Authors' Reply to "Comments on "On optimal control of spatially distributed systems""

Nader Motee[1] and Ali Jadbabaie[1]

We thank the author in [1] for careful reading of our paper and pointing out the errors. It is very unfortunate that both of us and the reviewers missed these errors. In [2], the notion of spatially-decaying (SD) matrices is introduced. These are classes of infinite matrices defined by the off-diagonal decay of their entries. Families of coupling characteristic functions (e.g. exponential and algebraic off-diagonal decay) are defined to quantify off-diagonal decay of matrices. The objective of [2] is to show that the unique solution of Lyapunov and Riccati equations with SD coefficients are SD. These results are reported in Theorems 5 and 6 in [2], respectively. The proofs of these theorems are mainly based on convergence properties of the corresponding differential Lyapunov and Riccati equations and the result of Corollary 1 in [2]. The statement of Corollary 1 in [2] asserts that if a sequence of operators in the Banach algebra of SD matrices is convergent in the sense of ℓ_p , then it is also convergent in the Banach algebra of SD matrices. As pointed in [1] Corollary 1 in [2] is not true in general, resulting in the collapse of the proofs of Theorems 5 and 6.

Even though some of the proofs as appeared in [2] are wrong, the statements of Theorem 5 in [2] and a modified version of Theorem 6 still hold under some additional assumptions on the coupling characteristic functions and the distance function. We assume that the following assumptions hold: the GRS-condition (2) which essentially requires that the coupling characteristic function decays sub-exponentially, a weak growth condition (5) on the coupling characteristic function, a growth condition (6) on the volume of a ball in the index set, and $\mathcal{A}$ being self-adjoint. We provide a proof for Theorem 5 in [2] which does not rely on Corollary 1 in [2]. A modified and more restrictive version of Theorem 6 in [2] remains valid. Essentially, we prove that the solution of the operator Riccati differential equation is in the Banach algebra on finite time intervals. Whether the result is true for infinite time, remains to be seen.

[1] The authors are with the Department of Electrical and Systems Engineering and GRASP Laboratory, University of Pennsylvania, 200 South 33rd Street, Philadelphia, PA 19104 USA ({motee, jadbabai}@seas.upenn.edu).

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A. *Brief Review of Notations*

Let $\mathbb{G}$ be a countable index set and $\mathrm{dis}(.,.)$ a distance function. In the following definition, we restrict the definition of a coupling characteristic function, as it is presented in [2], by imposing conditions (1) and (2).

Definition 1: Let $\rho : [0, \infty) \to [0, \infty)$ be a continuous concave function such that $\rho(0) = 0$. A coupling characteristic function χ is of the following form

$$\chi(x) = e^{\rho(x)} \tag{1}$$

and it satisfies $\chi(0) = 1$ and $\chi(x + y) \le \chi(x)\chi(y)$. A coupling characteristic function $\chi(x)$ is said to satisfy the GRS-condition (Gelfand-Raikov-Shilov condition) if

$$\lim_{n\to\infty} \chi(nx)^{\frac{1}{n}} = 1 \qquad \text{for all} \quad x \in \mathbb{R}^+ \tag{2}$$

or equivalently, we have

$$\lim_{x\to\infty} \frac{\rho(x)}{x} = 0. \tag{3}$$

Typical examples of such coupling characteristic functions are polynomial functions $\chi_\tau(x) = (1 + |x|)^\tau$, sub-exponential functions of the form $\chi_\tau(x) = e^{\tau|x|^\beta}$, and logarithmic functions of the form $\chi_\tau(x) = \Big(\log(e + |x|)\Big)^\tau$ for all $\tau > 0$ and $0 < \beta < 1$ (see [3] for more details). More general coupling characteristic functions can be obtained by multiplying simple coupling characteristic functions. A family of coupling characteristic functions can be defined as $\mathscr{C} = \Big\{\chi_\tau \mid \tau \ge 0\Big\}$. We assume that all coupling characteristic functions in $\mathscr{C}$ satisfy the GRS-condition (2).

