Nonlinear $\mathcal{H}_\infty$ Control: An Enhanced Quasi-LPV Approach

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Abstract

Being formed into a state-dependent linear representation, an input-affine nonlinear system can be considered as a quasi linear parameter varying plant where the parameters are actually functions of the state variables. Recognizing that a critical element in this process is the choice of linear state-dependent representation, a technique is proposed that incorporates this design freedom into the control selection process in an optimal fashion. It is shown that the $\mathcal{L}_2$-gain analysis problem can be recast as an infinite dimensional convex optimization problem which can then be solved by gridding the parameter space. An iterative scheme similar to the D-K iteration in μ synthesis is proposed to solve the synthesis problem. Numerical examples are presented to illustrate the effectiveness of this method.

Keywords: Nonlinear $\mathcal{H}_\infty$ control, Linear Parameter Varying (LPV) methods, Linear Matrix Inequalities (LMI), Frozen Riccati Equations.

1. Introduction

In the past decade, the nonlinear $\mathcal{H}_\infty$ control problem has been studied extensively in the state-space framework. The control objective is to attenuate the $\mathcal{L}_2$-gain from the exogenous inputs to the controlled outputs. Most tools developed [5, 7, 9] rely on the solution to the Hamilton-Jacobi Inequality (HJI). Unfortunately, there is no efficient algorithm to solve the HJE/HJI, which is a first order partial differential equation/inequality, for systems with more than a few states. Based on the well-developed linear theory, a few alternative approaches [2, 8] were developed to handle optimal regulation and $\mathcal{H}_\infty$ control problems for a class of input-affine nonlinear systems which can be put into the state-dependent linear like structure. Since the corresponding HJE/HJI is reduced to some algebraic equation or inequality, these techniques are computationally more appealing.

The Frozen Riccati Equation (FRE) approach, initially proposed by Cloutier *et al* [2], is the simplest among them in terms of computation complexity and implementation. It works well for many examples in spite of the lack of a sound theoretical justification to guarantee even mere stability. On the other hand, the LPV technique [10] synthesizes controllers that guarantee the performance when the underlying parameters have bounded rate variation. When applied to nonlinear systems, the varying parameters become functions of the states instead of "free" variables, which can cause severe conservatism. Our recent research [6] revealed that different linear representations used for the FRE design may lead to quite different performance levels for the same nonlinear plant. It has also been proven that there always exists such a representation that the FRE recovers the optimal control under some assumptions on the value function. Expecting the similar effect on other schemes that employ the same representation of nonlinear systems, we propose a quasi-LPV strategy that incorporates this design freedom into the control selection process.

In this paper, an infinite dimensional Linear Matrix Inequality (LMI) condition is derived to compute the best achievable $\mathcal{L}_2$-gain over all the linear representations of the dynamics and parameter-dependent scaling functions. More specific, the problem of $\mathcal{L}_2$-gain analysis can be formulated as an infinite dimensional convex optimization problem which can be solved by gridding the parameter space and using the recently developed interior-point methods. Unfortunately, the problem of state feedback synthesis is not convex in both design variables. The synthesis problem can be cast as an infinite dimensional Bilinear Matrix Inequality (BMI) problem. Because of the non-convexity of the condition in both design variables, there is no guarantee in finding the global optimum. However, analogous to the "$D-K$ iteration" for μ synthesis [11], an iterative procedure is presented that attempts to find the best achievable performance. A practical computation scheme is proposed along with the examples demonstrating that the $\mathcal{L}_2$-gain of the closed loop system can be greatly improved when the choice of the quasi-LPV representation for the nonlinear dynamics is considered as one of the design variables. It is our hope that the introduction of this extra design freedom in forming the quasi-LPV model can be viewed as an enhancement to the existing LPV methodology, and will reduce the conservatism to a minimum. Even though we will concentrate solely on state feedback $\mathcal{H}_\infty$ control in this paper, the same strategy can be applied to other control objectives as well.

The notation used is fairly standard. We denote $\mathbf{R}^+$ as the set of non-negative real numbers, and $\mathcal{C}^1(\mathbf{R}^m, \mathbf{R}^n)$ as the set of continuously differentiable functions that map $\mathbf{R}^m$ to $\mathbf{R}^n$. The Euclidean norm of a vector is denoted by $\|\cdot\|$, i.e., $\|z\|^2 = z^T z$ for $z \in \mathbf{R}^k$. The notation $\mathcal{L}_2$ will be used for bounded energy signals, i.e., $\mathcal{L}_2(0,T) = \{w : \int_0^T \|w(t)\|^2 dt < \infty\}$.

