A REDUCTION IN CONSERVATISM IN STABILITY AND $\mathbf{L}_2$ GAIN ANALYSIS OF TAKAGI-SUGENO FUZZY SYSTEMS VIA LINEAR MATRIX INEQUALITIES

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Abstract: In this paper, we use a state-dependent Lyapunov function to reduce the conservatism in stability analysis of Takagi-Sugeno (T-S) fuzzy systems via Linear Matrix Inequalities (LMIs). Our approach uses T-S fuzzy rules to define Lyapunov functions, and therefore substantially enlarges the class of Lyapunov functions which can be used to prove stability of T-S fuzzy systems. This approach greatly reduces the conservatism and generalizes previous results in this area which relied on finding a common positive definite Lyapunov function. Since the resulting Lyapunov function is state-dependent, this method requires *a priori* bounds on the variations of those states that are the argument of the membership functions. The effectiveness of this approach is finally presented via a simple numerical example.

Keywords: Fuzzy systems, Linear Matrix Inequalities, L_2 gain analysis.

1. INTRODUCTION

A great deal of attention has been focused on the stability analysis of Takagi Sugeno fuzzy systems. One major reason for this attention is the fact that Takagi-Sugeno fuzzy systems have been shown to be universal approximators, i.e, they can be used to approximate a large class of nonlinear systems [3]. Once the T-S model is obtained, recently developed techniques in robust control methodology can be used for analysis of T-S fuzzy systems.

Despite the fact that the global T-S model is nonlinear due to the dependence of the membership functions on the state, it is in a very special form, known as Polytopic Linear Differential Inclusions (PLDI) [2] . The only difference between a T-S fuzzy system and a PLDI is that in the analysis of PLDIs, the coefficients are assumed to be unknown where as in T-S fuzzy systems the coefficients are normalized membership functions. Several researchers have addressed the issue of stability and substantial amount of progress has been made in the last two years in stability analysis and design of T-S fuzzy control systems, [4-8,11-13,17]. Most of the above mentioned references utilize the recently developed interior-point convex optimization methods for solving Linear Matrix Inequalities [2,10]. However, all of these except [8] are based on finding a common positive definite matrix which satisfies a set of LMIs. These conditions are essentially equivalent to quadratic stability, i.e., stability provable using a single quadratic Lyapunov function. However, it is well-known that in a lot of cases, a common positive definite matrix does not exist,whereas the system is stable. To overcome this limitation, we use the machinery of Linear Parameter Varying stems

(LPV)[1,16] to overcome this conservatism. By letting the Lyapunov function to be parameter-dependent (where in this case the parameter is a function of state), the class of T-S systems for which the stability can be proven is greatly enlarged. The major disadvantage of this approach is the fact that an *a priori* bound on the variation of the states which are the argument of membership functions is assumed, and coming up with these upper bounds is some how *ad hoc*.

In this paper we focus only on analysis issues, such as stability and L_2 gain analysis. The structure of this paper is as follows: In section (2), we present an overview of dynamic Takagi Sugeno systems and LMI formulation of quadratic stability stability conditions. In section (3), we introduce fuzzy rules for defining the Lyapunov function and derive LMI conditions for stability and L_2 gain analysis. Our numerical results are presented in section (4), where we present a simple example to illustrate the effectiveness of the proposed stability conditions. Finally, conclusions and some future work are discussed in section (5).

2. TAKAGI-SUGENO FUZZY MODELS

A dynamic T-S fuzzy model is described by a set of fuzzy "IF... THEN" rules with fuzzy sets in the antecedents and dynamic LTI systems in the consequents. A generic T-S plant rule can be written as follows:

i^{th} Plant Rule: IF $x_1(t)$ is $\tilde{M}_{i1}$ and $\ldots, x_n(t)$ is $\tilde{M}_{in}$ THEN $\dot{x} = A_i x$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $i = \{1, \cdots, r\}$, r is the number of rules, $\tilde{M}_{ij}$ are input fuzzy sets, $A_i \in \mathbb{R}^{n\times n}$.
Using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(x) A_i x}{\sum_{i=1}^{r} w_i(x)} \tag{1}$$

where w_i is defined as

$$w_i(x) = \prod_{j=1}^{n} \mu_{ij}(x_j) \tag{2}$$

where μ_{ij} is the membership function of jth fuzzy set in the ith rule. Defining

$$\alpha_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)} \tag{3}$$

we can write (1) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x) \tag{4}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r} \alpha_i = 1$

The following stability theorem , essentially ensures simultaneous stabilization of the linear models at the antecedent of each rule [13].