Let $\mathscr{C}$ be a family of coupling characteristic functions. The class $\mathcal{S}_\tau^\infty(\mathscr{C})$ equipped with operator norm

$$|||\mathcal{A}|||_\tau = \max\Big\{\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \; \chi_\tau(\mathrm{dis}(k,i)), \;\sup_{i\in\mathbb{G}} \sum_{k\in\mathbb{G}} \|[\mathcal{A}]_{ki}\| \; \chi_\tau(\mathrm{dis}(k,i))\Big\} \tag{4}$$

is the set of all matrices $\mathcal{A}$ such that $|||\mathcal{A}|||_\tau$ is bounded. We emphasize that operator norm (4) is equivalent to the operator norm defined in Section IV-D in [2]. This is because χ_α is an increasing function of α and supremum of function χ_α over $[0, \tau)$ is χ_τ. The set $\mathcal{S}_\tau^\infty(\mathscr{C})$ corresponds to the class $\mathcal{A}_v^1$ in [3].

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I. Comments on the Counterexample in [1]

The counterexample considers a system with the Fourier transforms $\check{A} = 0, \check{B} = 10 - e^{-i\theta} - e^{i\theta}, \check{C} = 1, \check{D} = 0$. It is shown that the solution to the Fourier transformed Riccati equation is

$$\check{Q}(\theta) = \frac{1}{4\sqrt{6}} + \frac{1}{2\sqrt{6}} \sum_{k=1}^{\infty} e^{-\delta k} \cos k\theta$$

where $\delta = -\ln(5 - \sqrt{24})$. If the coupling characteristic function does not satisfy the GRS-condition, then Theorem 6 fails. For example if $\chi_\tau(x) = e^{\tau|x|}$ for some $\tau > 0$. As the counterexample shows

$$|||Q|||_\tau = \sum_{k\in\mathbb{Z}} |q_k| \; \chi_\tau(|k|) = \frac{1}{4\sqrt{6}} \sum_{k\in\mathbb{Z}} e^{-\delta|k|} \; e^{\tau|k|} = \frac{1}{4\sqrt{6}} \sum_{k\in\mathbb{Z}} e^{(\tau-\delta)|k|} = \infty$$

for all $\tau > \delta$.

However, under the assumption that $\mathscr{C}$ satisfies the GRS-condition, it follows that

$$Q \in \mathcal{S}_\tau^\infty(\mathscr{C}) \qquad \text{for all} \quad \tau > 0.$$

This is because an exponential function decays faster than any coupling characteristic function that satisfies the GRS-condition. For example, if $\chi_\tau(x) = e^{\tau|x|^\beta}$ for any $\tau > 0$ and some $0 < \beta < 1$ it follows that

$$|||Q|||_\tau = \sum_{k\in\mathbb{Z}} |q_k| \; \chi_\tau(|k|) = \frac{1}{4\sqrt{6}} \sum_{k\in\mathbb{Z}} e^{-\delta|k|} \; e^{\tau|k|^\beta} < \infty,$$

for all $\tau > 0$.

II. Modified Versions of Theorems 5 and 6 in [2]

In this section, we present modified versions of Theorems 5 and 6 in [2]. Our modifications and new proofs are based on Theorem 6 in [3]. Theorem 6 in [3] states that if a coupling characteristic function χ_τ satisfies the GRS-condition and the following weak growth condition

$$\chi_\tau(x) \geq (1 + |x|)^\delta \qquad \text{for some} \quad 0 < \delta \leq 1, \tag{5}$$

then the spectral radii of a self-adjoint operator with respect to $\|.\|_{2,2}$ (induced operator norm on ℓ_2) and $|||.|||_\tau$ are the same, i.e.,

$$\rho_{\mathcal{S}_\tau^\infty(\mathscr{C})}(\mathcal{A}) = \rho_{\ell_2}(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} = \mathcal{A}^* \in \mathcal{S}_\tau^\infty(\mathscr{C}).$$

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Assumption 1: The Banach algebra $\mathcal{S}_\tau^\infty(\mathscr{C})$ corresponds to a coupling characteristic function χ_τ that satisfies the GRS-condition and the weak growth condition (5).

