}}(A_c^{\mathrm{T}}M_qA_c-M_q)x+2x^{\mathrm{T}}A^{\mathrm{T}}M_qB(\mathrm{sat}(Kx)-Kx)\\
&\qquad+2\mathrm{sat}(Kx)^{\mathrm{T}}B^{\mathrm{T}}M_qw+2x^{\mathrm{T}}A^{\mathrm{T}}M_qw+w^{\mathrm{T}}M_q^{\mathrm{T}}w\\
&\quad\leqslant-\|x\|^2+a_12\|x\|L_u(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\qquad+2b_1\|x\|\|w\|+b_2\|w\|^2+2\mathrm{sat}(Kx)^{\mathrm{T}}B^{\mathrm{T}}M_qw\\
&\quad\leqslant-\|x\|^2+a_12\|x\|L_u(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)+\varepsilon_{xq}\|x\|^2\\
&\qquad+\beta_{wq2}\|w\|^2+\beta_{wq1}\|w\|,
\end{aligned}$$

where

$$A_c=A-\kappa BB^{\mathrm{T}}M_cA,\quad a_1=\sigma_{\max}(A^{\mathrm{T}}M_qB),$$
$$b_1=\sigma_{\max}(A^{\mathrm{T}}M_q),\quad b_2=\lambda_{\max}(M_q),$$

ε_{xq} is arbitrarily small, and β_{wq2}, β_{wq1} are positive. Note that Eqs. (16) and (12), and the inequality $2pq\leqslant\eta p^2+q^2/\eta$ ($\eta>0$) are used to derive the first and second inequalities above, respectively.

We now compute the difference of the cubic term in (27); our procedure is similar to, but is slightly different from that in [23]. First, consider the following:

$$\begin{aligned}
&x^{+\mathrm{T}}M_cx^+-x^{\mathrm{T}}M_cx\\
&\quad=(Ax+B\mathrm{sat}(Kx)+w)^{\mathrm{T}}M_c\\
&\qquad\times(Ax+B\mathrm{sat}(Kx)+w)-x^{\mathrm{T}}M_cx\\
&\quad\leqslant+2x^{\mathrm{T}}A^{\mathrm{T}}M_cB\mathrm{sat}(Kx)+2x^{\mathrm{T}}A^{\mathrm{T}}M_cw\\
&\qquad+\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cB\mathrm{sat}(Kx)\\
&\qquad+2\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cw+w^{\mathrm{T}}M_cw\\
&\quad=-\frac{1}{\kappa}Kx^{\mathrm{T}}\mathrm{sat}(Kx)-\frac{1}{\kappa}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\qquad+\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cB\mathrm{sat}(Kx)\\
&\qquad+2x^{\mathrm{T}}A^{\mathrm{T}}M_cw+2\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cw+w^{\mathrm{T}}M_cw\\
&\quad\leqslant-\frac{1}{\kappa}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)-\frac{1}{\kappa}Kx^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\qquad+\frac{1}{\kappa}\mathrm{sat}^{\mathrm{T}}(Kx)\mathrm{sat}(Kx)\\
&\qquad+2x^{\mathrm{T}}A^{\mathrm{T}}M_cw+2\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cw+w^{\mathrm{T}}M_cw\\
&\quad\leqslant-\frac{1}{\kappa}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)+2x^{\mathrm{T}}A^{\mathrm{T}}M_cw\\
&\qquad+2\mathrm{sat}^{\mathrm{T}}(Kx)B^{\mathrm{T}}M_cw+w^{\mathrm{T}}M_cw\\
&\quad\leqslant-\frac{1}{\kappa}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)+c_1\|x\|\|w\|+\beta_1\|w\|\\
&\qquad+\beta_2\|w\|^2,
\end{aligned}\tag{A.1}$$

where β_1 and β_2 are positive, and $c_1=2\sigma_{\max}(A^{\mathrm{T}}M_c)$. Eqs. (13) and (15) are used to derive the first and second inequalities, respectively. The last inequality ensues from the boundedness of the saturation function. We also employ two inequalities as used in [23]. For $b\geqslant a\geqslant 0$ and $l\geqslant 1$,

