COMPARISON OF NONLINEAR CONTROL DESIGNS FOR A DUCTED FAN MODEL

Jie Yu, Ali Jadbabaie, James Primbs, and Yun Huang

Control and Dynamical Systems, Mail Code 107-81, California Institute of Technology, Pasadena, CA 91125, USA
jyu, alij, jprimbs, yunhuang@cds.caltech.edu

Abstract: In this paper we compare different nonlinear control design methods by applying them to the planar model of a ducted fan engine. Those design can be divided into two steps. The first step requires the derivation of a Control Lyapunov Function (CLF), while the second involves using an existing CLF to generate a controller. The main premise of this paper is that by combining the best of these two phases, it is possible to find controllers that achieve superior performance when compared to those that apply each phase independently. All of the results are compared to the optimal solution which is approximated by solving a trajectory optimization problem with a sufficiently large time horizon.

Keywords: Nonlinear control design, Optimal Control, LPV Methods, Model Predictive Control, Ducted Fan, Control Lyapunov Functions

1. INTRODUCTION

For more than three decades, nonlinear stabilization has been a major part of control engineering research.Despite the fact that nonlinear control design is much harder than its linear counterpart, several state of the art techniques have been developed to tackle the problem of nonlinear stabilization. They can be roughly divided into two major groups. The first group is comprised of those methods that generate a control Lyapunov Function (CLF), while the second group takes an existing CLF and uses it to generate a stabilizing controller.

Motivated by the notion of combining the advantages of these two viewpoints into the design methodology, a workshop was organized at the American Control Conference in June 1997. Several nonlinear design techniques were applied to simple toy examples generated by the converse HJB method Doyle *et al.* (1996), and they were compared to the optimal solution which was known in advance. The purpose of this paper is to continue this comparison further, and add some other design methods to the existing ensemble. The methods are applied to a ducted fan, a research setup developed at Caltech as a testbed for nonlinear control design. In this paper we report the simulation results performed using various nonlinear methods on a simplified model of the fan, and experimental results will be presented in our future papers.

This paper is organized as follows: In section 2, we describe the state space model for the ducted fan engine. Section 3 deals with the methods that obtain a control Lyapunov function. In section 4, we describe methods that utilize the CLF to generate a stabilizing controller. A comparison of different methods is performed in section 5, and finally our conclusions are presented in section 6.

2. CALTECH DUCTED FAN MODEL

The Caltech Ducted Fan is a small flight control experiment whose dynamics are representative of either a Harier in hover mode or a thrust vectored aircraft such as F18-HARV or X-31 in forward flight Murray (1998). This system has been used for a number of studies and papers. In particular, a comparison of several linear and nonlinear controllers was performed in Kantner *et al.* (1995); Bodenheimer *et al.* (1996); Nieuwstadt and R.M.Murray (1996). In this section we describe the simple planar model of the fan shown in Figure 1 which ignores the stand dynamics.

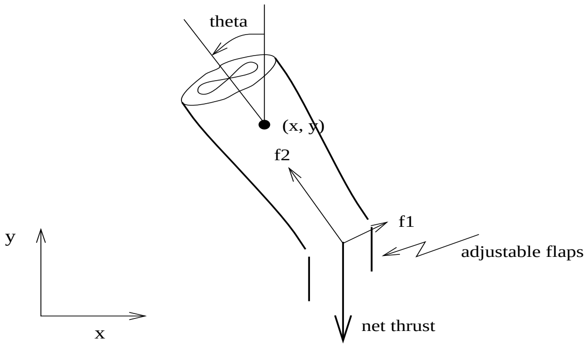

Fig. 1. Planar ducted fan model

Let (x, y, θ) denote the position and orientation of a point on the main axis of the fan that is distance l from the center of mass. We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r and a force f_2 parallel to the axis of the fan. Assuming m, J, and g to be the mass of the fan, the moment of inertia, and the gravitational constant respectively, the equations of motion can be written as follows:

$$\begin{aligned} m\ddot{x} &= -d\dot{x} + f_1\cos\theta - f_2\sin\theta \\ m\ddot{y} &= -d\dot{y} + f_1\sin\theta + f_2\cos\theta - mg \\ J\ddot{\theta} &= rf_1 \end{aligned} \tag{1}$$

where the drag terms are modeled as viscous friction with d being the viscous friction coefficient. The detailed data as : $m = 4.0kg$,$g = 9.8m/sec^2$,$J = 0.0475kgm^2$,$r = 0.26m$,$l = 0.00213m$,$d = 0.1Nsec$. It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let $u_1 = f_1$ and $u_2 = f_2 - mg$.