Remark 1: In [2] and this paper, we present our results for a countable index set $\mathbb{G}$ and an arbitrary distance function $\mathrm{dis}(.,.)$ on $\mathbb{G}$. Our proofs and results are mainly based on Theorem 6 in [3]. The result of this theorem is expressed for index set $\mathbb{Z}^d$ and distance function $\mathrm{dis}(m,n) = \|m-n\|$. In the following discussion which is adopted from Remark 2 in [3], we briefly show that the proof of Theorem 6 in [3] carries over to arbitrary countable index sets $\mathbb{G}$ endowed with a non-trivial metric $\mathrm{dis}(.,.)$. The key modification in the proof of Theorem 6 is when we apply Barnes's Lemma (Lemma 5 in [3]) in the proof of Lemma 9 in [3]. The statement of Barnes's Lemma in [3] is based on the results of Lemma 4.6 and Theorem 4.7 in [4]. In [4], Lemma 4.6 and Theorem 4.7 are proved for a countable index set $\mathbb{G}$ and an arbitrary distance function $\mathrm{dis}(.,.)$ under the following assumption on the distance function (see condition (4.1) in [4])

$$\exists C > 0 \quad \text{and} \quad \exists \beta > 0 \ \text{ such that } \ \mu(B(x,r)) \leq C\ r^\beta \ \text{ for all } \ x \in \mathbb{G} \tag{6}$$

in which $B(x,r)$ is a ball with center x and radius r defined as follows

$$B(x,r) = \left\{ y \in \mathbb{G} \;\middle|\; \mathrm{dis}(x,y) \leq r \right\}$$

and μ is a positive regular σ-finite Borel measure on $\mathbb{G}$. Condition (6) means that the volume of balls $B(x,r)$ grows polynomially in the radius r and independently of $x \in \mathbb{G}$. For more details, we refer to the Appendix.

Lemma 1: Suppose that assumption 1 holds and $\mathcal{T}$ is a strongly continuous semigroup on ℓ_2 and that for all $t \geq 0$, $\mathcal{T}(t)$ is self-adjoint. Assume that $\mathcal{T}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ on a closed interval of time. If the semigroup $\mathcal{T}(t)$ is exponentially stable, then there exist constants $M, \gamma > 0$ such that

$$|||\mathcal{T}(t)|||_\tau \ \leq \ M\ e^{-\gamma t}$$

for all $t \geq 0$.

Proof: Since $\mathcal{T}(t)$ is exponentially stable, we have

$$\|\mathcal{T}(t)\|_{2,2} \ \leq \ C\ e^{-\mu t} \quad \text{for all} \quad t \geq 0 \tag{7}$$

for some $C, \mu > 0$. Therefore, we can choose $t_0 > 0$ so that

$$\|\mathcal{T}(t_0)\|_{2,2} < 1.$$

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From assumption 1, $\mathcal{T}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ and self-adjoint for all $t \geq 0$ and according to Theorem 6 in [3], we have the following relationship between the spectral radii

$$\rho_{\mathcal{S}_\tau^\infty(\mathscr{C})}(\mathcal{T}(t_0)) = \rho_{\ell_2}(\mathcal{T}(t_0)) = \|\mathcal{T}(t_0)\|_{2,2} \tag{8}$$

where $\rho_{\mathcal{S}_\tau^\infty(\mathscr{C})}$ and ρ_{ℓ_2} are spectral radii in $\mathcal{S}_\tau^\infty(\mathscr{C})$ and ℓ_2, respectively. According to proposition 3.8 [5], the limit $\lim_{n\to\infty} \sqrt[n]{|||\mathcal{T}(t_0)^n|||_\tau}$ exists and

$$\rho_{\mathcal{S}_\tau^\infty(\mathscr{C})}(\mathcal{T}(t_0)) = \lim_{n\to\infty} \sqrt[n]{|||\mathcal{T}(t_0)^n|||_\tau} = \lim_{n\to\infty} \sqrt[n]{|||\mathcal{T}(nt_0)|||_\tau}. \tag{9}$$

The last equality in (9) follows from the semigroup properties. From (8) and (9), one can see that there exists a number $\lambda > 0$ such that

$$\lim_{n\to\infty} \sqrt[n]{|||\mathcal{T}(nt_0)|||_\tau} = \|\mathcal{T}(t_0)\|_{2,2} = e^{-\lambda} < 1.$$

