Observer-Based Controller Synthesis for Model-Based Fuzzy Systems via Linear Matrix Inequalities

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Abstract

In this chapter, we extend some recent results regarding the stability of continuous-time and discrete-time Takagi-Sugeno (T-S) fuzzy systems to the case where the system's states are not available for measurement. We introduce the notion of a fuzzy observer, and show that the separation of closed-loop observer/controller holds. In other words, we show that the observer and controller design problems can be formulated as two separate LMI feasibility problems. Finally, we present a numerical example to show the effectiveness of our approach and discuss some future research directions.

Key words: Takagi-Sugeno, fuzzy systems, LMIs, observer, separation property.

1 Introduction

There has recently been a rapidly growing interest in using Takagi-Sugeno (T-S) fuzzy models to approximate nonlinear systems. This interest relies on the fact that dynamic T-S models are easily obtained by linearization of the nonlinear plant around different operating points. Once the T-S fuzzy models are obtained, linear control methodology can be used to design local state feedback controllers for each linear model. Aggregation of the fuzzy rules results in a generally nonlinear model, but in a very special form known as a Polytopic Linear Differential Inclusion (PLDI) [1]. Fortunately, the stability conditions for these type of systems can be formulated in a Linear Matrix Inequality framework, which can then be solved using convex optimization methods. However, the resulting stability conditions might be conservative, because they require the existence of a common Lyapunov matrix.

In a recent paper, Petterson and Lennartson [2] have studied a similar problem and have come up with stability conditions that do not require a single Lyapunov matrix for the solution of the Linear Matrix Inequalities (LMIs), but has some extra assumptions on the different Lyapunov functions.However, these are analysis results only. Our approach is based on the LMI formulation of the stability conditions for closed-loop T-S systems given in [7, 3]. We extend these results to the case when states are not available for measurement and feedback by introducing the notion of a *fuzzy observers*.

This chapter is organized as follows: In Section 2, we give an overview of Takagi-Sugeno fuzzy systems and sufficient stability conditions for both continuous-time and discrete-time cases. Section 3 deals with the LMI formulation of the stability results given in Section 2. In Section 4, we introduce the notion of a fuzzy observer for both continuous-time and discrete-time systems. We also state and prove the separation property of the closed-loop observer and controller, i.e., we show that in both continuous-time and discrete-time cases, we can design the observer and controller gains by solving two separate sets of LMI feasibility problems. We present a numerical example in Section 5, to illustrate our approach. Finally, we present our conclusions in Section 6.

2 Takagi-Sugeno Models

2.1 Continuous-Time T-S Models

A continuous-time T-S model is represented by a set of fuzzy **If** $\cdots$ **Then** rules written as follows :

i^{th} Plant Rule: **IF** $x_1(t)$ is M_{i1} and $\ldots, x_n(t)$ is M_{in} **THEN** $\dot{x} = A_i x$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $i = \{1, \cdots, r\}$, r is the number of rules, M_{ij} are input fuzzy sets, and the matrices $A_i \in \mathbb{R}^{n\times n}$.

Using singleton fuzzifier, product inference and weighted average deffuzifier [4, 5], the aggregated fuzzy model can be written as follows:

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(x)(A_i x)}{\sum_{i=1}^{r} w_i(x)} \tag{1}$$

and w_i is defined as

$$w_i(x) = \prod_{j=1}^{n} \mu_{ij}(x_j) \tag{2}$$

where μ_{ij} is the membership function of jth fuzzy set in the ith rule. Now, defining

$$\alpha_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)} \tag{3}$$

we can write (1) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x) A_i x; \quad i = 1, \cdots, r \tag{4}$$

where $\alpha_i(x) > 0$ and

$$\sum_{i=1}^{r} \alpha_i(x) = 1$$

The interpretation of equation (4) is that the overall system is a “fuzzy” blending of the implications. It is evident that the system (4) is generally nonlinear due to the nonlinearity of α_is. In the next section, we present sufficient conditions based on Lyapunov stability theory for the stability of open-loop system (4). The following theorem, due to Sugeno and Tanaka, is first presented [6]:

Theorem 1 *The continuous-time T-S system (4) is globally asymptotically stable if there exists a common positive definite matrix $P > 0$ which satisfies the following inequalities:*

$$A_i^T P + P A_i < 0; \quad \forall\, i = 1, \cdots, r \tag{5}$$

where r is the number of T-S rules.