2. Preliminary LPV Results

We first review some results for LPV systems with bounded parameter variation rates. (See [10] and the references therein for details.)

Define the allowable parameter set as

$$\mathcal{F}_\mathcal{P}^\nu := \{\rho \in \mathcal{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho(t) \in \mathcal{P},\ |\dot{\rho}_i| \le \nu_i,\ i = 1, \cdots, m\}$$

where $\mathcal{P} \subset \mathbf{R}^m$ is a compact set, and $\nu_i \ge 0$.

2.1. Analysis Test

The LPV system is governed by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, and A, B, C and D are continuous functions of proper dimensions. The next lemma gives a sufficient condition for the system to have an $\mathcal{L}_2$-gain bounded by γ.

Lemma 2.1 *If there exists a $\mathcal{C}^1$ positive definite function $P : \mathbf{R}^m \to \mathbf{R}^{n\times n}$ such that*

$$\begin{bmatrix} \mathcal{M}(\rho) & P(\rho)B(\rho) \\ B^T(\rho)P(\rho) & -\gamma^2 I \end{bmatrix} < 0 \tag{2}$$

$$\mathcal{M}(\rho) = \sum_{i=1}^{m} \pm \nu_i \frac{\partial P}{\partial \rho_i} + P(\rho)A(\rho) + A^T(\rho)P(\rho) + C^T(\rho)C(\rho)$$

for all $\rho \in \mathcal{P}$, then the LPV system (1) has $\mathcal{L}_2$-gain $\leq \gamma$ for all the admissible parameter trajectories.

Note that (2) actually represents 2^m inequalities since $\sum_{i=1}^{m} \pm(\cdot)$ indicates that every combination of $+(\cdot)$ and $-(\cdot)$ should be included.

The proof of the lemma is omitted here. The key point in the proof is the use of a parameter-dependent Lyapunov function $V(x,t) = x^T(t)W(\rho(t))x(t)$. While the quadratic Lyapunov functions are used for the linear time-invariant systems, this modification captures the varying nature of the LPV plant.

2.2. State Feedback Synthesis

Next we consider the state-feedback synthesis problem where u is the control input:

$$\begin{bmatrix} \dot{x}(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & 0 & 0 \\ C_2(\rho(t)) & 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{3}$$

For simplicity of the formulas, the standard assumptions are made in the above model. The feed-through term from w to z is assumed to be zero and the D_{22} term where the control u enters is scaled to I. More complicated results can be derived if these assumptions are relaxed.

We want to find a continuous function $F(\rho)$ such that the closed loop system has $\mathcal{L}_2$ gain less than γ under the state feedback law $u(t) = F(\rho(t))x(t)$.

Lemma 2.2 *If there exists a continuous differentiable matrix function $X(\rho) > 0$ for all $\rho \in \mathcal{P}$ that satisfies*

$$\begin{bmatrix} \mathcal{M}(\rho) & X(\rho)C_1^T(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{4}$$

$$\mathcal{M}(\rho) = -\sum_{i=1}^{m} \pm \nu_i \frac{\partial X}{\partial \rho_i} + \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}(\rho)^T - B_2(\rho)B_2^T(\rho)$$

where $\hat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$, then under the state feedback law

$$u(t) = -(B_2(\rho(t))X^{-1}(\rho(t)) + C_2(\rho(t)))x(t) \tag{5}$$

the closed loop system (3) has $\mathcal{L}_2$ gain $\leq \gamma$ for all the allowable parameter trajectories $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

The above result is a natural generalization of the linear $\mathcal{H}_\infty$ theory to LPV plants. Instead of a quadratic Lyapunov function for the linear time invariant plant, a parameter dependent Lyapunov function $V(x,t) = x^T(t)X^{-1}(\rho(t))x(t)$ is assumed. As a result, we have to solve the parameter-dependent LMI (4), which is an infinite dimensional convex optimization problem.

