

Stable Flocking of Mobile Agents, Part I: Fixed Topology

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Abstract—This is the first of a two-part paper that investigates the stability properties of a system of multiple mobile agents with double integrator dynamics. In this first part we generate stable flocking motion for the group using a coordination control scheme which gives rise to smooth control laws for the agents. These control laws are a combination of attractive/repulsive and alignment forces, ensuring collision avoidance and cohesion of the group and an aggregate motion along a common heading direction. In this control scheme the topology of the control interconnections is fixed and time invariant. The control policy ensures that all agents eventually align with each other and have a common heading direction while at the same time avoid collisions and group into a tight formation.

I. Introduction

Over the last years, the problem of coordinating the motion of multiple autonomous agents, has attracted significant attention. Besides the links of this issue to problems in biology, social behavior, statistical physics, and computer graphics, to name a few, research was partly motivated by recent advances in communication and computation. Considerable effort has been directed in trying to understand how a group of autonomous moving creatures such as flocks of birds, schools of fish, crowds of people [34], [18], or man-made mobile autonomous agents, can cluster in formations without centralized coordination.

Similar problems have been studied in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups [1], [21], [37], [10], [7]. A computer model mimicking animal aggregation was proposed in [25]. Following the work in [25] several other computer models have appeared in the literature (cf. [11] and the references therein), and led to creation of a new area in computer graphics known as *artificial life* [25], [31]. At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [35], [33], [32], [19], [16], [28], [5], [14]. Related problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see

for example [15], [2], [22], [24], [6], [17], [8], [29], [12], [20], [4], [13], [36], [23].

The animal aggregation model of [25] aimed at generating computer animation of the motion of bird flocks and fish schools. It was based on three dimensional computational geometry of the sort normally used in computer animation or computer aided design. This flocking model consists of three steering behaviors which describe how an individual agent maneuvers based on the positions and velocities its nearby flockmates:

- **Separation:** steer to avoid crowding local flockmates.
- **Alignment:** steer towards the average heading of local flockmates.
- **Cohesion:** steer to move toward the average position of local flockmates.

The superposition of these three rules results in all agents moving in a formation, with a common heading while avoiding collisions.

Generalizations of this model include a leader follower strategy, in which one agent acted as a group leader and the other agents would just follow the aforementioned cohesion/separation/alignment rules, resulting in leader following. Vicsek *et al.* [35] proposed such a model in 1995. Although developed independently, Vicsek's model turns out to be a special case of [25], in which all agents move with the same speed (no dynamics), and only follow an alignment rule. In [35], each agent heading is updated as the average of the headings of agent itself with its nearest neighbors plus some additive noise. Numerical simulations in [35] indicate a coherent collective motion, in which the headings of all agents converge to a common value. This was quite a surprising result in the physics community and was followed by a series of papers [3], [33], [32], [27], [19]. The first rigorous proof of convergence for Vicsek's model (in the noise-free case) was given in [12].

Inspired by the results of [25], this paper introduces a set of control laws that give rise to flocking behavior and provides a system theoretic justification by combining results from classical control theory, mechanics and algebraic graph theory. In this first part of the

paper, we consider the case where the topology of the control interactions between the agents is fixed. Each agent regulates its position and orientation based on a fixed set of “neighbors”. In this case, the control inputs for the agent are smooth. The case where the set of neighbors may change in time, depending on the relative distances between the agent and its flockmates, is treated separately in Part II [30]. Here we show that under fixed control interconnection topology, the system of mobile agents is capable of coordinating itself so that all agents attain a common heading, they cluster to a tight formation. Collision free fashion can be guaranteed under sufficient network connectivity assumptions. The control laws that ensure cohesion and separation can be *decoupled* from alignment.

This paper is organized as follows: in section II we define the problem addressed in this paper and sketch the solution approach. In section III we give a brief introduction on algebraic graph theory. The purpose of section IV is to introduce the control scheme that triggers flocking and analyze the stability of the closed loop system. Results are verified in section V via numerical simulations. Section VI summarizes and highlights new research directions.

II. Problem Description

Consider N agents, moving on the plane with dynamics described by:

$$\dot{r}_i = v_i \quad (1a)$$

$$\dot{v}_i = u_i \quad i = 1, \dots, N, \quad (1b)$$

where $r_i = (x_i, y_i)^T$ is the position vector of agent i , $v_i = (\dot{x}_i, \dot{y}_i)^T$ is its velocity vector and $u_i = (u_{x_i}, u_{y_i})^T$ its control (acceleration) input. The heading of agent i , θ_i , is defined as:

$$\theta_i = \arctan(\dot{y}_i, \dot{x}_i). \quad (2)$$

Relative position vector between agents i and j is denoted $r_{ij} = r_i - r_j$.