According to the definition, for a small enough number $\varepsilon > 0$ for which $\varepsilon + e^{-\lambda} < 1$, another number $N_\varepsilon > 0$ exists such that

$$\sqrt[n]{|||\mathcal{T}(nt_0)|||_\tau} < \varepsilon + e^{-\lambda} < 1 \quad \text{for all} \quad n \geq N_\varepsilon.$$

This implies that

$$|||\mathcal{T}(nt_0)|||_\tau < e^{-\gamma n t_0} \quad \text{for all} \quad n \geq N_\varepsilon$$

in which $\gamma = -\frac{1}{t_0}\log(\varepsilon + e^{-\lambda})$. From our assumption that $\mathcal{T}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ on a finite interval of time and the semigroup properties, one can verify that

$$M_0 = \max_{t\in[0,N_\varepsilon t_0]} |||\mathcal{T}(t)|||_\tau < \infty$$

By choosing $M = M_0\ e^{\gamma N_\varepsilon t_0}$, we can conclude that

$$|||\mathcal{T}(t)|||_\tau \ \leq \ M\ e^{-\gamma t}$$

for all $t \geq 0$. ■

In the following, we apply the result of Lemma 1 to show that the unique solution of operator Lyapunov equation is in the Banach algebra.

Theorem 5 in [2]: Suppose that $\mathcal{A} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ is self-adjoint and the infinitesimal generator of an exponentially stable C_0-semigroup $e^{\mathcal{A}t}$ and that $\mathcal{Q} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ is a self-adjoint positive definite operator.

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Then the unique positive definite solution of operator Lyapunov equation (11) satisfies $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$.

Proof: From Wiener-Levy theorem it follows that $e^{\mathcal{A}t} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$ (cf. [3]). From our assumption that $\mathcal{A}$ is the infinitesimal generator of an exponentially stable C_0-semigroup and Lemma 1, it follows that

$$|||e^{\mathcal{A}t}|||_\tau \;\leq\; M\; e^{-\gamma t} \tag{10}$$

for some $M, \gamma > 0$. The Lyapunov equation

$$\langle \mathcal{A}\phi, \mathcal{P}\psi\rangle + \langle \mathcal{P}\phi, \mathcal{A}\psi\rangle + \langle \phi, \mathcal{Q}\psi\rangle = 0 \quad \text{for all} \quad \phi, \psi \in \ell_2(\mathbb{Z}) \tag{11}$$

has a unique solution given by

$$\mathcal{P}\phi = \int_0^\infty e^{\mathcal{A}^* t}\mathcal{Q}e^{\mathcal{A}t}\phi\; dt \quad \text{for all} \quad \phi \in \ell_2(\mathbb{Z}) \tag{12}$$

We can also view $\mathcal{P}$ as the strong limit of the solution $\mathcal{P}(t)$ of the following operator differential equation

$$\langle \phi, \frac{d}{dt}\mathcal{P}(t)\psi\rangle = \langle \mathcal{A}\phi, \mathcal{P}(t)\psi\rangle + \langle \mathcal{P}(t)\phi, \mathcal{A}\psi\rangle + \langle \phi, \mathcal{Q}\psi\rangle \quad \text{for all} \quad \phi, \psi \in \ell_2(\mathbb{Z}) \tag{13}$$

with initial condition $\mathcal{P}(0) = 0$ [6]. One can verify that

$$\mathcal{P}(t)\phi = \int_0^t e^{\mathcal{A}^* t}\mathcal{Q}e^{\mathcal{A}t}\phi\; ds \quad \text{for all} \quad \phi \in \ell_2(\mathbb{Z}) \tag{14}$$

It is shown [6], [7] that as

$$\lim_{t\to\infty} \mathcal{P}(t)\phi = \mathcal{P}\phi \quad \text{for all} \quad \phi \in \ell_2(\mathbb{Z})$$

In the following, we show that $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$. From (14), we have

$$[\mathcal{P}(t)]_{ij} = \int_0^t [e^{\mathcal{A}^* s}\mathcal{Q}e^{\mathcal{A}s}]_{ij}\; ds$$