2.3. Computation Considerations

An *ad hoc* computation scheme can be used to solve the state feedback problem stated in Lemma 2.2. First choose a set of $\mathcal{C}^1$ functions $\{f_i(\rho)\}_{i=1}^{M}$ as a basis for $X(\rho)$, i.e.

$$X(\rho) = \sum_{i=1}^{M} f_i(\rho)X_i \tag{6}$$

where $X_i \in \mathcal{S}^{n\times n}$ is the coefficient matrix for $f_i(\rho)$. If $X(\rho)$ in (4) is substituted with (6), the cHf state feedback problem then becomes the following optimization:

$$\min_{\{X_i\}_{i=1}^{M}} \gamma^2$$

subject to

$$\begin{bmatrix} \mathcal{M}(\rho) & \sum_{j=1}^{M} f_j(\rho)X_jC_1^T(\rho) & B_1(\rho) \\ C_1(\rho)\sum_{j=1}^{M} f_j(\rho)X_j & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{7}$$

$$\sum_{j=1}^{M} f_j(\rho)X_j > 0 \tag{8}$$

where

$$\begin{aligned} \mathcal{M}(\rho) \quad = \quad & -\sum_{i=1}^{m} \pm \nu_i \Big(\sum_{j=1}^{M} \frac{\partial f_j}{\partial \rho_i} X_j\Big) \\ & + \sum_{j=1}^{M} f_j(\rho)(\hat{A}(\rho)X_j + X_j\hat{A}(\rho)^T) - B_2(\rho)B_2^T(\rho) \end{aligned}$$

Note that both (7) and (8) are LMI's in terms of the matrix variables $\{X_i\}_{i=1}^{M}$ when the parameter ρ is fixed. To solve this infinite dimensional convex optimization problem, we can grid the parameter set $\mathcal{P}$ by L points $\{\rho_k\}_{k=1}^{L}$ in each dimension. Then evaluate the above LMI's at these points. Since (7) consists of 2^m constraints, a total of $(2^m + 1)L^m$ affine matrix inequalities in terms of M matrix variables $\{X_i\}$ have to be solved. An approximation of the density of the gridding points that guarantees the global solvability of the LMI's is given in [10].

This computation scheme has a few obvious limitations. Associated with gridding is the "curse of dimensionality". Fortunately, the number of varying parameters is usually not more than 3. Another significant problem is the lack of guidance in choosing the basis functions for $X(\rho)$. Despite these drawbacks, it provides us a tool to construct control laws for LPV systems that theoretically guarantees the performance for parameter trajectories with a priori variation rate.

3. Main Results

Gain scheduling has been a common practice to deal with nonlinear plants in many engineering applications. Traditionally, the designs are based on linear-invariant approximations of the nonlinear plant at several operation points. The recent development of the LPV techniques provide a systematic way to design controllers that schedule on varying parameters. Treating the nonlinear dynamics as the state-dependent linear system, we can directly apply the LPV technique to a large class of nonlinear systems. Based on the observation that the quasi-LPV model is not uniquely determined by the nonlinear dynamics, we propose a scheme that incorporates the freedom we have in choosing the LPV representation into the optimization problem for both $\mathcal{L}_2$-gain analysis and synthesis.

3.1. Quasi-LPV Model of Input-Affine Systems

We first establish the parameterization of all the state-dependent linear representations of some nonlinear dynamics.

Lemma 3.1 *Suppose a function $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuously differentiable with $f(0) = 0$. Then a continuous matrix-valued function $A_o(x) : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ can alway be found such that $f(x) = A_o(x)x$. Moreover, all the possible $A(x)$ satisfying $f(x) = A(x)x$ can be parameterized as*

$$A(x) = A_o(x) + N(x)$$

where $N : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ satisfies $N(x)x = 0$.

Note that $A(x)$ in the above lemma is not the linearization of $f(x)$. In fact, we have an infinite number of possible matrix representations $A(x)$ for a given $f(x)$. Following the lemma, the input-affine nonlinear dynamics can be formed into

$$\dot{x} = A(x)x + g(x)w,$$

which looks like a linear system except that the state matrices depend on the state variables instead of being constant as in the linear case. To keep the number of varying parameters in the state matrices A and g to a minimum, the dependence of the state matrices $A(x)$ and $g(x)$ on the state variables will be changed to $\rho(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^m)$ with $m \le n$, i.e. $A(x)$ and $g(x)$ become $A(\rho(x))$ and $g(\rho(x))$ respectively. For instance, $\rho(x)$ can simply represent part of the state variables.