The objective is for the whole group to move at a common speed and direction and maintain constant distances between agents. The control input for agent i is a combination of two components (Figure 1):

$$u_i = a_i + \alpha_i. \quad (3)$$

The first component, a_i , is derived from the field produced by an artificial potential function, V_i , that depends on the relative distances between agent i and its flockmates. This term is responsible for collision avoidance and cohesion in the group. The second component, α_i regulates the velocity vector of agent i to the weighted average of that of its flockmates.

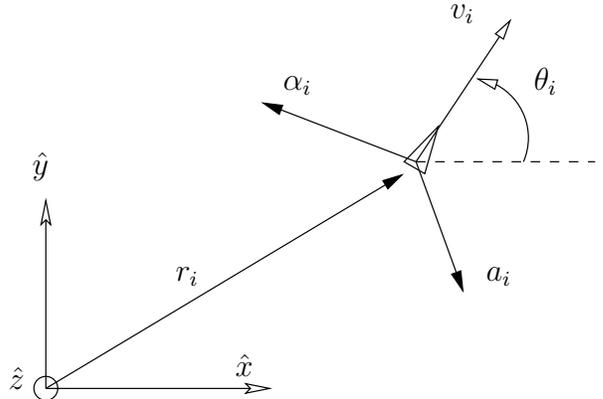


Fig. 1. Control forces acting on agent i .

III. Graph Theory Preliminaries

The following is a brief and selective introduction to algebraic graph theory. For more information, the reader is referred to [9].

A graph \mathcal{G} consists of a *vertex set* \mathcal{V} , and an *edge set* \mathcal{E} , where an *edge* is an unordered pair of distinct vertices in \mathcal{V} . If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are *adjacent*, or *neighbors* and we denote this by $x \sim y$. A graph is called *complete* if any two vertices are neighbors. A *path of length r* from vertex x to vertex y is a sequence of $r+1$ distinct vertices starting with x and ending with y , such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph \mathcal{G} , then \mathcal{G} is said to be *connected*. An *orientation* in a graph is the assignment of a direction to each edge, so that edge (i, j) is an arc from vertex i to vertex j . We denote \mathcal{G}^σ the graph \mathcal{G} with orientation σ . The *incidence matrix* $B(\mathcal{G}^\sigma)$ of a graph \mathcal{G}^σ is the matrix whose rows and columns are indexed by the vertices and edges of \mathcal{G} respectively, such that the i, j entry of $B(\mathcal{G})$ is equal to 1 if the edge j is incoming to vertex i , -1 if edge j is outgoing from vertex i , and 0 otherwise.

The symmetric matrix defined as:

$$L(\mathcal{G}) = B(\mathcal{G}^\sigma)B(\mathcal{G}^\sigma)^T$$

is called the Laplacian of \mathcal{G} and is independent of the choice of orientation σ . It is known that the Laplacian matrix captures many topological properties of the graph. Among those, it is the fact that L is always positive semidefinite and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. For a connected graph, L has a single zero eigenvalue, and the associated eigenvector is the n -dimensional vector of ones, $\mathbf{1}_n$. The second smallest eigenvalue of L , denoted λ_2 is known as the *algebraic connectivity* of the graph because it is directly related with the way the nodes are interconnected.

IV. Control Law with Fixed Topology

In this section we will refine the acceleration input of (3) into specific expressions for the components a_i and α_i . To represent the control interconnections between the agents we use a graph with a vertex corresponding to each agent. The edges capture the dependence of agent controllers on the state of other agents. Adjacency in the graph will thus induce a (logical) neighboring relation between agents. In Part II, this neighboring relation will also be associated with physical adjacency.

Definition IV.1 (Neighboring graph) *The neighboring graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is an undirected graph consisting of:*

- a set of vertices (nodes), $\mathcal{V} = \{n_1, \dots, n_N\}$, indexed by the agents in the group, and
- a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} \mid n_i \sim n_j\}$, containing unordered pairs of nodes that represent neighboring relations.