2.2 Continuous-Time T-S Controllers and Closed-Loop Stability

In the previous section, we discussed the open-loop T-S fuzzy systems as well as sufficient conditions for the stability of the open-loop system. Now, we introduce the notion of the Takagi-Sugeno controller in the same fashion as the T-S system. The controller consists of fuzzy **If ... Then** rules. Each rule is a local state-feedback controller, and the overall controller is obtained by the aggregation of local controllers. A generic non-autonomous T-S plant rule can be written as follows:

i^{th} Plant Rule: IF $x_1(t)$ is M_{i1} and $\ldots$ $x_n(t)$ is M_{in} THEN $\dot{x} = A_i x + B_i u$

The overall plant dynamics can be written as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{6}$$

in the same fashion, a generic T-S controller rule can be written as:

i^{th}Controller Rule: IF $x_1(t)$ is M_{i1} and $\ldots$ $x_n(t)$ is M_{in} THEN $u = -K_i x$

The overall controller, using the same inference method as before, is given as

$$u(t) = -\sum_{i=1}^{r} \alpha_i(x) K_i x(t) \tag{7}$$

where the α_is are defined in (3). Note that we are using the same fuzzy sets for the controller rules and the plant rules. Replacing (7) in (6), and keeping in mind that

$$\sum_{i=1}^{r} \alpha_i(x) = 1$$

we can write the closed-loop equation as follows:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{8}$$

The following theorem presents sufficient conditions for closed-loop stability [7].

Theorem 2 *The closed-loop Takagi-Sugeno fuzzy system (8) is globally asymptotically stable if there exists a common positive-definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) &< 0\,; \quad \forall\, i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} &< 0\,; \quad j < i \le r \end{aligned} \tag{9}$$

where G_{ij} is defined as

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i\,; \quad j < i \le r \tag{10}$$

Proof: The proof can be easily obtained by multiplying the first set of inequalities in (9) by α_i^2 and the second set of inequalities by $\alpha_i \alpha_j$ and summing them up. ■

2.3 Discrete-Time T-S Controllers

We can define the non-autonomous discrete-time T-S system in the same fashion as the continuous-time. The non-autonomous discrete-time T-S system can be written as:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x)(A_i x(k) + B_i u(k)) \tag{11}$$

We define the discrete-time T-S controller as a set of fuzzy implications. A generic implication can be written as

i^{th} Controller Rule: IF $x_1(k)$ is M_{i1} and $\ldots x_n(k)$ is M_{in} THEN $u(k) = -K_i x(k)$

where $K_i \in \mathbb{R}^{m\times n}$. The overall controller will be

$$u(k) = -\sum_{i=1}^{r} \alpha_i(x) K_i x(k) \tag{12}$$

Replacing (12) in (11) we obtain the following closed-loop equation

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x(k) \tag{13}$$

Sufficient conditions for the stability of the closed-loop system are given by the following theorem [7].

Theorem 3 *The closed-loop system (13) is globally asymptotically stable if there exists a common positive-definite matrix P that satisfies the following matrix inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P (A_i - B_i K_i) - P &< 0\,; \quad i = 1,\cdots,r \\ G_{ij}^T P G_{ij} - P &< 0\,; \quad i < j \le r \end{aligned} \tag{14}$$

where G_{ij} is the same as in (10).

Proof: The proof is similar to that of Theorem 2. ■

3 LMI Stability Conditions for T-S Fuzzy Systems

3.1 Continuous-Time Case

Sufficient stability conditions for open-loop continuous time T-S systems were derived in Theorem 1. These conditions, as discussed earlier, are LMIs in the matrix variable P. Note that equation (4) is the equation for a Polytopic Linear Differential Inclusions. [1].