The following quasi-LPV representation of the nonlinear input-affine system will be used for the synthesis problem:

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(\rho(x)) & B_1(\rho(x)) & B_2(\rho(x)) \\ C_1(\rho(x)) & 0 & 0 \\ C_2(\rho(x)) & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{9}$$

This is different from the true LPV model (3) only in that the underlying parameters ρ_i's are now functions of the state variables, i.e $\rho = \rho(x)$. Not really varying freely in time, the trajectory of the parameters is actually determined by the behavior of the closed-loop system. Aiming at attenuating the effect of the exogenous inputs for all the allowable parameter trajectories, the LPV treatment will inevitably introduce some conservatism due to the connection between the parameters and the states.

One of the difficulties when applying Lemma 2.2 is to determine the bound of the variation rate of ρ since it is no longer known a priori to us. One way to bypass this problem is to restrict X in Lemma 2.2 to be constant, i.e only the quadratic storage functions are considered. Since the variation rate ν plays no role in this case, the $\mathcal{L}_2$-gain is guaranteed for all state trajectories. More often, however, it is too conservative to restrict our search to quadratic functions. A more practical approach is to use the best estimate of the operating range of the chosen varying parameters, and an approximation on the parameter variation rates. These assumptions should be verified by simulations after the controller is designed.

As stated in Lemma 3.1, we have infinite number of choices when constructing the quasi-LPV model from the nonlinear dynamics. However, they are not equal in terms of the achievable performance when the LPV technique is applied. Here is an illustrative example.

Example 3.2

$$\begin{cases} \dot{x}_1 = x_2 + (x_2 - 1)u \\ \dot{x}_2 = x_1 + x_2 + w - (x_1 + 2)u \\ z = [x_1\ x_2\ u]^T \end{cases}$$

Though the dynamics are linear without control, it is not any easier than any other nonlinear problem if the HJI has to be solved. Since this example was created using the converse Hamilton-Jacobi method [3], we know that a closed loop $\mathcal{L}_2$ gain of 1 can be achieved by state feedback $u = x_1 + 2x_2$, and $V(x) = x_1^2 + x_2^2$ serves as a storage function. However, if the obvious choice of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is used to form the quasi-LPV model, no finite $\mathcal{L}_2$ gain can be achieved by state feedback since the pair $[A(x)\ B_2(x)]$ is not controllable on two lines in the state space: $x_2 = \frac{1}{2}(1+\sqrt{5})x_1 + 2 + \sqrt{5}$ and $x_2 = \frac{1}{2}(1-\sqrt{5})x_1 + 2 - \sqrt{5}$, and A is simply unstable. Fortunately, not every choice of $A(x)$ has this problem. If we are lucky enough to use this one:

$$A(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}x_2 - 1 \\ -\frac{1}{2}x_1 - 2 \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \end{bmatrix}$$

then it is easy to check that (4) is satisfied with any $\gamma > 1$ and $X(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This example demonstrates that the differences in performance levels achieved by LPV controllers can be huge due to different choices of $A(x)$. In our recent research on the frozen Riccati approach to the nonlinear optimal regulation problem, we were able to show that there always exists a representation $f(x) = A(x)x$ such that the solution to the frozen Riccati equation recovers the optimal control, provided that the optimal value function has a certain form [6]. This means if the right $A(x)$ is used, then one only needs to solve the algebraic Riccati equation at any instant to get the optimal control. The bad news is that we cannot find the optimal $A(x)$ without figuring out the optimal value function. For the nonlinear $\mathcal{H}_\infty$ problem, we don't have rigorous theoretical justification of the existence of such optimal $A(x)$. However, the difference in the achievable performance delivered by different $A(x)$ under the quasi-LPV framework is too big to ignore. It is usually the case that the obvious choice does not lead to the best performance. In the next section, we will propose a systematic way to gear the original choice of $A(x)$ toward those that can result in the best performance.