$$\begin{aligned} m\ddot{x} &= -mg\sin\theta - d\dot{x} + u_1\cos\theta - u_2\sin\theta \\ m\ddot{y} &= mg(\cos\theta - 1) - d\dot{y} + u_1\sin\theta + u_2\cos\theta \\ J\ddot{\theta} &= ru_1 \end{aligned} \tag{2}$$

These equations are referred to as the *planar ducted fan equations.* The following quadratic cost function was used for comparison of different design techniques:

$$\mathcal{J} = \int_0^\infty (\bar{x}^T(t)Q\bar{x}(t) + u^T(t)Ru(t))dt$$

where $\bar{x} = [x\,\dot{x}\,y\,\dot{y}\,\theta\,\dot{\theta}]^T$, $R = I$, and Q is chosen to be a diagonal matrix with the following diagonal terms:

$$Q = diag(\begin{bmatrix} 10 & 1 & 10 & 1 & 1 & 1 \end{bmatrix}) \tag{3}$$

We chose the following initial condition:

$$\begin{bmatrix} x & \dot{x} & y & \dot{y} & \theta & \dot{\theta} \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 0 & -0.9\pi/2 & 0 \end{bmatrix} \tag{4}$$

3. GENERATION OF CLFS

Consider the following nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. A *control Lyapunov function* (CLF) is a $\mathbf{C}^1$, proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that:

$$\inf_u [V_x(x)f(x) + V_x(x)g(x)u] < 0 \tag{6}$$

for all $x \neq 0$ Artstein (1983); Sontag (1983, 1989). It can be shown that the existence of a CLF for the system (5) is equivalent to the existence of a globally asymptotically stabilizing control law $u = k(x)$ which is smooth everywhere except possibly at $x = 0$ Artstein (1983).

In what follows, some of the widely used methods in nonlinear control design are presented , and interpreted in the context of the search for a control Lyapunov function.

3.1 *Jacobian Linearization*

Perhaps the simplest method of deriving a CLF is to use the Jacobian linearization of the system and generate a CLF by solving an LQR problem. In the case of the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

A and B are assumed to be :

$$A = \frac{\partial f(x)}{\partial x}|_{x=0} \quad B = \frac{\partial g(x)}{\partial x}|_{x=0}$$

Obviously the obtained CLF $V(x) = x^T Px$ by solving corresponding Riccati Equation will be valid only in a region around the equilibrium. Therefore, we should not expect good performance from initial conditions far from the origin. This is indeed the case as simulation results show

3.2 *Global Linearization*

The basic idea behind this approach is to model a nonlinear system as a Polytopic Linear Differential Inclusion (PLDI) Boyd *et al.* (1994). The

PLDI describing the planar ducted fan model can be written as:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{2} \alpha_i(t)(A_i x + B_i u) \\ u &= -Kx \end{aligned} \tag{7}$$

The problem of quadratic stability of the obtained PLDI, can be recast as an LMI feasibility problem which can be solved very efficiently using interior point methods.

Minimize

$$tr(Z)$$

Subject to:

$$\begin{gathered} Y > 0 \\ \begin{bmatrix} N & YQ^{1/2} & X^T R^{1/2} \\ Q^{1/2}Y & -I & 0 \\ R^{1/2}X & 0 & -I \end{bmatrix} < 0 \\ \begin{bmatrix} Z & x_0^T \\ x_0 & Y \end{bmatrix} > 0 \\ i = 1, 2 \end{gathered}$$

where $N = YA_i^T + A_i Y - B_i X - X^T B_i^T$, $Y = P^{-1}$, and $X = KY$ are the change of variables made to recast the matrix inequalities as LMIs Boyd *et al.* (1994). Q and R are chosen as before, and A_1, B_1 are obtained by linearization of the ducted fan model at the origin and A_2 and B_2 are chosen such that the dynamics lie in the convex hull described by (7). Using the LMI formulation of the LQR problem for PLDIs Boyd *et al.* (1994), we can find a CLF (given by $V(x) = x^T P x$) for the ducted fan model for positive values of θ. However, a global constant quadratic CLF does not exist.