On the other hand, the closed-loop case is different. Theorem 2 provides sufficient conditions for the stability of the closed loop. But these Lyapunov inequalities are not LMIs in P and K_i, since they contain the product of P and K_i. However, using a clever change of variables due to Bernussou *et. al.* [8], we can recast the matrix inequalities in (9) as LMIs. In fact, let:

$$\begin{aligned} P^{-1} &= Y \\ X_i &= K_i Y. \end{aligned} \tag{15}$$

Then pre-multiplying and post-multiplying the inequalities in (9) by Y and using the above change of variable, we obtain the following LMIs [7]:

$$\begin{aligned} 0 &< Y \\ 0 &< YA_i^T + A_iY - B_iX_i - X_i^TB_i^T; \quad \forall\, i = 1, \cdots, r \\ 0 &< Y(A_i + A_j)^T + (A_i + A_j)Y - (B_iX_j + B_jX_i) - (B_iX_j + B_jX_i)^T; \quad j < i \le r \end{aligned} \tag{16}$$

If the above LMIs have a solution, stability of the closed-loop T-S system is guaranteed. We can find the T-S controller gains by reversing the transformations in (15), i.e.

$$K_i = X_iY^{-1}$$

Again, we point out the fact that the resulting T-S controller is conservative, because we are searching for a common quadratic Lyapunov function. In the next section we derive the LMI conditions for stability of discrete-time T-S fuzzy systems.

3.2 Discrete-Time Case

The closed-loop stability conditions in (14) can be recast as the following LMIs [7, 3].

$$\begin{aligned} Y &> 0 \\ \begin{bmatrix} Y & (A_iY - B_iX_i)^T \\ (A_iY - B_iX_i) & Y \end{bmatrix} &> 0\,; \quad i = 1, \cdots, r \\ \begin{bmatrix} Y & [(A_i + A_j)Y - M_{ij}]^T \\ (A_i + A_j)Y - M_{ij} & Y \end{bmatrix} &> 0\,; \quad j < i \le r \end{aligned} \tag{17}$$

where, Y, and X_i are defined in (15), and M_{ij} is given by

$$M_{ij} = B_iX_j + B_jX_i \tag{18}$$

If the LMIs are feasible, the controller gains can be obtained from

$$K_i = X_i Y^{-1}.$$

Once the controller gains are obtained, we can write the control action as (7) for the continuous-time case and as (12) in the discrete-time case. In the next section, we generalize our design methodology and present T-S output feedback controllers using an asymptotic observer methodology.

4 Fuzzy Observers

4.1 Why Output Feedback?

So far, we have developed a systematic framework for the design of T-S state feedback controllers. An implicit assumption in all previous sections was that the states are available for measurement. However, we know that measuring the states can be physically difficult and costly. Moreover, sensors are often subject to noise and failure. This motivates the question:"How can we design output feedback controllers for T-S fuzzy systems?"

We already know from classical control theory that using an observer, we can estimate the states of an observable LTI system from output measurements the output. In fact, we even know how to estimate the states of linear time-invariant (LTI) system in the presence of additive noise in the system, and measurement noise in the output, using a Kalman filter [9]. Our attempt is to generalize the observer methodology to the case of a PLDI instead of a single LTI system, or more specifically, to the case of T-S fuzzy systems. We present a new approach, which is to design an observer based on fuzzy implications, with fuzzy sets in the antecedents, and an asymptotic observer in the consequents. Each fuzzy rule is responsible for observing the states of a locally linear subsystem. The following section will describe the observer design in the continuous-time case [10]