3.2. $\mathcal{L}_2$-Gain Analysis

The following theorem is a direct application of the non-uniqueness of the quasi-LPV model. Consider the following model

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{10}$$

Assume the underlying parameter ρ varies in the allowable set

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho \in \mathcal{C}^1(\mathbf{R}^+, \mathbf{R}^m) : \rho \in \mathcal{P},\ \underline{\nu}_i(\rho) \le \dot{\rho}_i \le \overline{\nu}_i(\rho), \\ i = 1, \cdots, m\} \tag{11}$$

where $\mathcal{P} \subset \mathbf{R}^m$ is a compact set. Note that the bounds on the rate variation of the parameter are allowed to vary with the parameter value.

Theorem 3.3 *If there exists a positive $P(\rho(x)) > 0$ and a $W(\rho(x))$ with $W(\rho(x))x = 0$ that satisfy the following matrix inequality:*

$$\begin{bmatrix} \mathcal{M}(\rho) & P(\rho)B(\rho) \\ B^T(\rho)P(\rho) & -\gamma^2 I \end{bmatrix} < 0 \tag{12}$$

$$\mathcal{M}(\rho) = \sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho)\tfrac{\partial P}{\partial \rho_i} + P(\rho)A(\rho) + A^T(\rho)P(\rho) \\ +W(\rho) + W^T(\rho) + C^T(\rho)C(\rho) \tag{13}$$

for all $\rho \in \mathcal{P}$, then the nonlinear system (10) has $\mathcal{L}_2$-gain less than γ.

The notation $\sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho)$ in (12) means that every combination of $\overline{\nu}_i(\rho)$ and $\underline{\nu}_i(\rho)$ should be included in the inequality. For instanece, when $m = 2$, $\overline{\nu}_1(\rho) + \overline{\nu}_2(\rho)$, $\overline{\nu}_1(\rho) + \underline{\nu}_2(\rho)$, $\underline{\nu}_1(\rho) + \overline{\nu}_2(\rho)$ and $\underline{\nu}_1(\rho) + \underline{\nu}_2(\rho)$ should be checked individually. In other words, (12) actually represents 2^m inequalities.

Proof.* Let

$$N(\rho) = P^{-1}(\rho)W(\rho).$$

Substituting $W(\rho)$ with $P(\rho)N(\rho)$, (13) becomes

$$\begin{aligned} \mathcal{M}(\rho) &= \sum_{i=1}^{m} \underline{\overline{\nu}}_i(\rho)\frac{\partial P}{\partial \rho_i} + P(\rho)(A(\rho) + N(\rho)) \\ &\quad +(A(\rho) + N(\rho))^T P(\rho) + C^T(\rho)C(\rho) \end{aligned}$$

Since $W(\rho(x))x = 0$, we have $N(\rho(x))x = 0$ from (3.2). Therefore (12) is just the $\mathcal{L}_2$ analysis constraints of the same nonlinear system for a different quasi-LPV representation $\hat{A}(\rho) = A(\rho) + N(\rho)$. □

Recognizing the freedom we have in choosing $A(\rho)$ for a nonlinear plant, we introduce an extra design variable $W(\rho)$ to the standard LPV analysis matrix inequalities (2). The interesting point here is the way $W(\rho)$ entering the LMI, which preserves the convexity of the constraint in both design variables $P(\rho)$ and $W(\rho)$. The same computational considerations previously reviewed can be directly applied. The only extra requirement is $W(\rho(x))x = 0$, which can be enforced by choosing a set of basis functions that guarantee it.

The purpose of introducing this extra design variable $W(\rho)$ is to reduce the conservatism of using the LPV method to handle the nonlinear system. As we pointed out earlier, the achievable performance level calculated by the LPV method is greatly dependent on the choice of $A(\rho)$. Most often, however, it is not at all trivial to pick a good $A(x)$. The free design variable $W(\rho)$ provides a systematic way to make the selection process tractable.