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Consider the following summation

$$\begin{aligned}\sum_j \left\|[\mathcal{P}(t)]_{ij}\right\| \chi_\tau(\mathrm{dis}(i,j)) &= \sum_j \left\|\int_0^t [e^{\mathcal{A}^* s}\mathcal{Q}e^{\mathcal{A}s}]_{ij}\, ds\right\| \chi_\tau(\mathrm{dis}(i,j)) \\ &\leq \sum_j \int_0^t \left\|[e^{\mathcal{A}^* s}\mathcal{Q}e^{\mathcal{A}s}]_{ij}\right\| \chi_\tau(\mathrm{dis}(i,j))\, ds \\ &\leq \int_0^t \sum_j \left\|[e^{\mathcal{A}^* s}\mathcal{Q}e^{\mathcal{A}s}]_{ij}\right\| \chi_\tau(\mathrm{dis}(i,j))\, ds \qquad (15)\end{aligned}$$

Inequality (15) follows by the monotonicity of the integral. From this, it follows that

$$\sup_i \sum_j \left\|[\mathcal{P}(t)]_{ij}\right\| \chi_\tau(\mathrm{dis}(i,j)) \leq \int_0^t |||e^{\mathcal{A}^* s}|||_\tau\, |||\mathcal{Q}|||_\tau\, |||e^{\mathcal{A}s}|||_\tau\, ds$$

Similarly, we can do the same and obtain the same inequality for $\sup_j \sum_i$. Thus, we have

$$|||\mathcal{P}(t)|||_\tau \leq \int_0^t |||e^{\mathcal{A}^* s}|||_\tau\, |||\mathcal{Q}|||_\tau\, |||e^{\mathcal{A}s}|||_\tau\, ds \qquad (16)$$

Therefore, from (10) we have that

$$|||e^{\mathcal{A}t}|||_\tau \leq M\, e^{-\gamma t}$$

for some $M, \gamma > 0$. We apply this inequality to (16) to obtain

$$|||\mathcal{P}(t)|||_\tau \leq M^2\, |||\mathcal{Q}|||_\tau \int_0^t e^{-2\gamma s}\, ds = M^2\, |||\mathcal{Q}|||_\tau\, \frac{1}{2\gamma}\left(1 - e^{-2\gamma t}\right) \qquad (17)$$

This implies that $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$. A similar analysis shows that

$$|||\mathcal{P}(t_n) - \mathcal{P}(t_m)|||_\tau \leq M^2\, |||\mathcal{Q}|||_\tau \left(\frac{1 - e^{-2\gamma(t_n - t_m)}}{2\gamma}\right) e^{-2\gamma t_m}$$

in which $t_n > t_m$. This implies that the sequence $\{\mathcal{P}(t_n)\}_{n=0}^\infty$ is a Cauchy sequence in Banach algebra $\mathcal{S}_\tau^\infty(\mathscr{C})$ for all sequences $\{t_n\}_{n=0}^\infty$ in the interval $[0, \infty)$. We conclude that $\mathcal{P}(t)$ is also convergent in $\mathcal{S}_\tau^\infty(\mathscr{C})$ and that $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$. ■

Remark 2: In reference [8], it is shown that under some additional assumptions on $\mathcal{A}$ the result of Lemma 1 and Theorem 5 can be extended to the case where $\mathcal{A}$ is not self-adjoint. According to Theorem 3.3 of [8], if $\mathcal{A}$ satisfies the following condition (cf. inequality (3.1) in [8])

$$\|\mathcal{A}^n\|_{2,w_0} \leq P(n) \max\left(\|\mathcal{A}^n\|_{2,2}, \|(\mathcal{A}^*)^n\|_{2,2}\right) \quad \text{for all} \quad n \geq 1$$

where $P(n) \geq 1$ is a polynomial, w_0 is the constant function with value 1, and

$$\|\mathcal{A}\|_{2,w_0} := \sup_{k\in\mathbb{G}} \|[\mathcal{A}]_{k,.}\|_2 + \sup_{i\in\mathbb{G}} \|[\mathcal{A}]_{.,i}\|_2$$

then

$$\rho_{\mathcal{S}^\infty_\tau(\mathscr{C})}(\mathcal{A}) = \max\{\rho_{\ell_2}(\mathcal{A}), \rho_{\ell_2}(\mathcal{A}^*)\}$$

We refer to Theorem 3.3 in [8] for a complete discussion on this. Therefore, under some additional assumptions the result of Theorem 6 in [3] can be extended to the case where $\mathcal{A}$ is not self-adjoint.