4.2 Continuous-Time T-S Fuzzy Observers

Consider the closed-loop fuzzy system described by r plant rules and r controller rules as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(y)(A_i x(t) + B_i u(t))$$

$$y(t) \quad = \quad \sum_{i=1}^{r} \alpha_i(y) C_i x(t) \tag{19}$$

We define a *fuzzy observer* as a set of T-S If $\cdots$ Then rules which estimate the states of the system (19). A generic observer rule can be written as

ith Rule: If $y_1(t)$ is M_{i1} and $\cdots$ $y_p(t)$ is M_{ip} THEN: $$\dot{\hat{x}} = A_i \hat{x} + B_i u + L_i (y - \hat{y})$$

where p is the number of measured outputs, $y_i = C_i x$ is the output of ith T-S plant rule, $\hat{y}$ is the global output estimate, and $L_i \in \mathbb{R}^{n \times p}$ is the local observer gain matrix. The deffuzified global output estimate can be written as:

$$\hat{y}(t) = \sum_{j=1}^{r} \alpha_j(y) C_j \hat{x}(t)$$

where α_is are the normalized membership functions as in (3). The aggregation of all fuzzy implications results in the following state equations:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \alpha_i(y)(A_i \hat{x} + B_i u) + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(y) \alpha_j(y) L_i C_j (x - \hat{x}) \tag{20}$$

Since $\sum_{j=1}^{r} \alpha_j(y) = 1$, we can write equation (20) as

$$\dot{\hat{x}} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(y) \alpha_j(y) \left[(A_i - L_i C_j) \hat{x} + B_i u + L_i C_j x \right]. \tag{21}$$

Note that we wrote the normalized membership functions as a function of y instead of x since the antecedents are measured output variables, not states. The controller is also based on the estimate of the state rather than the state itself, i.e., we have

$$u(t) = -\sum_{j=1}^{r} \alpha_j(y) K_j \hat{x}(t) \tag{22}$$

Substituting (22) in (4) we get the following equation for the closed-loop system

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(y) \alpha_j(y)(A_i x - B_i K_j \hat{x}) \tag{23}$$

Defining the state estimation error as

$$\tilde{x} = x - \hat{x}$$

and subtracting (23) from (21) we get

$$\dot{\tilde{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)(A_i - L_iC_j)\tilde{x} \tag{24}$$

To guarantee that the estimation error goes to zero asymptotically, we can use theorem 2. The observer dynamics is stable if a common positive-definite matrix P_2 exists such that the following matrix inequalities are satisfied:

$$\begin{aligned} (A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i) &< 0\,; \quad i = 1, \cdots, r \\ H_{ij}^T P_2 + P_2 H_{ij} &< 0\,; \quad j < i \leq r \end{aligned} \tag{25}$$

where H_{ij} is defined as:

$$H_{ij} = A_i - L_iC_j + A_j - L_jC_i. \tag{26}$$

Although the inequalities in (25) are not LMIs, they can be recast as LMIs by the following change of variables:

$$W_i = P_2 L_i \tag{27}$$

Using the above variable change in (25) and utilizing the LMI Lemma [12], we obtain the following LMIs in P_2 and W_i:

$$\begin{aligned} P_2 &> 0 \\ A_i^T P_2 + P_2 A_i - W_iC_i - C_i^T W_i^T &< 0\,; \quad i = 1, \cdots, r \\ (A_i + A_j)^T P_2 + P_2(A_i + A_j) - (W_iC_j + W_jC_i) - (W_iC_j + W_jC_i)^T &< 0\,; \quad j < i \leq r \end{aligned} \tag{28}$$

The observer gains are obtained from:

$$L_i = P_2^{-1} W_i \tag{29}$$

By augmenting the states of the system with the state estimation error, we obtain the following $2n$ dimensional state equations for the observer/controller closed-loop system:

$$\left[\begin{array}{c} \dot{x} \\ \hline \dot{\tilde{x}} \end{array}\right] = \left[\begin{array}{c|c} \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j(A_i - B_iK_j) & \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j B_iK_j \\ \hline 0 & \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j(A_i - L_iC_j) \end{array}\right] \left[\begin{array}{c} x \\ \hline \tilde{x} \end{array}\right]$$

$$y \quad = \quad \left[\begin{array}{c|c} \sum_{j=1}^{r} \alpha_j C_j & 0 \end{array} \right] \left[\begin{array}{c} x \\ \hline \tilde{x} \end{array} \right] \qquad (30)$$

We then have the following theorem for the stability of the closed-loop observer/controller system.