Assumption IV.2 *Graph \mathcal{G} is connected.*

Since \mathcal{G} is constant with respect to time, the above assumption ensures that \mathcal{G} will remain connected for all time. The set of all neighbors of agent i is called the neighboring set, denoted: $\mathcal{N}_i \triangleq \{j \mid i \sim j\} \subseteq \{1, \dots, N\} \setminus \{i\}$. Cohesion and separation is achieved using artificial potential fields [26]. In fact, although cohesion is ensured for a connected graph, the fixed topology of the graph cannot guarantee collision avoidance unless the neighboring graph is complete - when two agents are not linked, they cannot be aware of being close to each other. Cohesion and separation forces exerted to a pair of neighboring agents are generated by a potential function V_{ij} (Figure 2) which satisfies:

Definition IV.3 (Potential function) *Potential V_{ij} is a differentiable, nonnegative, radially unbounded function of the distance $\|r_{ij}\|$ between agents i and j , such that*

1. $V_{ij}(\|r_{ij}\|) \rightarrow \infty$ as $\|r_{ij}\| \rightarrow 0$,
2. V_{ij} attains its unique minimum when agents i and j are located at a desired distance.

Having defined V_{ij} we can express the total potential of agent i as

$$V_i = \sum_{j \in \mathcal{N}_i} V_{ij}(\|r_{ij}\|), \quad (4)$$

The control law u_i can then be defined as:

$$u_i = - \underbrace{\sum_{j \in \mathcal{N}_i} (v_i - v_j)}_{\alpha_i} - \underbrace{\sum_{j \in \mathcal{N}_i} \nabla_{r_i} V_{ij}}_{a_i}. \quad (5)$$

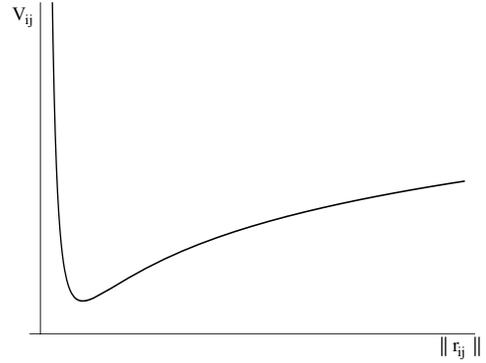


Fig. 2. Example of an inter-agent potential function.

Consider the following positive semi-definite function

$$W = \frac{1}{2} \sum_{i=1}^N (V_i + v_i^T v_i).$$

The level sets of W ,

$$\Omega = \{(v_i, r_{ij}) \mid W \leq c\} \quad (6)$$

define compact sets in the space of agent velocities and relative distances. This is because the set $\{r_{ij}, v_i\}$ such that $W \leq c$, for $c > 0$ is closed by continuity. Boundedness, on the other hand, follows from connectivity: from $W \leq c$ we have that $V_{ij} \leq c$. Connectivity ensures that a path connecting nodes i and j has length at most $N - 1$. Thus $\|r_{ij}\| \leq V_{ij}^{-1}(c(N - 1))$. Similarly, $v_i^T v_i \leq c$ yielding $\|v\|_i \leq \sqrt{c}$.

Due to V_i being symmetric with respect to r_{ij} and the fact that $r_{ij} = -r_{ji}$,

$$\frac{\partial V_{ij}}{\partial r_{ij}} = \frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j}, \quad (7)$$

and therefore it follows:

$$\frac{d}{dt} \sum_{i=1}^N \frac{1}{2} V_i = \sum_{i=1}^N \nabla_{r_i} V_i \cdot v_i.$$

Theorem IV.4 (Flocking in a fixed network)

Consider a system of N mobile agents with dynamics (1), each steered by control law (5) and assume that the neighboring graph is connected. Then all agent velocity vectors become asymptotically the same, collisions between interconnected agents are avoided and the system approaches a configuration that minimizes all agent potentials.

Proof: Taking the time derivative of W , we have:

$$\dot{W} = \frac{1}{2} \sum_{i=1}^N \dot{V}_i - \sum_{i=1}^N v_i^T \left(\sum_{j \sim i} (v_i - v_j) + \nabla_{r_i} V_i \right) \quad (8)$$

which due to the symmetric nature of V_{ij} , simplifies to

$$\begin{aligned}\dot{W} &= \sum_{i=1}^N v_i^T \nabla_{r_i} V_i - \sum_{i=1}^N v_i^T \left(\sum_{j \sim i} (v_i - v_j) + \nabla_{r_i} V_i \right) \\ &= - \sum_{i=1}^N v_i^T \sum_{j \sim i} (v_i - v_j) = -v^T (L \otimes I_2) v\end{aligned}$$

where v is the stack vector of all agent (three dimensional) velocity vectors, L is the Laplacian of the neighboring graph and \otimes denotes the Kronecker matrix product. Writing the quadratic form explicitly,

$$\dot{W} = -v_x^T L v_x - v_y^T L v_y \quad (9)$$

where v_x and v_y are the stack vectors of the components of the agent velocities along \hat{x} and \hat{y} directions (Figure 1), respectively.