Modified Version of Theorem 6 in [2]: Assume that $\mathcal{A}, \mathcal{B}, \mathcal{Q} \in \mathcal{S}^\infty_\tau(\mathscr{C})$ and $\mathcal{Q} \succeq 0$. Moreover, assume that conditions of Theorem 2 in [2] hold. Then solution of the differential Riccati equation

$$\frac{d}{dt}\mathcal{P}(t) = \mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} + \mathcal{Q} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t)$$

where $\mathcal{P}(0) = \mathcal{P}_0 \in \mathcal{S}^\infty_\tau(\mathscr{C})$ is hermitian and non-negative, satisfies $\mathcal{P}(t) \in \mathcal{S}^\infty_\tau(\mathscr{C})$ for all $t \in [0, T_N]$ and all $N \geq 1$ where

$$T_N = \sum_{i=0}^{N} \delta_i$$

and

$$\delta_{i+1} = \frac{1}{4(|||\mathcal{A}|||_\tau + r_i|||\mathcal{B}|||_\tau)}\ , \quad r_i = 2|||\mathcal{P}(\delta_i)|||_\tau$$

in which $\delta_0 = 0$.

Proof: In the following, we will follow similar lines of proof as in Lemma 2.2 in Chapter IV-1 of [7]. The differential Riccati equation is equivalent to the following integral equation:

$$\gamma(\mathcal{P})(t)\phi = \mathcal{P}(t)\phi$$

where

$$\gamma(\mathcal{P})(t)\phi = \mathcal{P}(0)\phi + \int_0^t \Big(\mathcal{A}^*\mathcal{P}(s) + \mathcal{P}(s)\mathcal{A} + \mathcal{Q} - \mathcal{P}(s)\mathcal{B}\mathcal{B}^*\mathcal{P}(s)\Big)\phi \ ds$$

Consider the ball

$$B_{r_0,\delta_1} = \Big\{X \in C_s\big([\delta_0, \delta_1]; \Sigma(\ell_2(\mathbb{Z}))\big) \ \Big| \ |||X||| \ \leq \ r_0\Big\}$$

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in which $\Sigma(\ell_2(\mathbb{Z}))$ is the Banach space of all bounded hermitian operators on $\ell_2(\mathbb{Z})$ and $C_s\big([\delta_0,\delta_1];\Sigma(\ell_2(\mathbb{Z}))\big)$ is the space of all strongly continuous mappings $X:[\delta_0,\delta_1]\rightarrow\Sigma(\ell_2(\mathbb{Z}))$ endowed with the norm

$$|||X||| = \sup_{s\in[\delta_0,\delta_1]} |||X(s)|||_\tau.$$

In the following, we show that for all $p>0$ there exist $\delta_1=\delta(p)>0$ and $r_0=r(p)>0$ such that

$$|||\mathcal{P}_0|||_\tau \le p \quad\Rightarrow\quad \gamma(B_{r_0,\delta_1})\subset B_{r_0,\delta_1}$$

and

$$|||\ \gamma(\mathcal{P})-\gamma(\mathcal{P}')\ |||\ \le\ \frac{1}{2}\ |||\mathcal{P}-\mathcal{P}'|||$$

for all $\mathcal{P},\mathcal{P}'\in B_{r_0,\delta_1}$. Let $\mathcal{P}\in B_{r_0,\delta_1}$, then we have

$$\begin{aligned} |||\gamma(\mathcal{P})||| \ &\le\ |||\mathcal{P}(0)|||_\tau+\int_{\delta_0}^{\delta_1}\Big(2|||\mathcal{A}|||_\tau\ |||\mathcal{P}(s)|||+|||\mathcal{Q}|||_\tau+|||\mathcal{B}|||_\tau^2\ |||\mathcal{P}(s)|||^2\Big)ds \\ &\le\ |||\mathcal{P}(0)|||_\tau+\delta_1\Big(|||\mathcal{B}|||_\tau^2\ r_0^2+2|||\mathcal{A}|||_\tau\ r_0+|||\mathcal{Q}|||_\tau\Big) \end{aligned} \tag{18}$$