Theorem 1. The closed loop fuzzy system (4) is globally asymptotically stable if there exist a common positive definite matrix P which satisfies the following Lyapunov inequalities:

$$\begin{aligned} &A_i^T P + P A_i < 0 \\ &i = 1, \cdots, r \end{aligned} \tag{5}$$

Proof: The proof can be easily obtained by multiplying both sides of the inequalities by α_i's and summing them up. ■
Although this condition guarantees stability, it can be very conservative, i.e., we can easily find T-S fuzzy systems, or in a broader sense, PLDIs which are stable, but are not quadratically stable, i.e., their stability can not be proven with a single quadratic function. One way of dealing with this problem, is to use the approach given in [8], that is, to use piecewise quadratic Lyapunov functions and then use the $\mathcal{S}$ procedure [2] to come up with stability conditions.

3. LYAPUNOV FUNCTION DESCRIBED BY T-S FUZZY RULES

In this section, we introduce T-S fuzzy rules that describe the Lyapunov function. Each T-S rule has fuzzy sets in the antecedents and quadratic Lyapunov functions in the consequents. A generic rule for the Lyapunov function can be written as follows:

i^{th} Rule for the Lyapunov Function:
IF $x_1(t)$ is $\tilde{M}_{i1}$ and $\ldots, x_n(t)$ is $\tilde{M}_{in}$ **THEN**

$$V = x' P_i x$$

. Using the same inference method as before, the Lyapunov function can be written as :

$$V = \sum_{j=1}^{r} x' P(x) x \tag{6}$$

Where

$$P(x) = \sum_{i=1}^{r} \alpha_i(x) P_i$$

Before obtaining the stability result, we need the following definition [16]: Let $\mathcal{P} \subset \mathbf{R}^{\mathbf{s}}$ be a compact set and $\nu_i, i = 1, \cdots, s$ be nonnegative numbers, then we define the **parameter ν-variation set** as:

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho(t) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i| \leq \nu_i, \forall t \geq 0\} \quad (7)$$

where $\rho_i(t)$ are a subset of the states which the membership functions depend on. In other words it is assumed that the state variables which describe the membership functions have bounded variations, and ν_i are upper bounds on the derivative of the states. Note that in a majority of practical applications usually membership functions in the antecedents depend on one or two state variables. The following theorem describes sufficient conditions for stability.

Theorem 2. Let $\{\rho_i(t)\}_{i=1}^{s} \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $1 \leq s \leq n$ be the state variables in the antecedents of T-S rules, then we have the following:
The T-S fuzzy system (4) is globally asymptotically stable if there exist positive definite matrices $P_1, \cdots, P_r$ which satisfy the following inequalities:

$$\begin{aligned} \mathcal{L}_i + \sum_{k=1}^{s}\sum_{l=1}^{r} \pm\nu_k\gamma_{kl}P_l &< 0 \\ \mathcal{M}_{ij} + \sum_{k=1}^{s}\sum_{l=1}^{r} \pm\nu_k\gamma_{kl}P_l &< 0 \\ P_i &> 0 \\ i = 1, \cdots, r \quad j < i \leq r & \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathcal{L}_i &= A_i^T P_i + P_i A_i \\ \mathcal{M}_{ij} &= A_i^T P_j + A_j^T P_i + P_j A_i + P_i A_j \end{aligned} \quad (9)$$

and

$$\gamma_{kl} > |\frac{\partial \alpha_l}{\partial \rho_k}|$$

is an upper bound on the variations of the membership functions, and $\pm$ indicates that all possible combinations of $+(.)$ and $-(.)$ should be considered.