For a connected graph \mathcal{G} , L is positive semidefinite and the eigenvector associated with the single zero eigenvalue is $\mathbf{1}$. Thus $\dot{W} = 0$ implies that both v_x and v_y belong to $\text{span}\{\mathbf{1}\}$. This means that all agent velocities have the same components and are therefore equal. It follows immediately that $\dot{r}_{ij} = 0, \forall (i, j) \in N \times N$.

Application of LaSalle's invariant principle establishes convergence of system trajectories to $S = \{v \mid \dot{W} = 0\}$. In S , the agent velocity dynamics become:

$$\dot{v} = - \begin{bmatrix} \nabla_{r_1} V_1 \\ \vdots \\ \nabla_{r_N} V_N \end{bmatrix} = -(B \otimes I_2) \begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} \quad (10)$$

which can be expanded to

$$\dot{v}_x = -B[\nabla_{r_{ij}} V_{ij}]_x, \quad \dot{v}_y = -B[\nabla_{r_{ij}} V_{ij}]_y.$$

Thus, \dot{v}_x and \dot{v}_y belong in the range of the incidence matrix B . For a connected graph, $\text{range}(B) = \text{span}\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x, \dot{v}_y \in \text{span}\{\mathbf{1}\}^\perp. \quad (11)$$

In an invariant set within S ,

$$v_x, v_y \in \text{span}\{\mathbf{1}\} \Rightarrow \dot{v}_x, \dot{v}_y \in \text{span}\{\mathbf{1}\}. \quad (12)$$

Combining (11) and (12),

$$\dot{v}_x, \dot{v}_y \in \text{span}\{\mathbf{1}\} \cap \text{span}\{\mathbf{1}\}^\perp \equiv \{0\}.$$

Thus, in steady state agent velocities must not change. Furthermore, from (10) it follows that in steady state the potential V_i of each agent i is minimized. Interconnected agents cannot collide since this will result in $V_i \rightarrow \infty$ and the system departing Ω , which is a contradiction since Ω is positively invariant. ■

Remark IV.5 *Collision avoidance between all agents can only be guaranteed with this control scheme when all agents are interconnected to each other. This requires the neighboring graph to be complete.*

V. Simulations

In this Section we verify numerically the stability results obtained in Section IV. In the simulation example, the group consists of ten mobile agents with identical second order dynamics. Initial positions were generated randomly within a ball of radius $R_0 = 2.5$ [m] centered at the origin. Initial velocities were also selected randomly with arbitrary directions and magnitude in the $(0, 1)$ [m/s] range. The interconnection graph was also generated in random, with the only requirement being that it is connected.

Figures 3-8 depict snapshots of the system's evolution within a time frame of 100 simulation seconds. The corresponding time instant is given below each Figure. The position of each agent is represented by a small dot and the neighboring relations by line segments connecting them. Velocity vectors are depicted as arrows, with their base point being the position of the corresponding agent. Dotted lines show the trajectory trails for each agent. Simulation verifies that the system converges to an invariant set that corresponds to a tight formation and a common heading direction. The shape of the formation which the group converges to, is determined by the artificial potential functions.

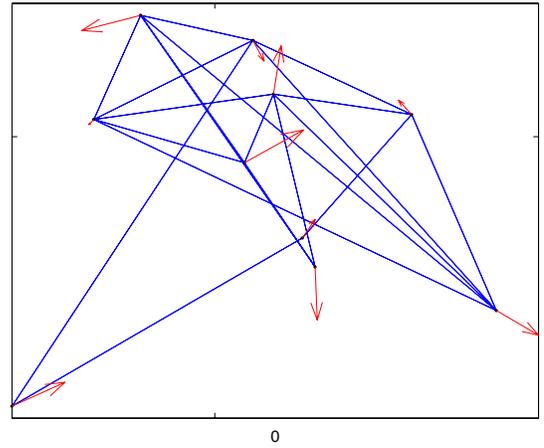


Fig. 3. *Initial configuration.*

VI. Conclusions

In this paper we demonstrate how a group of autonomous mobile agents, can cooperate to exhibit a flocking behavior. Flocking requires all the agents to have a common heading and stay close to each other while avoiding collisions. We model flocking and introduce local controllers that establish a stable, coordinated flocking motion. These local controllers rely on a fixed communication (or sensing) network, that allows the exchange of state information between interconnected agents. The stability of the group motion follows from the connectivity properties of the underlying