Example 3.4 *Consider the TORA example [1] which was introduced as a benchmark for nonlinear control design. The non-dimensional equations of motion are given by*

$$\dot{x} = f(x) + B_1(x)w + B_2(x)u$$

where

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1-\epsilon^2\cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1-\epsilon^2\cos^2 x_3} \end{bmatrix},$$

$$B_1(x) = \begin{bmatrix} 0 \\ \frac{1}{1-\epsilon^2\cos^2 x_3} \\ 0 \\ \frac{-\epsilon\cos x_3}{1-\epsilon^2\cos^2 x_3} \end{bmatrix}, \quad B_2(x) = \begin{bmatrix} 0 \\ \frac{-\epsilon\cos x_3}{1-\epsilon^2\cos^2 x_3} \\ 0 \\ \frac{1}{1-\epsilon^2\cos^2 x_3} \end{bmatrix}.$$

and $\epsilon = 0.3$ is the parameter. It can be shown that the constant state feedback controller $u = -Kx$ with

$$K = [-0.2380\ 0.2248\ 1.0000\ 1.4638].$$

is stabilizing for $x_3 \in [-\pi/2, \pi/2]$ when $w = 0$. Our objective is to compute the $\mathcal{L}_2$-gain of the closed system from the disturbance w to the regulated output given by

$$z = [0.5x_1\ \ x_3\ \ u]^T.$$

A natural choice of the quasi-LPV model is

$$A(\rho) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\epsilon^2\cos^2 x_3} & 0 & 0 & \frac{\epsilon x_4 \sin x_3}{1-\epsilon^2\cos^2 x_3} \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon\cos x_3}{1-\epsilon^2\cos^2 x_3} & 0 & 0 & \frac{\epsilon^2 x_4 \cos x_3 \sin x_3}{1-\epsilon^2\cos^2 x_3} \end{bmatrix}$$

where $\rho = [x_3\ x_4]$ is used as the varying parameter. Restricting the operating range of (x_3, x_4) to be $\mathcal{P} = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-0.5, 0.5]$, we grid $\mathcal{P}$ uniformly with 9 points on x_3 and 7 points on x_4. The range of x_4 is decided based on the simulation of the closed-loop system with the given state feedback. Since x_4 is the derivative of x_3, we are able to use the exact variation rate of x_3, which should incur no conservatism at all. Also from the simulation, a rate bound of 0.5 is tentatively set for x_4. Different basis functions for $P(\rho)$ are used for comparison:

$$\beta_1 = \{1, \frac{2}{\pi}x_3, (3(\frac{2}{\pi}x_3)^2 - 1)/2\}$$

$$\beta_2 = \{1, \tfrac{2}{\pi}x_3, (3(\tfrac{2}{\pi}x_3)^2 - 1)/2, (5(\tfrac{2}{\pi}x_3)^3 - 3(\tfrac{2}{\pi}x_3))/2, \\ 35(\tfrac{2}{\pi}x_3)^4 - 30(\tfrac{2}{\pi}x_3)^2 + 3)/2\}$$

$$\beta_3 = \{1, \frac{2}{\pi}x_3, (3(\frac{2}{\pi}x_3)^2 - 1)/2, 2x_4, (3(2x_4)^2 - 1)/2, x_3x_4\}$$

Method	Constant P	Parameter-dependent P		
		β_1	β_2	β_3
LPV w/o $W(\rho)$	not feasible	73.6	73.2	71.4
LPV w/ $W(\rho)$	66.4	8.2	8.0	7.4

Table 1: $\mathcal{L}_2$-gain of the closed-loop system of Example 3.4 with the given state feedback law

Note that the above basis are Legendre polynomials on the operating regions. The basis in β_1 and β_2 are functions of only x_3, thus the LPV algorithm should yield a guaranteed bound on the $\mathcal{L}_2$-gain since the exact variation rate is used for x_3.

To impose the constraint of $W(\rho(x))x = 0$, we assume that $W(\rho(x))$ is of the following form

$$W(\rho(x)) = V(\rho(x))[0\ 0\ x_4\ \ -x_3]$$

where $F(\rho)$ is a 4×1 vector function. Picking a basis of $\{g_i(\rho)\} = \{1,\ x_3,\ x_4\}$ for $V(\rho)$, we have

$$W(\rho(x)) = \sum_{i=1}^{3} g_i(\rho) V_i [0\ 0\ x_4\ \ -x_3].$$

with $V_i \in \mathbf{R}^4$.

Table 1 lists the $\mathcal{L}_2$-gain computed using both the standard and new LPV algorithms. The LMI's (12) from the standard LPV method on the gridding points are not feasible for constant P. With the extra variable $W(\rho)$ in the LMI's, we obtain significantly less conservative numbers. Also notice that allowing P to be parameter-dependent greatly reduces conservatism. When β_1 is used, the coefficient matrices Z_i returned by the optimization are as follows:

$$Z_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ Z_2 = \begin{bmatrix} 2.70 \\ -1.85 \\ -0.68 \\ 1.12 \end{bmatrix},\ Z_3 = \begin{bmatrix} -0.11 \\ 1.48 \\ -0.32 \\ -0.08 \end{bmatrix}.$$

Similar results are obtained for other basis functions.