$$(b-a)^l\leqslant b^l-b^{l-1}a.\tag{A.2}$$

Given any $\varepsilon>0$ and $l>1$, there exists $\beta_\varepsilon>0$ such that

$$(m+v)^l\leqslant m^l+\varepsilon m^{l-1/2}+\beta_\varepsilon v^{l-1/2}\tag{A.3}$$

for all $m\geqslant 0$, and for all $v\geqslant 0$ with a known upper bound. Using (A.1) and inequalities of the forms in (A.2) and (A.3), we can compute the difference of the cubic term as follows:

$$\begin{aligned}
&(x^{+\mathrm{T}}M_cx^+)^{3/2}\\
&\quad\leqslant\Bigg(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|+\beta_1\|w\|+\beta_2\|w\|^2\\
&\qquad-\frac{1}{\kappa}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\Bigg)^{3/2}\\
&\quad\leqslant(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|+\beta_1\|w\|+\beta_2\|w\|^2)^{3/2}\\
&\qquad-\frac{1}{\kappa}(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|+\beta_1\|w\|\\
&\qquad+\beta_2\|w\|^2)^{1/2}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\quad\leqslant(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|+\beta_1\|w\|+\beta_2\|w\|^2)^{3/2}\\
&\qquad-\frac{1}{\kappa}(x^{\mathrm{T}}M_cx)^{1/2}(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\quad\leqslant(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|)^{3/2}\\
&\qquad+\varepsilon_1(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|)\\
&\qquad+\beta_3(\beta_1\|w\|+\beta_2\|w\|^2)\\
&\qquad-\frac{a_2}{\kappa}\|x\|(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\quad\leqslant(x^{\mathrm{T}}M_cx+c_1\|x\|\|w\|)^{3/2}+\varepsilon_2\|x\|^2\\
&\qquad+\beta_4\|w\|+\beta_5\|w\|^2-\frac{a_2}{\kappa}\|x\|(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)\\
&\quad\leqslant\left[(x^{\mathrm{T}}M_cx)^{3/2}\left(1+\frac{c_1\|x\|\|w\|}{x^{\mathrm{T}}M_cx}\right)^{3/2}\right]\\
&\qquad+\varepsilon_2\|x\|^2+\beta_4\|w\|+\beta_5\|w\|^2\\
&\qquad-\frac{a_2}{\kappa}\|x\|(Kx)^{\mathrm{T}}\mathrm{sat}(Kx)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left[(x^{\mathrm{T}} M_c x)^{3/2} \left(1 + \frac{c_1 \|x\| \|w\|}{x^{\mathrm{T}} M_c x} \right)^2 \right] \\
&\quad + \varepsilon_2 \|x\|^2 + \beta_4 \|w\| + \beta_5 \|w\|^2 \\
&\quad - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx) \\
&\leqslant \left[(x^{\mathrm{T}} M_c x)^{3/2} \left(1 + \frac{2c_1 \|x\| \|w\|}{x^{\mathrm{T}} M_c x} + \frac{c_1^2 \|x\|^2 \|w\|^2}{(x^{\mathrm{T}} M_c x)^2} \right) \right] \\
&\quad + \varepsilon_2 \|x\|^2 + \beta_4 \|w\| + \beta_5 \|w\|^2 \\
&\quad - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx) \\
&\leqslant (x^{\mathrm{T}} M_c x)^{3/2} + (x^{\mathrm{T}} M_c x)^{1/2} 2c_1 \|x\| \|w\| \\
&\quad + \frac{c_1^2 \|x\|^2 \|w\|^2}{(x^{\mathrm{T}} M_c x)^{1/2}} + \varepsilon_2 \|x\|^2 \\
&\quad + \beta_4 \|w\| + \beta_5 \|w\|^2 - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx) \\
&\leqslant (x^{\mathrm{T}} M_c x)^{3/2} + 2c_1 c_2 \|w\| \|x\|^2 \\
&\quad + \frac{c_1^2}{c_3} \delta \|x\| \|w\| + \varepsilon_2 \|x\|^2 \\
&\quad + \beta_4 \|w\| + \beta_5 \|w\|^2 - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx) \\
&\leqslant (x^{\mathrm{T}} M_c x)^{3/2} + 2c_1 c_2 \|w\| \|x\|^2 + \varepsilon_{xc} \|x\|^2 \\
&\quad + \beta_{wc1} \|w\| + \beta_{wc2} \|w\|^2 - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx)
\end{aligned}
$$