in which $\delta_0=0$. By the following selection

$$r_0=2|||\mathcal{P}(\delta_0)|||_\tau \quad\text{and}\quad \delta_1\Big(|||\mathcal{B}|||_\tau^2\ r_0^2+2|||\mathcal{A}|||_\tau\ r_0+|||\mathcal{Q}|||_\tau\Big)\ \le\ \frac{r_0}{2} \tag{19}$$

it follows that

$$|||\gamma(\mathcal{P})|||\le r_0$$

for all $\mathcal{P}\in B_{r_0,\delta_1}$. It also follows that

$$\begin{aligned} &|||\gamma(\mathcal{P})-\gamma(\mathcal{P}')||| = |||\int_{\delta_0}^{\delta_1}\Big(\mathcal{A}^*(\mathcal{P}(s)-\mathcal{P}'(s))+(\mathcal{P}(s)-\mathcal{P}'(s))\mathcal{A} \\ &\qquad\qquad +\ (\mathcal{P}(s)-\mathcal{P}'(s))\mathcal{B}\mathcal{B}^*\mathcal{P}'(s)+\mathcal{P}(s)\mathcal{B}\mathcal{B}^*(\mathcal{P}'(s)-\mathcal{P}(s))\Big)ds||| \\ &\le\ \delta_1\Big(2|||\mathcal{A}|||_\tau+2r|||\mathcal{B}|||_\tau^2\Big)\ |||\mathcal{P}-\mathcal{P}'||| \end{aligned}$$

If we choose r_0 and δ_0 such that

$$\delta_1\Big(2|||\mathcal{A}|||_\tau+2r_0|||\mathcal{B}|||_\tau^2\Big)\ \le\ \frac{1}{2} \tag{20}$$

then it follows that

$$|||\ \gamma(\mathcal{P})-\gamma(\mathcal{P}')\ |||\ \le\ \frac{1}{2}\ |||\mathcal{P}-\mathcal{P}'|||$$

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for all $\mathcal{P}, \mathcal{P}' \in B_{r_0,\delta_1}$.

Therefore, if we choose $r_0 = 2|||\mathcal{P}(\delta_0)|||_\tau$ and δ_1 such that (19) and (20) hold, then γ is a $\frac{1}{2}$-contraction in B_{r_0,δ_1}, and there exists a unique solution $\mathcal{P}$ in B_{r_0,δ_1}. Now, we can repeat the procedure with the new initial condition $\mathcal{P}(\delta_1)$ to obtain $r_1 = 2|||\mathcal{P}(\delta_1)|||_\tau$ and δ_2. Similarly, it can be shown that γ is a $\frac{1}{2}$-contraction in the ball

$$B_{r_1,\delta_2} = \Big\{ X \in C_s\big([\delta_1, \delta_2]; \Sigma(\ell_2(\mathbb{Z}))\big) \ \Big| \ |||X||| \ \leq \ r_1 \Big\}.$$

and that these exists a unique solution $\mathcal{P}$ on the interval $[\delta_1, \delta_2]$. This procedure can be resumed until the time instant $T_N = \sum_{i=0}^{N} \delta_i$ for all $N \geq 1$. Thus, we can conclude that $\mathcal{P}(t) \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \in [0, T_N]$. ■

We note that the series $\sum_{i=0}^{N} \delta_i$ when $N \to \infty$ may be divergent and that the time instant T can be infinite. In this case, the result of Theorem 6 is $\mathcal{P} \in \mathcal{S}_\tau^\infty(\mathscr{C})$ for all $t \geq 0$.

III. Appendix

As it is explained in Remark 2, the proof of Theorem 6 in [3] carries over to arbitrary countable index sets $\mathbb{G}$ endowed with a non-trivial metric $\mathrm{dis}(.,.)$. The key modification in the proof of Theorem 6 is when we apply Barnes's Lemma in Lemma 9 in [3]. Under the assumption (6), we can use the results of Lemma 4.6 and Theorem 4.7 in [4] to adjust the proof of Lemma 9 in [3] to an arbitrary countable index sets $\mathbb{G}$ endowed with a non-trivial metric $\mathrm{dis}(.,.)$. For completeness of our discussion, we show necessary steps to adjust the proof of Lemma 9 in [3].