Proof: The proof simply follows from selecting a Lyapunov function as in (6), and computing the derivative of the Lyapunov function along the trajectory of the T-S fuzzy system in (4), and bounding the derivative of the membership functions with respect to time with $\nu_k\gamma_{kl}$. The details follow the proof of Lemma 3.1 in [16] and the proof of theorem 1. ■

Remark 1. Note that since the membership functions are known, γ_{kj} can be easily computed, since the membership functions are usually triangular or Gaussian shaped.

Remark 2. Unfortunately there is no systematic way of picking the bounds ν_i, for the state variables that are arguments of input membership functions. However, the bounds can be picked initially to be large and then reduced until the conditions in Theorem 2 are violated. In the next section, we use these stability conditions to obtain less conservative bounds on L_2 gain of T-S fuzzy systems.

4. L_2 GAIN ANALYSIS OF T-S FUZZY SYSTEMS

consider the following T-S fuzzy system:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i w(t)) \\ z(t) &= \sum_{i=1}^{r} \alpha_i(x)(C_i x + D_i w(t)) \end{aligned} \quad (10)$$

Where $w(t)$ is in $L_2[0,\infty)$. The L_2 gain analysis problem is to check whether the induced L_2 norm of the above system, from input $w(t)$ to the output $z(t)$ is less than some γ or not. This can be checked by the following theorem:

Theorem 3. The T-S fuzzy system (10) has an L_2 gain less than γ if the following LMIs are feasible:

$$\begin{bmatrix} \mathcal{L}_i + \sum_{k=1}^{s}\sum_{j=1}^{r} \pm\nu_k\gamma_{kj}P_j & P_iB_i & C_i^T \\ B_i^T P_i & -I & D_i^T \\ C_i & D_i & -\gamma^2 I \end{bmatrix} < 0$$

$$\begin{bmatrix} \mathcal{M}_{ij} + \sum_{k=1}^{s}\sum_{j=1}^{r} \pm\nu_k\gamma_{kj}P_j & P_iB_j & P_jB_i & C_i^T \\ B_j^T P_i & -I & 0 & 0 \\ B_i^T P_j & 0 & -I & D_i^T \\ C_i & 0 & D_i & -\gamma^2 I \end{bmatrix} < 0$$

$$P_i > 0$$

$$i = 1, \cdots, r \quad j < i \leq r \quad (11)$$

Proof: The proof follows from Theorem 2, and the KYP lemma. See [1,16] for the proof in case of LPV systems. ■

5. NUMERICAL EXAMPLES

In this section we present a numerical example to illustrate the advantage of using parameter dependent Lyapunov functions rather than a single quadratic. However, the example is just an academic example without any physical motivation.

5.1 *Stability Analysis*

Consider the following T-S fuzzy model consisting of two rules.
If $x_1(t)$ is *small* Then $\dot{x}(t) = A_1 x(t)$

If $x_1(t)$ is *large* Then $\dot{x}(t) = A_2 x(t)$

Where *small* and *large* are fuzzy sets described by the following two triangular membership functions:

$$\mu_{small}(x_1) = 1 - |\frac{x_1}{3}| \quad \mu_{large}(x_1) = |\frac{x_1}{3}|$$

and A_1 and A_2 are given as follows:

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

Unfortunately, A single quadratic Lyapunov function that proves stability of the above system does not exist [9] . However, simulation results indicate that all trajectories for the above fuzzy model converge to zero. Simulation results are depicted in figures 1 and 2. We define the following T-S rules for the Lyapunov function candidate:

If $x_1(t)$ is *small* **Then** $V = x^T P_1 x$

If $x_1(t)$ is *large* **Then** $V = x^T P_2 x$

In order to use the result of theorem (2), we need to assume bounds on the derivative of the state variable x_1 as well as derivative of membership functions μ_1 and μ_2. Note that the membership functions depend only on one state therefore $s = 1$ in Theorem 2. The bounds on membership functions, i.e., γ_1 and γ_2 are easily computed to be equal to 0.33. For the rate of variation of x_1, we pick a positive number, and then check in the simulation to see if the bound is realistic. Finding an upper bound on the variation of the state on which the membership functions depend on is the biggest bottle neck of this method and other LPV based methods. However by picking an initial large number as an estimate on the upper bound and then iteratively reducing the upper bound, this problem can be solved.