Theorem 4 *The closed-loop observer/controller system (30) is globally asymptotically stable, if there exists a common, positive-definite matrix $\tilde{P}$ such that the following Lyapunov inequalities are satisfied:*

$$\begin{aligned} A_{ii}^T \tilde{P} + \tilde{P} A_{ii} &< 0\,; i = 1, \quad \cdots, r \\ (A_{ij} + A_{ji})^T \tilde{P} + \tilde{P}(A_{ij} + A_{ji}) &< 0\,; \quad j < i \leq r \end{aligned} \qquad (31)$$

where A_{ij} can be defined as

$$A_{ij} = \begin{bmatrix} A_i - B_i K_j & B_i K_j \\ 0 & A_i - L_i C_j \end{bmatrix} \qquad (32)$$

Proof: The proof follows directly from Theorem 2. ■

Note that the above matrix inequalities are not LMIs in $\tilde{P}$, K_is, and L_is. We would like to know if, with the same change of variables as in (15) and (27), we can rewrite the inequalities in (31) as LMIs. In fact, we would like to check if we can extend the separation property of the observer/controller of a single LTI system to the case of (30). We will show in the next section, that in the case of (30), we indeed have the separation property, resulting in two separate sets of LMIs for the observer and the controller [10].

4.3 Separation Property of Observer/Controller

To show that the separation property holds, we have to prove that $\tilde{P}$, the common positive-definite solution of the inequalities in (31), is a block diagonal matrix with $\lambda P = \lambda Y^{-1}$ and P_2 as diagonal elements. Where P is the positive-definite solution of inequalities in (9), and P_2 is the solution of (25), and $\lambda > 0$. We now express our main result as the separation property in the following theorem:

Theorem 5 *(Main Result: Separation Theorem for T-S fuzzy systems): The closed-loop system (30) is globally asymptotically stable if the inequalities in (9) and (25) are satisfied independently.*

Proof: We choose $\tilde{P}$ as a block diagonal matrix with λP and P_2 as the block diagonal elements, i.e. we have the following:

$$\tilde{P} = \left[\begin{array}{c|c} \lambda P & 0 \\ \hline 0 & P_2 \end{array}\right] \tag{33}$$

We show that there always exists a $\lambda > 0$ such that $\tilde{P}$ satisfies the inequalities in (31), provided (9) and (25) are satisfied. Substituting for $\tilde{P}$ and A_{ij} in (31) we obtain the following:

$$\left[\begin{array}{c|c} \lambda\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] & \lambda P(B_iK_i) \\ \hline \lambda(B_iK_i)^T P & (A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i) \end{array}\right] < 0 \tag{34}$$

Using the LMI lemma [12], (34) is negative-definite if and only if the following conditions are satisfied:

$$\begin{aligned} \lambda\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] &< 0 \\ \lambda P(B_iK_i)\left[(A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i)\right]^{-1}(B_iK_i)^T P & \\ - \left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right] &> 0 \end{aligned} \tag{35}$$

Since (9) is satisfied, the first inequality is already true. The second condition is satisfied for any $\lambda > 0$ such that

$$\lambda \min_{1 \le i \le r} \mu_i > \max_{1 \le i \le r} \nu_i$$

where

$$\mu_i = \lambda_{min}\{P(B_iK_i)\left[(A_i - L_iC_i)^T P_2 + P_2(A_i - L_iC_i)\right]^{-1}(B_iK_i)^T P\}$$

and

$$\nu_i = \lambda_{max}\left[(A_i - B_iK_i)^T P + P(A_i - B_iK_i)\right]$$

and where $\lambda_{min}, \lambda_{max}$ are the minimum and maximum eigenvalues. Since (9) and (25) are already satisfied, such λ always exists. Using the same argument, we can show also show that the second set of inequalities in (31) is satisfied. Therefore, the two sets of inequalities can be solved independently, and the separation property holds. ■

In the next section, we present the dual case of discrete-time Takagi-Sugeno observers.