3.3 *Frozen Riccati Equation (FRE) Method*

This method was first introduced by Cloutier *et. al.* in Cloutier *et al.* (1996). The basic idea behind this method, sometimes called *State Dependent Riccati Equations*, is to solve the Riccati equations online, at each time step. Although results are often promising, there are no rigorous justifications for even maintaining mere stability. Nevertheless, the simplicity of the implementation makes the FRE approach a plausible alternative in some applications. To apply this method, the planar ducted fan model is written as:

$$\dot{x} = A(x)x + g(x)u \tag{8}$$

At each *frozen state* the Riccati equation is solved, and then the resulting state feedback controller is feedback to the system, that is, the state feedback nonlinear control law is obtained by solving the following:

$$\begin{aligned} 0 &= A(x)^T P(x) + P(x)A(x) \\ &\quad - P(x)g(x)g^T(x)P(x) + Q \\ u &= -g^T(x)P(x)x \end{aligned} \tag{9}$$

The quantity $V(x) = x^T P(x) x$ generated by this technique is in general only a local CLF. Furthermore, one of the major drawbacks of this method is the lack of a systematic procedure for selecting, among the infinite possibilities, a single parameterization for $f(x)$ (in the form of equation (8)) which achieves stability and acceptable performance. However, in the case of the ducted fan model, the *obvious parameterization* of $f(x)$ appears to work in simulation studies.

3.4 *Linear Parameter Varying (LPV) Methods*

In this technique, the following so-called quasi-LPV representation of a nonlinear input-affine system is used to design a state feedback controller

$$\dot{x} = A(\rho(x))x + B(\rho(x))u \tag{10}$$

Assume the underlying parameter ρ varies in the allowable set

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho \in \mathcal{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho \in \mathcal{P},\ \underline{\nu}_i \le \dot{\rho}_i \le \overline{\nu}_i, \}$$

where $\mathcal{P} \subset \mathbf{R}^m$ is a compact set. If there exists a positive definite $X(\rho)$ such that the following inequality is satisfied

$$\begin{bmatrix} -\sum_{i=1}^{m} \overline{\underline{\nu}}_i \frac{\partial X}{\partial \rho_i} + L & XC^T \\ CX & -I \end{bmatrix} < 0 \tag{11}$$

for all $\rho \in \mathcal{P}$ where $L = AX + XA^T - BB^T$, $C(\rho) = Q^{\frac{1}{2}}(\rho(x))$, then the closed-loop system is stable with the state feedback

$$u(x) = -B^T(\rho(x))X^{-1}(\rho(x))x.$$

Moreover, an upper bound on the optimal value function $V^*(x)$ (which also serves as a CLF) is given by:

$$V(x) = x^T X^{-1}(\rho(x))x \ge V^*(x).$$

The notation $\sum_{i=1}^{m} \overline{\underline{\nu}}_i(\rho)$ in (11) means that every combination of $\overline{\nu}_i(\rho)$ and $\underline{\nu}_i(\rho)$ should be included in the inequality. For instance, when $m = 2$, $\overline{\nu}_1(\rho) + \overline{\nu}_2(\rho)$, $\overline{\nu}_1(\rho) + \underline{\nu}_2(\rho)$, $\underline{\nu}_1(\rho) + \overline{\nu}_2(\rho)$ and $\underline{\nu}_1(\rho) + \underline{\nu}_2(\rho)$ should be checked individually. In other words, (11) actually represents 2^m inequalities. Additionally, solving (11) involves gridding the parameter space $\mathcal{P}$ and choosing a finite set of basis for $X(\rho)$ (see Wu *et al.* (1996) for details).

For the ducted fan, $\rho = \theta$ is chosen as the varying parameter, and the operation range $\mathcal{P} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The bound on the rate variation on θ is set to 10, i.e., $|\dot{\theta}| \leq 10$. Both $A(\rho)$ and $B(\rho)$ are the same as in the model used for the FRE method. A set of basis functions is chosen to compute $X(\rho)$, i.e., $X(\rho) = \sum_{i=1}^{5} f_i(\rho)X_i$ where X_i's are symmetric coefficient matrices and $\{f_i(\rho)\}$ are fifth order Legendre polynomials on $\mathcal{P}$:
$\{f_i(\rho)\} = \{1,\ \frac{2}{\pi}\theta,\ (3(\frac{2}{\pi}\theta)^2 - 1)/2,$
$(5(\frac{2}{\pi}\theta)^3 - 3(\frac{2}{\pi}\theta))/2, (35(\frac{2}{\pi}\theta)^4 - 30(\frac{2}{\pi}\theta)^2 + 3)/2\}$