Assuming $|\dot{x}_1(t)| < 3$, and solving the LMIs in (8), the following values are obtained for P_1 and P_2:

$$P_1 = \begin{bmatrix} 11.1173 & -3.0405 \\ -3.0405 & 26.2707 \end{bmatrix} \quad P_2 = \begin{bmatrix} 32.5864 & 1.7079 \\ 1.7079 & 6.9862 \end{bmatrix}$$

Simulation results validate the assumption that $|\dot{x}_1(t)| < 3$. Next, we present another example to compute the L_2 gain of a T-S fuzzy system using the result of Theorem 3.

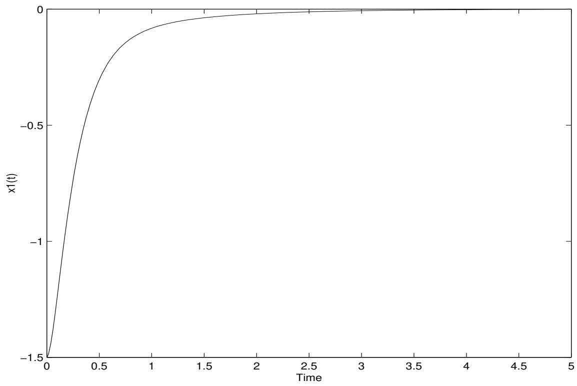

Fig. 1. Time response of $x_1(t)$.

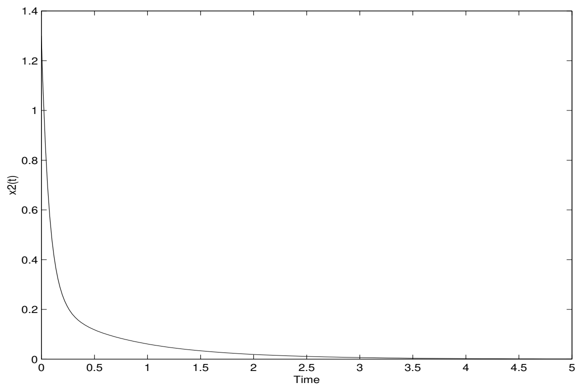

Fig. 2. Time Response of $x_2(t)$.

5.2 L_2 *Gain Analysis*

In the previous section, stability of the T-S system was proven using a membership function- dependent Lyapunov equation. We use the result of Theorem 3 to prove an upper bound on the L_2 gain of the following T-S fuzzy system:

If $x_1(t)$ is *small* Then $\dot{x}(t) = A_1 x(t) + B_1 w(t)$

If $x_1(t)$ is *large* Then $\dot{x}(t) = A_2 x(t) + B_2 w(t)$

Where A_1 and A_2 are the same as before and B_1 and B_2 are chosen to be the following:

$$B_1 = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \tag{12}$$

While a single quadratic Lyapunov function can not prove the stability, a state dependent Lyapunov function described by the following T-S fuzzy rules proves that the L_2 gain of the system from the disturbance $w(t)$ to the output $y(t) = x_1(t)$ is less than $\gamma = 0.703$. The bounds used for the derivative of the membership functions and $\dot{x}_1(t)$ were the same as in section 5.1.

6. CONCLUSIONS

This paper was an attempt to extend the stability analysis results for T-S fuzzy systems from a single quadratic Lyapunov function to parameter dependent Lyapunov functions. Since the Lyapunov function is described by T-S "If ... Then" rules, it depends on the membership functions and therefore on part of state variables. Hence, *a priori* bounds on the time derivative of normalized membership functions must be picked in advance. The major disadvantage of this approach is that it is not clear how to pick these bounds systematically other than using trial and error methods, so obviously there is a lot of work to be done in this area. However, the conservatism of stability conditions will be drastically reduced due to a much richer class of Lyapunov functions.

One of the major problems with LPV methods is picking the basis functions for the Lyapunov functions and using a T-S fuzzy model to describe the Lyapunov function seems to be a good alternative. The other advantage of using this type of analysis is that we don't need to solve the LMIs pointwise as it is required in LPV methods, since the T-S fuzzy systems are polytopic, and checking the stability of the vertices is sufficient for stability. Therefore, this method is less tedious than griding the parameter space and solving LMIs pointwise. Next step in this direction would be to extend the current results to state feedback design for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization.

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