Guaranteed-Cost Control of Polynomial Nonlinear Systems

Ali Jadbabaie[†], Chaouki T. Abdallah[*], and Peter Dorato[*]
[†]Control and Dynamical Systems
Division of Engineering and Applied Science
California Institute of Technology
Mail Code 107-81
Pasadena, CA 91125
alij@cds.caltech.edu
[*]Department of Electrical and Computer Engineering
University of New Mexico
Albuquerque, NM 87131, USA
{chaouki,peter}@eece.unm.edu

Abstract

This paper deals with the control of a class of nonlinear systems which are affine in control and the drift and control vector field are polynomials. The states are normalized in the closed interval $[0, 1]$. THe normalized system is then transformed to a Bernstein polynomial basis. Bernstien polynomials have this property that they form a partition of unity. therefore the dynamics of the nonlinear system can be written as a Polytopic Linear Differential Inclusion (PLDI). Once the PLDI is obtained, gain-scheduled controllers are designed using a guaranteed cost framework. This methods is further illustrated by a simple numerical example.

1 Introduction

One of the most popular methods of dealing with nonlinear systems is to use linear robust control methodologies. In this approach, the *trouble making* nonlinearities are assumed to be uncertain parameters. If the nonlinearity appears in a very special format, the nonlinear system can be written as a Polytpic Linear Differential Inclusion, i.e., as a convex combination of linear systems with nonlinearities being the coefficients of this convex combination. The idea of global linearization, implicit in early Soviet literature on the absolute stability problem, is the basis of these methods [5, 1, 3]. However, approximating a nonlinear system as a PLDI usually requires over bounding the nonlinearities with sector bounds.This could result in potentially severe conservatism. Since there are many trajectories of the PLDI which are not a trajectory of the nonlinear system. This is in addition to the conservatism due to search for a single quadratic Lyapunov function.

Fortunately, the first type of conservatism can be overcome for a special class of nonlinear systems. If the control and drift vector fields are assumed to be polynomial, a normalization of the states to the closed interval of [0,1] as well as choosing a polynomial basis known as Bernstein polynomial,transforms the nonlinear system into a pLDI framework without th eneed to bound the nonlinearities. Once the polynomial expansion model is obtained, a guaranteed-cost framework [3, 4] is used to design the controller using Linear Matrix Inequality methods and recently obtained relaxed stability conditions. Furthermore, if the drift vector firled is a constant. The number of LMIs would be reduced drastically.

This paper is organized as follows: first a brief introduction to PLDIs is presented in section 2. Section 3 deals with Bernstein polynomials and their properties. In section 4, stability conditions for PLDIs with gain-scheduled state feedback is presented and the special case where the control vectors are the same is considered. In section 5, we present a simple numerical example to illustrate this method. Finally, our conclusions and some drawbacks of thi smethod are presented in section 6.

2 Polytopic Linear Differential Inclusions(PLDIs)

A PLDI can be written as follows:

$$\dot{x} = A(t, x)x \tag{1}$$

where $A(t, x)$ can be written as:

$$A(t, x)=\alpha_1(t, x)A_1 + \alpha_2(t, x)A_2 + \cdots + \alpha_r(t, x)A_r \tag{2}$$

where $\{A_1, \cdots, A_r\}$ are known matrices and $\alpha_1, \ldots, \alpha_r$ are positive scalars which satisfy $\sum_{i=1}^{r} \alpha_i(t, x) = 1$. Using global linearization [1], we can use PLDIs to study properties of nonlinear time varying systems. In fact, consider the system

$$\dot{x}=f(t, x, u) \tag{3}$$

If the Jacobian of the system matrix $A(t, x) = \frac{\partial f}{\partial x}$ lies in the convex hull defined in (2), then every trajectory of the nonlinear system is also a trajectory of the LDI defined by Ω (See [1] for more details).