As we know, the $\mathcal{H}_\infty$ norm of the linearized system at the origin provides a lower bound for the best global $\mathcal{L}_2$-gain for the nonlinear system. Not to our surprise, the linearized closed-loop system with the given state feedback law indeed has a lower $\mathcal{H}_\infty$ norm of 6.6 at the origin. In fact, the best achievable $\mathcal{H}_\infty$ norm of the original system near zero is 4.6.

3.3. State Feedback $\mathcal{H}_\infty$ Synthesis: $X - N$ Iteration

Consider the quasi-LPV model (9). Assume the underlying parameter ρ varies in the allowable set $\mathcal{F}_{\mathcal{P}}^{\nu}$ (11). The next theorem is based on Lemma 2.2 and Lemma 3.1.

Theorem 3.5 *If we can find an $X(\rho(x)) > 0$ and an $N(\rho(x))$ with $N(\rho(x))x = 0$ that satisfy the following matrix inequality:*

$$\begin{bmatrix} \mathcal{M}(\rho) & X(\rho)C_1^T(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \qquad (14)$$

$$\begin{aligned} \mathcal{M}(\rho) \quad &= \quad -\sum_{i=1}^{m} \underline{\nu}_i(\rho) \frac{\partial X}{\partial \rho_i} + (\hat{A}(\rho) + N(\rho))X(\rho) \\ &\quad + X(\rho)(\hat{A}(\rho) + N(\rho))^T - B_2(\rho)B_2^T(\rho) \end{aligned}$$

with $\hat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$ for all $\rho \in \mathcal{P}$, then the closed loop system (9) has $\mathcal{L}_2$-gain $\leq \gamma$ under the state feedback law:

$$u(x) = -(B_2(\rho(x))X^{-1}(\rho(x)) + C_2(\rho(x)))x. \qquad (15)$$

Compared to the standard LPV result stated in Lemma 2.2, a new matrix-valued function $N(\rho(x))$ is added to take advantage of the non-uniqueness of the quasi-LPV model of the nonlinear dynamics. The best γ level associated with any particular $A(\rho)$ from Lemma 2.2 serves as an upper bound of the best achievable $\mathcal{L}_2$-gain of the closed-loop nonlinear system.

Note that the constraint (14) is not affine in $X(\rho)$ and $N(\rho)$ in contrast to the analysis result. For any value of ρ, (14) is a bilinear matrix inequality (BMI) in X and N. Because the BMI can be reformulated as a nonconvex programming problem, no tractable algorithm is believed to exist for a general BMI. However, there are computational procedures suggested for solving the BMI via a global optimization [4], though they exhibit exponential growth.

Even though minimizing γ over both X and N under the BMI constraint (14) is not a convex optimization, when we fix X or N in (14) and try to minimize γ over the other, it is a convex problem. Similar to the "$D - K$" iteration for μ synthesis, an "$X - N$" iteration is proposed as a practical scheme to achieve the best closed loop performance:

1. Find a quasi-LPV representation (9) for the nonlinear open loop system.
2. Choose the varying parameters ρ, the range of the parameter $\mathcal{P}$ and the variation rate bound ν.
3. Calculate the best achievable γ level for the current quasi-LPV model (Lemma 2.2). If the performance is satisfactory, stop.
4. Relax γ slightly, solve (4) for a feasible $X(\rho) > 0$.
5. Calculate the best γ over all $N(\rho)$ with $X(\rho)$ fixed from the previous step (Theorem 3.5). If the new γ level is satisfactory or unchanged, stop.
6. Relax γ slightly, solve (14) with $X(\rho)$ from step 4 for a feasible $N(\rho)$.
7. Replace $A(\rho)$ with $A(\rho) + N(\rho)$. Go to step 3.