4.4 Discrete-Time T-S Fuzzy Observers

We can define the T-S fuzzy observer in the same fashion as the continuous-time [11]. A generic rule for the discrete-time T-S fuzzy observer is:

ith Rule: IF $y_1(k)$ is M_{i1} and $\cdots$ $y_p(k)$ is M_{ip} THEN:

$$\hat{x}(k+1) = A_i x(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))$$

where p is the number of measured outputs, and $y_i(k) = C_i x(k)$ is the output of each T-S plant rule, $\hat{y}$ is the global output estimate, and $L_i \in \mathbb{R}^{n\times p}$ is the local observer gain matrix. The deffuzified output estimate can be written as:

$$\hat{y}(k) = \sum_{j=1}^{r} \alpha_j C_j \hat{x}(k)$$

where α_i are the normalized membership functions as in (3). The overall output can also be written in a similar manner:

$$y(k) = \sum_{j=1}^{r} \alpha_j C_j x(k)$$

The aggregation of all fuzzy implications results in the following state equation:

$$\hat{x}(k+1) = \sum_{i=1}^{r} \alpha_i(y)(A_i\hat{x} + B_i u) + \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y) L_i C_j (x - \hat{x}). \tag{36}$$

Since

$$\sum_{j=1}^{r} \alpha_j(y) = 1$$

we can write equation (36) as

$$\hat{x}(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)\left[(A_i - L_i C_j)\hat{x} + B_i u + L_i C_j x\right] \tag{37}$$

By defining the estimation error as before, we can write the estimation error $\tilde{x}(k)$ as follows:

$$\dot{\tilde{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y)\alpha_j(y)(A_i - L_i C_j)\tilde{x} \tag{38}$$

To guarantee that the estimation error goes to zero asymptotically, we can use Theorem 3. The observer dynamics is stable if a common positive-definite matrix P_2 exists such that the following matrix inequalities are satisfied:

$$\begin{aligned}(A_i - L_iC_i)^T P_2(A_i - L_iC_i) - P_2 &< 0\,; \quad i = 1, \cdots, r \\ H_{ij}^T P_2 H_{ij} - P_2 &< 0\,; \quad j < i \leq r\end{aligned} \tag{39}$$

where H_{ij} is defined as in (26). Although the inequalities in (39) are not LMIs, they can be recast as LMIs using the change of variables of equation (27). Using the above variable change and also utilizing the LMI lemma, we obtain the following LMIs in P_2 and W_i:

$$\begin{aligned}P_2 &> 0 \\ \left[\begin{array}{c|c} P_2 & (P_2A_i - W_iC_i)^T \\ \hline P_2A_i - W_iC_i & P_2 \end{array}\right] &> 0\,; \quad i = 1, \cdots, r \\ \left[\begin{array}{c|c} P_2 & (P_2(A_i + A_j) - W_iC_j + W_jC_i)^T \\ \hline P_2(A_i + A_j) - W_iC_j + W_jC_i & P_2 \end{array}\right] &> 0\,; \quad j < i \leq r\end{aligned} \tag{40}$$

The closed-loop observer/controller system can be written as:

$$\begin{aligned}\left[\begin{array}{c} x(k+1) \\ \hline \tilde{x}(k+1) \end{array}\right] &= \left[\begin{array}{c|c} \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j(A_i - B_iK_j) & \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j B_iK_j \\ \hline 0 & \sum_{i=1}^r \sum_{j=1}^r \alpha_i\alpha_j(A_i - L_iC_j) \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tilde{x}(k) \end{array}\right] \\ y &= \left[\begin{array}{c|c} \sum_{j=1}^r \alpha_jC_j & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tilde{x}(k) \end{array}\right]\end{aligned} \tag{41}$$

Using Theorem 3, the system (41) is globally asymptotically stable, if there exists a positive definite matrix $\tilde{P} > 0$ such that :

$$\begin{aligned}A_{ii}^T \tilde{P} A_{ii} - \tilde{P} &< 0\,; \quad i = 1, \cdots, r \\ (A_{ij} + A_{ji})^T \tilde{P} (A_{ij} + A_{ji}) - \tilde{P} &< 0\,; \quad j < i \leq r\end{aligned} \tag{42}$$

where A_{ij} is the same as in (32).