where $c_2 = \sqrt{\lambda_{\max}(M_c)}$, $c_3 = \sqrt{\lambda_{\min}(M_c)}$, β_j's ($j = 1, \ldots, 5$), β_{wc1} and β_{wc2} are positive, and ε_1, ε_2 and ε_{xc} can be arbitrarily small. Eqs. (A.2) and (A.3) are used to derive the second and fourth inequalities, respectively. Finally, we have

$$
\begin{aligned}
&W_c(Ax + B\mathrm{sat}(Kx) + w) - W_c(x) \\
&\quad \leqslant 2c_1 c_2 \|w\| \|x\|^2 + \varepsilon_{xc} \|x\|^2 + \beta_{wc1} \|w\| \\
&\quad\quad + \beta_{wc2} \|w\|^2 - \frac{a_2}{\kappa} \|x\| (Kx)^{\mathrm{T}} \mathrm{sat}(Kx).
\end{aligned}
$$

Appendix B. Deriving the upper bounds on M_1 and M_3 in the proof of Theorem 6

The upper bounds on M_1 and M_3 given in (33) and (36) are obtained as follows:

$$
\begin{aligned}
M_1 &= (x^*(N;x) + A^{N-1} w)^{\mathrm{T}} Q (x^*(N;x) + A^{N-1} w) \\
&\quad + \mathrm{sat}(Kx)^{\mathrm{T}} R \mathrm{sat}(Kx) \\
&\leqslant (x^*(N;x) + A^{N-1} w)^{\mathrm{T}} Q (x^*(N;x) + A^{N-1} w) \\
&\quad + (\kappa B^{\mathrm{T}} M_c A x^*(N;x))^{\mathrm{T}} R (\kappa B^{\mathrm{T}} M_c A x^*(N;x)) \\
&= x^*(N;x)^{\mathrm{T}} Q x^*(N;x) + 2x^*(N;x)^{\mathrm{T}} \\
&\quad\quad \times Q A^{N-1} w + w^{\mathrm{T}} A^{N-1^{\mathrm{T}}} Q A^{N-1} w \\
&\quad\quad + x^*(N;x)^{\mathrm{T}} \kappa^2 A^{\mathrm{T}} M_c B R B^{\mathrm{T}} M_c A x^*(N;x) \\
&\leqslant \varepsilon_1 \|x^*(N;x)\|^2 + \beta_1 \|w\|^2
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \sum_{i=1}^{N-1} \{ (x^*(i;x) + A^{i-1} w)^{\mathrm{T}} Q (x^*(i;x) \\
&\quad + A^{i-1} w) - x^*(i;x)^{\mathrm{T}} Q x^*(i;x) \} \\
&= \sum_{i=1}^{N-1} \{ 2x^*(i;x)^{\mathrm{T}} Q A^{i-1} w + w^{\mathrm{T}} A^{i-1^{\mathrm{T}}} Q A^{i-1} w \} \\
&= \sum_{i=1}^{N-1} \{ 2(A^i x + W_{ci} U_i)^{\mathrm{T}} Q A^{i-1} w \\
&\quad + w^{\mathrm{T}} A^{i-1^{\mathrm{T}}} Q A^{i-1} w \} \\
&= \sum_{i=1}^{N-1} \{ 2(x^{\mathrm{T}} A^{i^{\mathrm{T}}} + U_i^{\mathrm{T}} W_{ci}^{\mathrm{T}}) Q A^{i-1} w \\
&\quad + w^{\mathrm{T}} A^{i-1^{\mathrm{T}}} Q A^{i-1} w \} \\
&= \sum_{i=1}^{N-1} \{ 2x^{\mathrm{T}} A^{i^{\mathrm{T}}} Q A^{i-1} w + 2U_i^{\mathrm{T}} W_{ci}^{\mathrm{T}} Q A^{i-1} w \\
&\quad + w^{\mathrm{T}} A^{i-1^{\mathrm{T}}} Q A^{i-1} w \} \\
&\leqslant \varepsilon_3 \|x\|^2 + \beta_4 \|w\|^2 + \beta_5 \|w\|.
\end{aligned}
$$

References

[1] D. Angeli, Intrinsic robustness of global asymptotic stability, Systems Control Lett. 38 (1999) 297–307.