4. CLF BASED CONTROL SCHEMES

4.1 *Sontag's formula*

Perhaps the most important formula for producing a stabilizing controller based on the existence of a CLF was introduced in Sontag (1989) and has come to be known as Sontag's formula. Below, we present a slight variation of Sontag's original formula:

$$u_s = \begin{cases} -\left[\dfrac{V_x f + \sqrt{M}}{V_x g g^T V_x^T}\right] g^T V_x^T & V_x g \neq 0 \\ 0 & V_x g = 0 \end{cases} \quad (12)$$

where $M = (V_x f)^2 + q(x)(V_x g g^T V_x^T)$, $V_x = \frac{\partial V}{\partial x}$. While this formula has obtained recognition for possessing many desirable properties (particularly related to the continuity of the control law Sontag (1989)), it also has an interpretation in terms of optimal control. Sontag's formula, in essence, uses the directional information given by the CLF, V, and scales it properly to solve the Hamilton-Jacobi-Bellman (HJB) equation. That is, Sontag's formula can be "derived" by assuming the control action to be of the form:

$$u = -\frac{\lambda(x)}{2} g^T V_x^T$$

and determining λ by solving the HJB equation pointwise with $\lambda(x)V_x$ substituting for the gradient of the value function. In particular, if V has level curves that agree with those of the value function, then Sontag's formula produces the optimal controller. On the other hand, when a CLF does not closely resemble the value function, poor performance may result Freeman and Primbs (1996). Results from Sontag's formula with the CLF from LPV can be found in the comparisons section.

4.2 *MPC+CLF*

This section deals with combining the CLF obtained using methods described in the previous section with a Model Predictive Control (MPC) strategy. First, however we review some results on MPC.

4.2.1. *Model Predictive Control (MPC)* In Model Predictive Control (cf. García *et al.* (1989); Mayne and Michalska (1990)), the current control at state x and time t is obtained by determining on–line the optimal control $\hat{u}$ over the interval $[t, t+T]$ respecting the following objective:

$$\inf_{u(\cdot)} \int_{t}^{t+T_h} (q(x(\tau)) + u^T(\tau)u(\tau))d\tau + \Phi(x(t+T))$$
$$s.t.\ \dot{x} = f(x) + g(x)u$$

The resulting control trajectory is implemented until a new state update occurs, usually at pre-specified sampling intervals of time T_s. Repeating these calculations yields a feedback control law. This type of implementation is commonly referred to as receding or moving horizon. As is evident from this sort of control scheme, obtaining a reduced value of the performance index is of utmost importance.

Model predictive control has traditionally been plagued by stability questions. For nonlinear systems, these have been addressed in various ways, including the use of end constraints (which require that the state be identically zero at the end of the horizon T), and by various other methods Mayne and Michalska (1990); Michalska and Mayne (1993) Unfortunately, these do not always result in satisfactory solutions when considered in the framework of practical implementation. Often such additional constraints introduce a number of problems related to the feasibility of the on-line optimization. Hence, while receding horizon control exploits the ability to compute on-line, this also introduces many difficulties ranging from stability to feasibility of the on-line optimization. Nevertheless, as computer power increases, receding horizon control and on-line computation become increasingly attractive.

4.2.2. *MPC extensions of CLF formulas* In Freeman and Kokotović (1995), it was proposed to use a CLF in conjunction with the following optimization to determine a control law:

Pointwise Min-Norm

$$minimize\ u^T u \quad (13)$$
$$subject\ to\ V_x(f + gu) \leq -\sigma(x) \quad (14)$$

where σ is some positive definite function and the optimization is solved pointwise (i.e. for each x). This formula pointwise minimizes the control energy used while requiring that V be a Lyapunov function for the closed loop system and decrease by at least σ at every point. Sontag's formula (12) results by choosing:

$$\sigma_s = \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}$$

In Primbs *et al.* (1998) the following MPC extension of pointwise min-norm controllers was proposed:

MPC+CLF

$$minimize \int_t^{t+T_h} (q(x) + u^T u) dt \qquad (15)$$

$$s.t.\ V_x(f + gu(t)) \leq -\epsilon\sigma(x) \qquad (16)$$

$$V(x(t+T_h)) \leq V(x_\sigma(t+T_h)) \qquad (17)$$

where $1 \geq \epsilon > 0$ and x_σ represents the state trajectory from the pointwise min-norm controller with parameter $\sigma(x)$. The constraint (16) requires that V is a Lyapunov function for the closed loop system and is applied over the sampling interval $[t, t+T_s]$ in which the optimizing control trajectory is implemented. The second constraint (17) obviates the need for a terminal weight, using the information from the CLF instead.