As in the continuous-time case, we can show that the Lyapunov matrix $\tilde{P}$ is indeed block diagonal, i.e., the discrete-time version of Theorem 5 holds, and the observer and controller gains can be found via separate LMI feasibility problems. A proof of the separation property in the discrete-time case is given in [11].

5 Numerical Example

We present a numerical example to illustrate the results obtained in this paper. We use a two-rule T-S fuzzy model which approximates the motion of an inverted pendulum on a cart. This system has been studied in [7, 13]. The T-S fuzzy rules are obtained by approximation of the nonlinear system around 0° and 88°. The T-S rules can be written as:

Plant Rule 1: If y is *around* 0 Then $\dot{x} = A_1x + B_1u$

Plant rule 2: If y is *around* $\pm\pi/2$ Then $\dot{x} = A_2x + B_2u$

Controller Rule 1: If y is *around* 0 Then $u = -K_1x$

Controller Rule 2: If y is *around* $\pm\pi/2$ Then $u = -K_2x$

Observer Rule 1:If y is *around* 0 Then $\dot{\hat{x}} = A_1\hat{x} + B_1u + L_1C(x - \hat{x})$

Observer Rule 2: If y is *around* $\pm\pi/2$ Then $\dot{\hat{x}} = A_2\hat{x} + B_2u + L_2C(x - \hat{x})$

where $y = x_1$ is the measured angle from the vertical point and A_1, A_2, B_1, B_2, C are given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 17.3 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -0.177 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 \\ 9.45 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix} \qquad (43) \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (44) \end{aligned}$$

The membership functions μ_1 and μ_2 for the two fuzzy sets *close to zero*, and *close to* $\pm\pi/2$ are plotted in Figure 1. We design the observer and controller gains by local pole placement, and look for common Lyapunov matrices P, and P_2. We place the closed-loop poles of the system at $-2, -2$, and the poles of the observer dynamics at $-6, -6.5$ respectively. The observer and controller gains are:

$$\begin{aligned} K_1 &= \begin{bmatrix} -120.67 & -66.67 \end{bmatrix} \quad K_2 = \begin{bmatrix} -2551.6 & -764.0 \end{bmatrix} \\ L_1 &= \begin{bmatrix} 12.5 & 57.3 \end{bmatrix}^T \quad L_2 = \begin{bmatrix} 12.5 & 50.0 \end{bmatrix}^T \qquad (45) \end{aligned}$$

Fortunately, the LMIs are feasible and we can find positive-definite Lyapunov matrices P and P_2 as solutions to the LMIs in (16) and (28). The simulation results for the states of the system

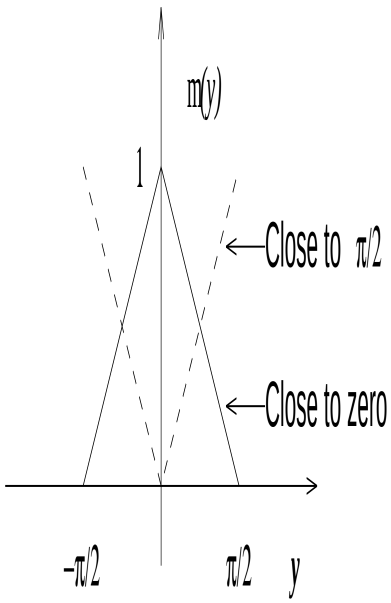

Figure 1: Membership functions for for the angle.