3 Bernstein Polynomials

We proceed by giving a brief description of Bernstein polynomials: Bernstein polynomials of degree n are defined as:

$$B_{i,n}x = \binom{n}{i} x^i (1 - x)^{n-i}$$

for $i = 1, \cdots n$ and where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

note that there are $n+1$ nth-order Bernstein polynomials and that we set $B_{i,n} = 0$ if $i < 0$ or $i > n$. It is easy to generate these polynomials as the Bernstein polynomials of degree 1 are: $B_{0,1}(x) = 1 - x$ and $B_{1,1}(x) = x$, those of degree 2 are $B_{0,2}(x) = (1-x)^2$, $B_{1,2}(x) = 2x(1-x)$, and $B_{2,2}(x) = x^2$. In fact, we can obtain a recursive formula of the Bernstein polynomials as,

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x) \tag{4}$$

The important property of Bernstein polynomials which makes them useful in teh context of PLDIs is the fact that they are all non-negative over the interval $[0,1]$, and one can show that $\sum_{i=0}^{n} = B_{i,n}(x) = 1$, i.e., they form a partition of unity. Most importantly, Bernstein polynomials of order n form a basis for polynomials of degree less than or equal to n. Although we limit our discussion to univariate Bernstein polynimials, the results presented can be extended to multivariate Bernstein Polynimials with some modifications.

Our control approach is based on normalizing the states to the closed interval $[0,1]$, and then writing the nonlinear system as a convex combination of linear systems with Bernstein polynomials being the coefficients of these convex combinations. Once the nonlinear system is written as convex combination of linear systems with coefficients of this convex combination being Bernstein polynomials, we can use recently developed LMI methods to design gain-scheduling type controllers for the nonlinear system. This way of modeling has recently become quite popular. The key point of this modeling approach is that once linear models are obtained, linear control methodology can be used to design controllers for each linear model. The overall controller for the original nonlinear system is obtained by aggregating the linear models. Stability conditions for these systems were first given in [7] in the context of model-based fuzzy systems. These conditions required the existence of a common Lyapunov matrix which would simultaneously satisfy a set of Lyapunov Matrix Inequalities. It was later shown in [8] that these stability conditions can be relaxed and that they can be transformed into Linear Matrix Inequalities which are efficiently solvable using interior-point convex optimization methods [1, 6]. While the approach of the authors in [8] is based on approximation of the nonlinear system as a fuzzy blending of local linear models, the representation of the polynomial nonlinear systems as convex combination of linear models is an exact representation. However, the techniques used for guaranteeing stability is based on [8], and for guaranteed-cost performance, the results are based on [4].

4 LMI-Based Stability Conditions

As it was mentioned earlier, we are interested in nonlinear systems in the following form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ &= A(x)x + g(x)u \end{aligned} \tag{5}$$

With all entries of $A(x)$ and $B(x)$ being polynomials. Furthermore,we assume that the states are already normalized to the closed interval [0,1]. We first write (5) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{6}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r} \alpha_i = 1$ are Bernstein polynomials. The following structure is picked for the gain-scheduled controller:

$$u = -\sum_{j=1}^{r} \alpha_j(x) K_j x \tag{7}$$

Replacing (7) in (6), we obtain the following equation for the closed loop system:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{8}$$

Stability conditions are given in the following theorem.

Theorem 1 *[8]:The closed-loop system (8) is globally asymptotically stable, if the pairs (A_i, B_i) are stabilizable , and there exist a common positive-definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) &< 0 \quad i = 1, \cdots, r \\ G_{ij}^T P + P G_{ij} &< 0 \quad j < i \le r \\ P &> 0 \end{aligned} \tag{9}$$

where G_{ij} is defined as

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i$$

.