This formulation of MPC avoids the stability problems of standard MPC by incorporating the stability properties of the CLF. This allows the horizon to be varied on-line and the optimization to be preempted before the optimizing solution is found, all without losing stability. When no horizon is present, it reduces to the pointwise min-norm problem, which also serves as a feasible trajectory for the MPC problem when a horizon is present. In order to test this hybrid strategy on the planar model of the ducted fan, we used the CLF obtained by the Quasi-LPV method described in section 3.4. Simulations were performed for two different time horizons, $T_h = 0.1$ and $T_h = 0.3$ are resulted in trajectories quite similar to the optimal trajectory. Note that as we increase the time horizon, the response improves as expected.

5. COMPARISONS

As was mentioned earlier, by choosing a large time horizon we found the optimal cost for the quadratic cost $\mathcal{J}$ from the chosen initial condition by solving a single trajectory optimization. Values of the cost function for all of the methods described in this paper are given in the following table.

A review of Table 1 leads to some interesting observations. At the top is MPC without the use

Table 1. Values of the cost function $\mathcal{J}$ using different methods for the ducted fan example.

Method	Cost
MPC w/o CLF ($T_h = 0.3, T_s = 0.05$)	unstable
Jacobian Linearization+LQR	unstable
FRE	2801
Global Linearization+LQR using LMIs	2423
Quasi-LPV	1795
CLF from LPV+Sontag	1761
CLF from LPV+MPC ($T_h = 0.1s, T_s = 0.05$)	1572
CLF from LPV+MPC($T_h = 0.3s, T_s = 0.05$)	1451
'Optimal'	1362

of a terminal weight or a CLF. Despite the fact that it exploits on-line computation to directly address the performance aspect of the optimal control problem, it lacks a stability guarantee, and in this case is actually unstable. Not so surprising a result is that JL+LQR is found to be unstable. This illustrates the true nonlinear nature of the problem and indicates that nonlinear techniques are needed. Only slightly more sophisticated than JL+LQR is the FRE technique. Even though it also lacks global stability guarantees, it is stabilizing from the chosen initial condition, although with a rather poor cost.

Next, we find that the Global Linearization techniques provide a guarantee of stability and reduce the cost even further. When even more off-line computation is thrown at the problem as in LPV techniques, the cost reduces even further. Note that in LPV and FRE techniques, a design choice is involved in the selection of a state dependent representation. Although no systematic procedure was used (nor does one exist), the results obtained here were the best of the state dependent representations that were tested.

The results from LPV on down were all qualitatively similar to the optimal trajectory. Applying Sontag's formula with the aid of the CLF from LPV results in a cost and trajectory very similar to that obtained from the standard LPV implementation. When this approach was extended to an on-line MPC implementation, a significant drop in the cost was observed. As the horizon was increased from $T_h = 0.1$ to $T_h = 0.3$, the cost steadily decreased, providing the lowest cost observed in any of the simulations.

To summarize, in general the following trends were observed. While not uniformly true, the more detailed and sophisticated techniques, which generally involve extensive off-line analysis, tended to outperform the simpler, less theoretically sound techniques. Extensive computation was also found to be extremely beneficial, especially when employed in an on-line manner, but only when used under the guidance of a solid theoretical framework. This accounts for the difference between the near optimal MPC+CLF controllers and the unstable standard MPC approach. A question of increasing importance will be whether on-line techniques such as MPC+CLF can realistically be implemented in a real time environment such as the Caltech ducted fan experiment. This will be a topic of ongoing interest as this research continues.

Note that since the dynamics of the planar fan model are not static feedback linearizable, we did not include that here. Also, there are a variety of other methods that generate a CLF such as back-stepping and forwarding which were not explored because these methods require the nonlinear system to be transformed to a special structure.

6. CONCLUSIONS

The purpose of this paper was to perform a comparison between several existing nonlinear design methods on the planar model of a ducted fan. The design schemes were divided into two groups. The first dealt with different methods of generating Control Lyapunov Functions (CLFs). Despite the fact that the CLF generating methods have their standard method of generating a controller, it was shown via simulation that if the CLF methodology is combined with an MPC framework, the hybrid method works better than any of the individual ones. Simulation results were performed for a specific initial condition which incorporated the true nonlinear nature of the problem. Further research in this direction is under progress, and our goal is to do a similar comparison with more sophisticated models and eventually with the Caltech ducted fan experiment.

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