$x(1)$ and $x(2)$ as well as the estimation error $x(3)$, and $x(4)$ are depicted in Figures 2 through 5. Although we were able to solve for positive-definite Lyapunov matrices P and P_2 using local pole placement, this might not always be possible. This is the reason why we need to design for performance in addition to stability. The details are discussed in [13]

6 Conclusion

The purpose of this chapter was to extend the current methods of designing stabilizing state feedback controllers for T-S fuzzy systems to the output feedback case for continuous-time and discrete-time. We stated and proved a separation theorem which makes it possible to design for the observer and controller gains separately. Future research can be done in this area by extending these results to the case where performance is needed in addition to stability. Using a guaranteed-cost framework developed in [13], we can design an observer/controller system which minimizes an upper bound on a quadratic performance index . Another extension would be to include additive noise and develop a Kalman filter for T-S fuzzy systems.

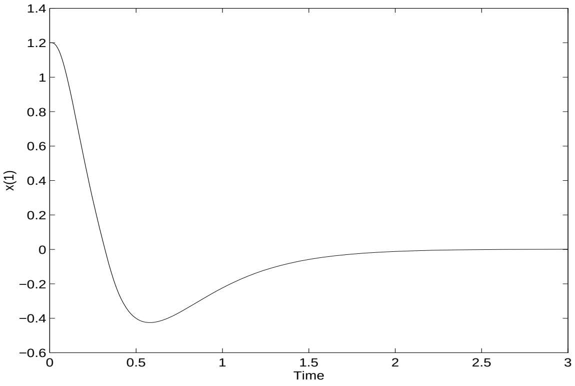

Figure 2: Initial condition response of the pendulum angle.

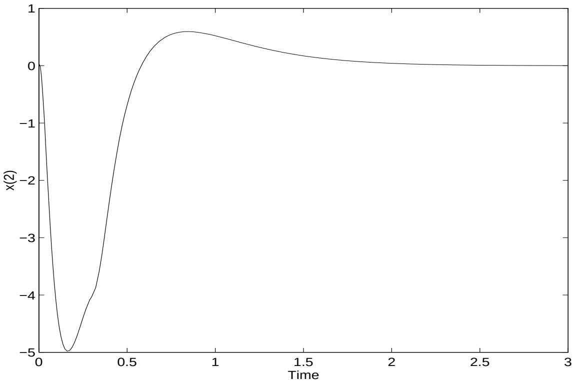

Figure 3: Initial Condition response of the angular velocity.

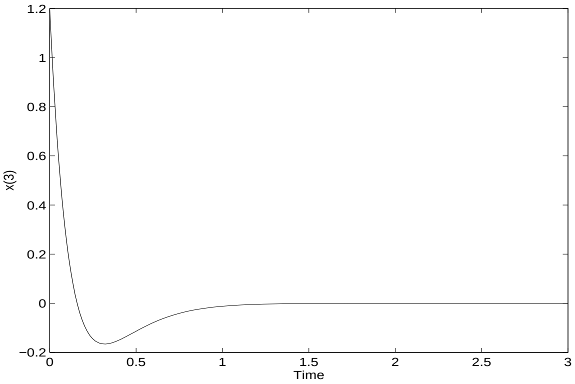

Figure 4: Estimation error for angle.

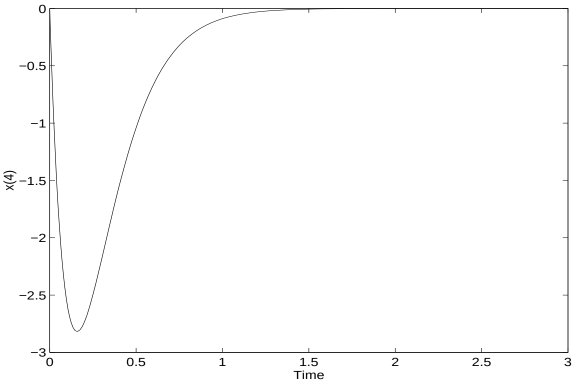

Figure 5: Estimation error for angular velocity.

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