[2] D. Angeli, E.D. Sontag, Y. Wang, A characterization of integral input-to-state stability, IEEE Trans. Automat. Control 45 (2000) 1082–1097.

[3] X. Bao, Z. Lin, E.D. Sontag, Finite gain stabilization of discrete-time linear systems subject to actuator saturation, Automatica 36 (2000) 269–277.

[4] A. Bemporad, M. Morari, V. Dua, E.N. Pistikopoulos, The explicit linear quadratic regulator for constrained systems, Automatica 38 (2002) 3–20.

[5] Y. Chitour, W. Liu, E.D. Sontag, On finite gain stabilizability of linear systems subject to input saturation, SIAM J. Control Optim. 34 (1996) 1190–1219.

[6] J. Choi, On the stabilization of linear discrete time system subject to input saturation, Systems Control Lett. 36 (1999) 241–244.

[7] J. Choi, K.S. Lee, Mixed constrained infinite horizon linear quadratic optimal control, Asian J. Control 5 (2003) 412–418.

[8] C. De Persis, R. De Santis, A.S. Morse, Supervisory control with state-dependent dwell-time logic and constraints, Automatica 40 (2004) 269–275.

[9] G. Grimm, M. Messina, A. Teel, S. Tuna, Model predictive control: for want of a local control Lyapunov function, all is not lost, IEEE Trans. Automat. Control 50 (2005) 546–558.

[10] J. Hespanha, D. Liberzon, A.S. Morse, Supervision of integral-input-to-state stabilizing controllers, Automatica 38 (2002) 1327–1335.

[11] A. Isidori, Nonlinear Control Systems, Vol. 2, Springer, London, 1999.

[12] A. Jadbabaie, C. De Persis, T.-W. Yoon, A globally stabilizng receding horizon controller for neutrally stable linear systems with input constraints, in: Proceedings of the CDC, 2002.

[13] A. Jadbabaie, A.S. Morse, On the ISS property for receding horizon control of constrained linear systems, in: Proceedings of the 2002 IFAC Congress, July 2002.

[14] A. Jadbabaie, J. Yu, J. Hauser, Unconstrained receding horizon control of nonlinear systems, IEEE Trans. Automat. Control 46 (2001) 776–783.

[15] Z.-P. Jiang, Y. Wang, Input-to-state stability for discrete-time nonlinear systems, Automatica 37 (2001) 857–869.
[16] C.M. Kellet, A.R. Teel, Smooth Lyapunov function and robustness of stability for difference inclusions, Systems Control Lett. 52 (2004) 395–405.
[17] J.-S. Kim, Supervisory control of globally stabilizable nonlinear systems and its application to input-constrained neutrally stable linear plants, Ph.D. Dissertation, Korea University, http://acorn.snu.ac.kr/~jskim/phdtext.pdf, 2005.
[18] E.C. Kerrigan, J.M. Maciejowski, On robust optimization and the optimal control of constrained linear systems with bounded state disturbances, in: Proceedings of the ECC, 2003.
[19] H.K. Khalil, Nonlinear Systems, third ed., Prentice Hall, Englewood Cliffs, 2002.
[20] D. Limon, T. Alamo, E.F. Camacho, Input to state stable MPC for constrained discrete time nonlinear systems with bounded additive uncertainties, in: Proceedings of CDC, 2002.
[21] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica 36 (2000) 789–814.
[22] M.M. Seron, G.C. Goodwin, J.A. De Dona, Characterisation of receding horizon control for constrained linear systems, Asian J. Control 5 (2003) 271–286.
[23] G. Shi, A. Saberi, A.A. Stoorvogel, On the Lp(lp) stabilization of open-loop neutrally stable linear plants with input subject to amplitude saturation, Internat. J. Robust Nonlinear Control 13 (2003).
[24] M. Vidyasagar, Nonlinear Systems Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1993.