Remark 1 *If $B_i = B$ for all values of i, i.e., the polytope describing $g(x)$ is a singleton, the second set of inequalities in terms of G_{ij} are redundant, and the stability condition reduces to the following:*

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) &< 0 \quad i = 1, \cdots, r \\ P &> 0 \end{aligned} \tag{10}$$

Pre-multiplying and post-multiplying both sides of the inequalities in (9) by P^{-1} and using the following change of variables

$$\begin{aligned} Y &= P^{-1} \\ X_i &= K_i Y \end{aligned} \tag{11}$$

we obtain the following LMIs [4]:

$$\begin{aligned} Y A_i^T + A_i Y - B_i X_i - X_i^T B_i^T &< 0 \quad i = 1, \cdots, r \\ Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T &< 0 \\ j < i &\le r \\ Y &> 0 \end{aligned} \tag{12}$$

where M_{ij} is defined as:

$$M_{ij} = B_i X_j + B_j X_i \tag{13}$$

The feasibility of the above LMIs guarantees stability, but in most practical problems, stability is just a primary goal and performance is also usually required. Next, we develop a guaranteed-cost framework for the design of nonlinear controllers [4].

5 Guaranteed-Cost LQ Performance

Consider the problem of minimizing the quadratic performance index:

$$J = \mathbb{E}_{x_0} \int_0^{\infty} (x(t)^T Q x(t) + u(t)^T R u(t)) dt \tag{14}$$

subject to : The PLDI in (6). It was shown in [4] that this problem can be transformed into the following optimization problem:

minimize: $tr(P)$

Subject to:

$$\begin{aligned} (A_i - B_i K_i)^T P + P(A_i - B_i K_i) + Q + \sum_{i=1}^{r} K_i^T R K_i &< 0 \\ G_{ij}^T P + P G_{ij} + Q + \sum_{i=1}^{r} K_i^T R K_i &< 0 \\ i = 1, \cdots, r \quad j < i &\le r \end{aligned} \tag{15}$$

where M_{ij} is the same as in (13). Using the change of variables in (11) and utilizing theSchur Complement lemma [1, 2], the inequalities in (15) can be transformed into the following LMIs:

$$\begin{aligned} &\begin{bmatrix} N_i & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} < 0 \\ &\begin{bmatrix} O_{ij} & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} < 0 \\ &Y > 0 \\ &i = 1, \cdots, r \quad j < i \le r \end{aligned} \tag{16}$$

Where N_i and O_ij are defind as follows:

$$\begin{aligned} N_i &= YA_i^T + A_i Y - B_i X_i - X_i^T B_i^T \\ O_{ij} &= Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T \end{aligned} \tag{17}$$

To obtain the least possible upper-bound using a quadratic Lyapunov function, we have to solve the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in(16)
This is a convex optimization problem which can be solved in polynomial time [6] using any of the available LMI toolboxes. To make it possible to use *Matlab*'s LMI Toolbox, we introduce an artificial variable Z, which is an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form

Min $tr(Z)$
Subject To LMIs in(16), and

$$\begin{bmatrix} Z & I_{n\times n} \\ I_{n\times n} & Y \end{bmatrix} > 0 \tag{18}$$

If the above LMIs are feasible, we can calculate the controller gains as

$$K_i = X_i Y^{-1}$$

The global controller can then be obtained as in (7).
Remark 2 *If the B_is are all the same, the second set of LMIs in (16) are redundant.*

6 Numerical Example

In this section we present a numerical example to illustrate the procedure discussed in th eprevious sections. Consider the following third order nonlinear system:

$$\begin{aligned} \dot{x} &= A(x)x + Bu \\ u &= -k(x)x \\ A(x) &= \begin{bmatrix} -2x_1^2 & -x_1^2 - 2x_1 - 2 - 2x_2^2 + x_2 & 0 \\ -0.75x_1 & -2.75x_1^2 + 2x_1 - 0.7x_2 - 0.5x_2^2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{aligned} \tag{19}$$

Since the entries in the $A(x)$ are all polynomials of at most second degree, we use the following second order Bernstein Polynomials as the polynomial basis:

$$\{\frac{1}{2}x_1^2, x_1(1-x_1), \frac{1}{2}(1-x_1)^2, \frac{1}{2}x_2^2, x_2(1-x_2), \frac{1}{2}(1-x_2)^2\}$$

Since B is a constant vector, there are only six LMIs which need to be solved. By solving the corresponding optimization problem discussed inthe previous section, we obtain values for $\{K_i\}_{i=1}^{6}$. After obtaining values for controller gains, the control action is computed using (7). Simulation results are depictude in Figure 1 and 2.