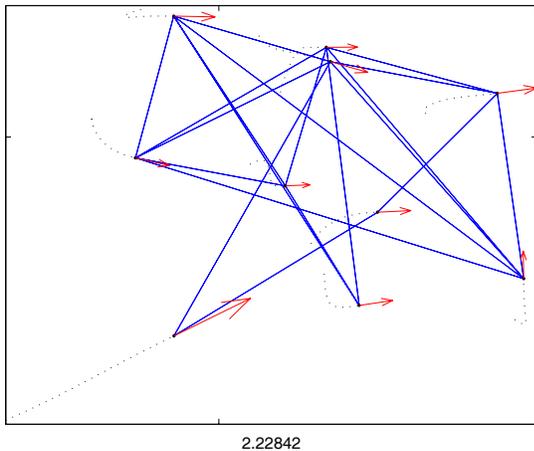


Fig. 4. Initial maneuvering may bring agents close.

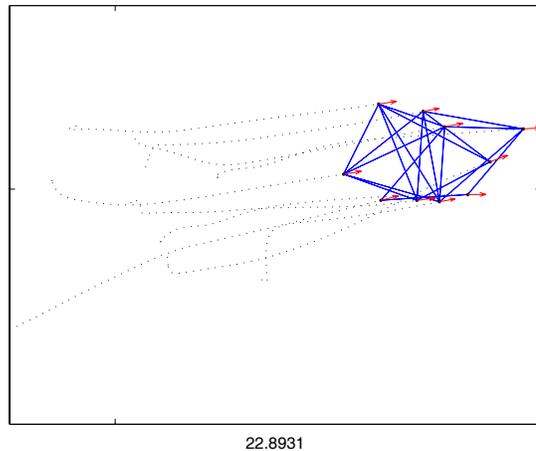


Fig. 6. Velocity vectors converge.

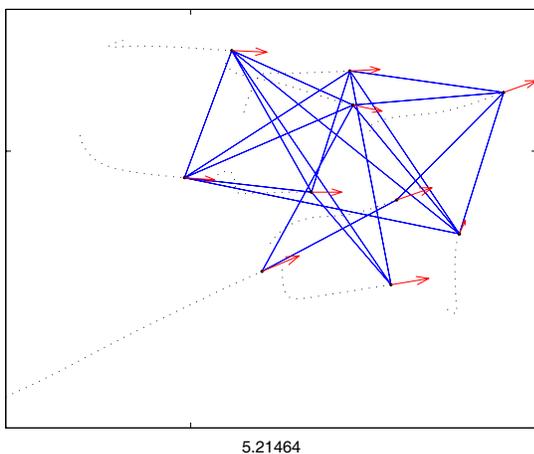


Fig. 5. Potential forces ensure cohesion.

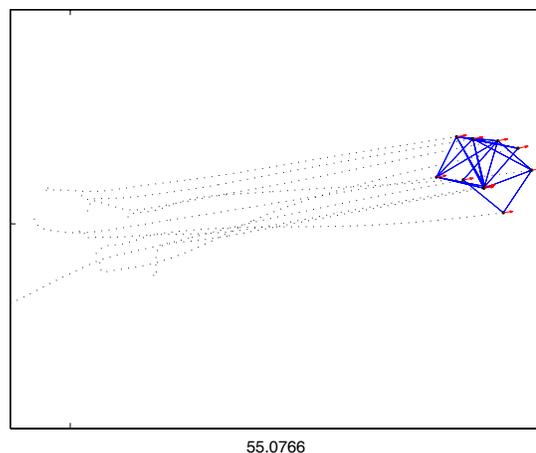


Fig. 7. The group moves in the same direction.

network topology. The case where the interconnection topology is dynamic, is treated separately in [30].

Acknowledgment

The authors would like to thank Glenn Vinnicombe for his help in the mathematical formulation.

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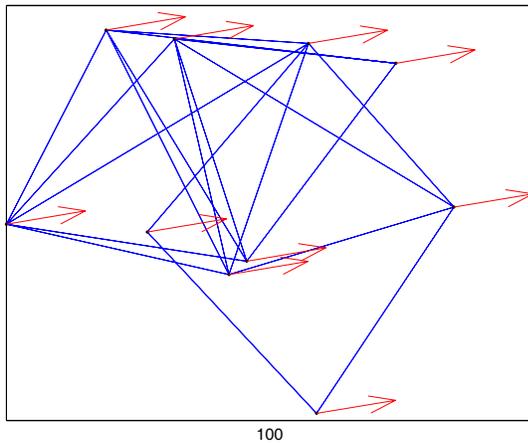


Fig. 8. Steady state.

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