Lemma 9 in [3]: Under the hypotheses of Theorem 6 and with v_n as in Lemma 8, the following identities hold for every $A = A^* \in \mathcal{A}_v^1$:

$$\lim_{n\to\infty} \|A\|_{\mathcal{A}^1_{v_n}} = \|A\|_{\mathcal{A}^1}, \text{ and} \tag{21}$$

$$\rho_{\mathcal{A}^1_v}(A) = \rho_{\mathcal{A}^1}(A) = \|A\|_{op} \tag{22}$$

Proof: (a) Let $\epsilon > 0$. For self-adjoint $A \in \mathcal{A}_v^1$ we have

$$\|A\|_{\mathcal{A}^1_{v_n}} = \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} |a_{ki}| \ v_n(\mathrm{dis}(k,i)).$$

Since by construction of v_n we have $v_n(x) = e^{-n} v(x)$ for $\|x\| \geq \beta_n$, there is some $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{G}} \sum_{\{i\in\mathbb{G} \ | \ \mathrm{dis}(k,i)\geq\beta_{n_0}\}} |a_{ki}| \ v_{n_0}(\mathrm{dis}(k,i)) \ \leq \ e^{-n_0} \|A\|_{\mathcal{A}^1_v} < \epsilon.$$

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By monotonicity $v_{n+1} \leq v_n \leq v$ for all n we therefore obtain that for all $n \geq n_0$

$$\sup_{k\in\mathbb{G}} \sum_{\{i\in\mathbb{G} \ | \ \mathrm{dis}(k,i)\geq\beta_{n_0}\}} |a_{ki}| \ v_n(\mathrm{dis}(k,i)) < \epsilon.$$

If $\|x\| \leq \beta_{n_0}$, then v_n converges to 1 uniformly, so for $n \geq n_1 = n_1(\epsilon)$ we have that

$$\sup_{k\in\mathbb{G}} \sum_{\{i\in\mathbb{G} \ | \ \mathrm{dis}(k,i)\leq\beta_{n_0}\}} |a_{ki}| \ v_n(\mathrm{dis}(k,i)) \leq (1+\epsilon) \ \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} |a_{ki}|.$$

Combining these estimates we obtain that for $n \geq \max(n_0, n_1)$

$$\|A\|_{\mathcal{A}^1_{v_n}} \leq \epsilon + (1+\epsilon)\|A\|_{\mathcal{A}^1}.$$

We conclude that

$$\lim_{n\to\infty} \|A\|_{\mathcal{A}^1_{v_n}} \leq \|A\|_{\mathcal{A}^1} \qquad \forall A \in \mathcal{A}^1_v.$$

The inverse inequality is obvious, since $v_n \geq 1$ for all n. (b) Using step (a) and the equivalence of the weights v and v_n, we then have

$$\rho_{\mathcal{A}^1_v}(A)^k = \rho_{\mathcal{A}^1_v}(A^k) = \rho_{\mathcal{A}^1_{v_n}}(A^k) \leq \|A^k\|_{\mathcal{A}^1_{v_n}} \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\rho_{\mathcal{A}^1_v}(A)^k \leq \lim_{n\to\infty} \|A^k\|_{\mathcal{A}^1_{v_n}} = \|A^k\|_{\mathcal{A}^1} \quad \forall k \in \mathbb{N}.$$

and so by taking k-th roots we have

$$\rho_{\mathcal{A}^1_v}(A) \leq \lim_{k\to\infty} \|A^k\|^{1/k}_{\mathcal{A}^1} = \rho_{\mathcal{A}^1}(A).$$

Since $v(x) \geq C\ (1+|x|)^\delta = \tau_\delta(x)$ and $\delta > 0$, we have the inclusion $\mathcal{A}^1_v \subseteq \mathcal{A}^1_{\tau_\delta}$, and according to Lemma 4.6 and Theorem 4.7 in [4] it follows that $\rho_{\mathcal{A}^1}(A) = \|A\|_{op}$. Consequently, $\rho_{\mathcal{A}^1_v}(A) = \|A\|_{op}$, as desired. ∎

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