7 Conclusions

In this paper we developed a guaranteed cost framework for design of controllers for a class of polynomial nonlinear systems. Using a Bernstein polynomial expansion teh nonlinear plant was represented as a PLDI. Gain-scheduling type controllers were designed using LMI based optimization methods which would mini ize an upper bound on a quadratic performance index. The results can be extended to the case where the entries of the $A(x)$ matrix are multivariate polynomials using the multivariate expansion of Bersnstein polynomials. The proposed method suffers from potentially severe conservatism due to picking a single quadratic Lyapunov function. Also the *apriori* assumtion that state vriables are normalized to the [0,1] interval might not be always valid.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory,*

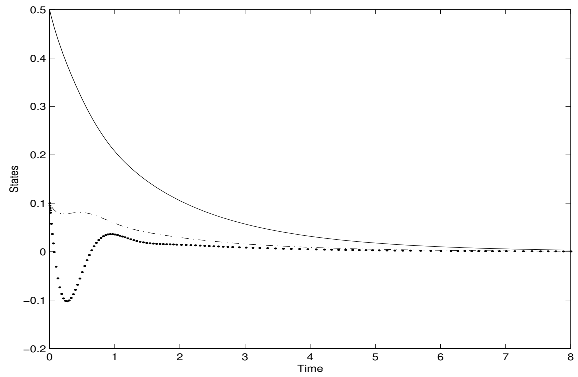

Figure 1: Time reponse of state variables.(x_1 =solid line, x_2 = dashed line, x_3 = dotted line)

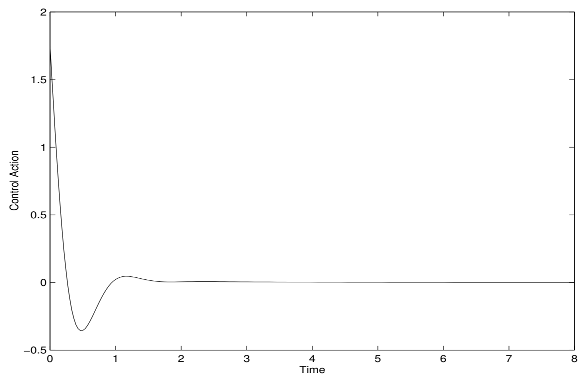

Figure 2: Control Action vs, time.

SIAM series in Applied Mathematics, Vol. 15, Philadelphia, PA., 1994.

[2] P. Dorato, C. Abdallah, and V. Cerone, *Linear Quadratic Control: An Introduction.* Prentice–Hall, Englewood Cliffs, NJ, 1995.

[3] A. Jadbabaie, C. T. Abdallah, P. Dorato and D. Famularo, "Robust, Non-fragile and Optimal Controller Design via Linear Matrix Inequalities," *Proc. American Cont. Conf.*, Philadelphia, PA, june 1998.

[4] A. Jadbabaie, M. Jamshidi, and A. Titli, Guaranteed-Cost Design of Continuous Time Takagi-Sugeno Fuzzy Controllers via LMIs, *to be presented at Fuzz IEEE 1998*, Anchorage, Alaska, May 1998.

[5] A. I. Lur'e and V. N. Postnikov, "On the Theory of Stability of Control systems," *Applied mathematics and Mechanics*, Vol 8, No. 3, 1944. In Russian.

[6] Y. Nesterov, and A. Nemirovsky, *Interior-Point Polynomial Methods in Convex Programming*, SIAM Series in Applied Mathematics, Philadelphia, PA, 1994.

[7] K. Tanaka and M. Sugeno, Stability analysis and design of fuzzy control systems, *Fuzzy Sets and systems*, Vol. 45, No. 2, pp. 135-156, 1992.

[8] H. O. Wang, K. Tanaka, and M. F. Griffin, An Approach to Fuzzy Control of Nonlinear Systems: Stability and Design Issues, *IEEE Trans. Fuzzy Syst.,* Vol. 4, No.1, pp.14-23, 1996.