Note that this strategy is not guaranteed to converge to the global optimum because it is not convex in both X and N. At step 4 and 6, we try to avoid using $X(\rho)$ or $N(\rho)$ associated with the minimum γ at the previous step, because it usually does not lead to the best overall γ level. From our experience, $X(\rho)$ and $N(\rho)$ that correspond to a relaxed γ are more likely to bring γ further down at the next step.

The practical computation scheme described in section 2.3, which involves gridding the parameter space $\mathcal{P}$ and employing basis functions for $X(\rho)$, can be used to solve LMI (4) in steps 3 and 4. At steps 5 and 6, the BMI (14) is reduced to an LMI in $N(\rho)$ since $X(\rho)$ is fixed. Selecting a set of basis functions for $N(\rho)$ that reflect the requirement of $N(\rho(x))x = 0$, we can get an approximate solution to (14) by solving the collection of the LMI's evaluated at the grid points of $\mathcal{P}$.

LPV method	Best γ (synthesis)	X	Best γ (analysis)
Standard	2.4	$\begin{bmatrix} 0.95 & 0.00 \\ 0.00 & 0.09 \end{bmatrix}$	1.7
Enhanced	1.0	$\begin{bmatrix} 0.99 & 0.00 \\ 0.00 & 0.95 \end{bmatrix}$	1.0

Table 2: Performance comparison of the standard and the enhanced LPV synthesis for Example 3.6

Example 3.6

$$\begin{cases} \dot{x}_1 &=& x_1 + 2x_1x_2^2 + w_1 + 2u \\ \dot{x}_2 &=& 4x_1^2 - x_2 + w_2 + 2x_1u \\ z &=& [x_1 \ x_2 \ u]^T \end{cases}$$

The control object is to find a state feedback that minimizes the $\mathcal{L}_2$-gain of the closed loop system. Again, this example was created so that we know that the closed-loop $\mathcal{L}_2$-gain of 1 can be achieved by the state feedback $u = -2x_1 - 2x_1x_2$ and $V(x) = x_1^2 + x_2^2$ serves as a storage function.

We begin with an arbitrary choice of $A(x)$:

$$A(x) = \begin{bmatrix} 1 & 2x_1x_2 \\ 4x_1 & -1 \end{bmatrix}$$

and other state space matrices needed in model (9):

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 2 \\ 2x_1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Same as the in the previous example, $\rho = [x_1 \ x_2]^T$ is used as the varying parameter. $\mathcal{P} := [-10, 10] \times [-10, 10]$ is uniformly gridded. To avoid the problem caused by the rate variation estimation, we only consider the quadratic storage functions, i.e., X is restricted to be constant. Notice in the two dimensional case that all the $N(x)$ satisfying $N(x)x = 0$ can be parameterized as $N(x) = \Phi(x)[x_2 \ \ -x_1]$ where $\Phi(x) : \mathbf{R}^2 \to \mathbf{R}^{2\times 1}$. A set of basis functions $\{1, \ x_1, \ x_2\}$ for $\Phi(x)$ is chosen for this example.

The computation results are shown in Table 2. With the initial choice of $A(\rho)$, the best γ level over all the positive definite scaling functions X obtained by the standard LPV scheme is 2.4, though the $\mathcal{L}_2$-gain analysis (Theorem 3.3) of the closed-loop system gives a better bound of 1.7. However, starting from the same initial $A(\rho)$, γ is reduced to 1.0 after a single "$X - N$ iteration."

4. Conclusions

It is shown through examples that the performance of the quasi-LPV controller is greatly dependent upon the choice of the model. The standard LPV analysis and state feedback synthesis tests are enhanced by incorporating a new design variable which represents the parameterization of the quasi-LPV model. Under this new framework, the $\mathcal{L}_2$-gain analysis is reformulated as a minimization problem over an affine matrix inequality constraint, and an approximate solution can be obtained by a tractable computation which involves solving a collection of LMI's. In contrast, the state feedback synthesis problem appears to be a "bilinear" constraint which is not jointly convex in both unknown variables. However, an iterative algorithm is proposed which divides it into two convex optimization problems. Although not guaranteed to converge to the global optimum, its effectiveness is demonstrated by a numerical example.

Even though conservatism is reduced by the enhanced quasi-LPV scheme, the optimal achievable performance still may not be obtained. In order to limit the computational load, the number of underlying parameters can be much less than the number of states. Consequently, the search space of $A(x)$ is rather limited in comparison to the complete parameterization.

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