Distributed Geodesic Control Laws for flocking of Nonholonomic Agents†

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***Abstract*— We study the problem of flocking and velocity alignment in a group of kinematic nonholonomic agents in 2 and 3 dimensions. By analyzing the velocity vectors of agents on a circle (for planar motion) or sphere (for 3D motion), we develop a geodesic control law that minimizes a misalignment potential and results in velocity alignment and flocking. The proposed control laws are distributed and will provably result in flocking when the underlying proximity graph which represents the neighborhood relation among agents is connected. We further show that flocking is possible even when the topology of the proximity graph changes over time, so long as a weaker notion of joint connectivity over time is preserved.**

I. Introduction

Cooperative control of multiple autonomous agents has become a very active part of control theory research. The main underlying theme of this line of research is to analyze and/or synthesize spatially distributed control architectures that can be used for motion coordination of large groups of autonomous vehicles. Each vehicle is assumed to be capable of local sensing and communication, and there is often a global objective, such as swarming, or reaching a stable formation, etc. A non-exhaustive list of relevant research in control theory and robotics includes [2], [4]–[6], [9], [13], [16]–[19], [21], [22], [30].

On the other hand, such problems of distributed coordination have also been studied in areas as diverse as statistical physics and dynamical systems (in the context of synchronization of oscillators and alignment of self propelled particles [25], [29]), in biology, and ecology, and in computer graphics in the context of artificial life and simulation of social aggregation phenomena.

Most of the above cited research on distributed control of multi-vehicle systems has been focused on fully actuated systems [20], [27], or planar under-actuated systems [16], [23], [28]. Our goal here is to develop motion coordination algorithms that can be used for distributed control of a group of nonholonomic vehicles in 2 and 3 dimensions. Using results of Bullo *et al.* [3] we develop *geodesic control laws* that result in flocking and velocity alignment for nonholonomic agents in 3 dimensions.

In order to introduce the idea of a geodesic control law to the reader, we start with the special case of planar motion in section III. We will show that the planar version of such a control law (where the velocity vector is restricted to stay on a circle) is exactly the well-known Kuramoto model of coupled nonlinear oscillators [14], [23], [24]. Such a control law is a gradient controller that minimizes a potential function which represents the aggregate "misalignment energy" between all agents. In section V we return to the general case of 3D motion and we develop control laws that result in stable coordination and velocity alignment of a group of agents with a fixed connectivity graph. In section VI we show that flocking is possible even when the topology of the proximity graph changes over time. Finally, in section VII we provide simulations that show the effectiveness of the designed controllers. But, let us review the concepts of graph theory that we use in this paper for stability analysis.

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II. Graph Theory Preliminaries

In this section we introduce some standard graph theoretic notation and terminology. For more information, the interested reader is referred to [10].

An (undirected) graph $\mathbb{G}$ consists of a vertex set, $\mathcal{V}$, and an edge set $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathbb{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be adjacent, or neighbors and we denote this by writing $x \sim y$. The number of neighbors of each vertex is its valence. A path of length r from vertex x to vertex y is a sequence of $r+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathbb{G}$, then $\mathbb{G}$ is said to be connected. If there is such a path on a directed graph ignoring the direction of the edges, then the graph is weakly connected.

The adjacency matrix $A(\mathbb{G}) = [a_{ij}]$ of an (undirected) graph $\mathbb{G}$ is a symmetric matrix with rows and columns indexed by the vertices of $\mathbb{G}$, such that $a_{ij} = 1$ if vertex i and vertex j are neighbors and $a_{ij} = 0$, otherwise. The valence matrix $D(\mathbb{G})$ of a graph $\mathbb{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valence of vertex i. The (un)directed graph of a (symmetric) matrix is a graph whose adjacency matrix is constructed by replacing all nonzero entries of the matrix with 1. Matrix A has *property SC* if and only if $|A|$ is the adjacency matrix of a strictly connected graph.

The symmetric singular matrix defined as:

$$L(\mathbb{G}) = D(\mathbb{G}) - A(\mathbb{G})$$

is called the Laplacian of $\mathbb{G}$. The Laplacian matrix captures many topological properties of the graph. The Laplacian L is a positive semidefinite M-matrix (a matrix whose off-diagonal entries are all nonpositive) and the algebraic multiplicity of its zero eigenvalue (i.e., the dimension of its kernel) is equal to the number of connected components in the graph. The n-dimensional eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}$.

Given an orientation of the edges of a graph, we can define the *incidence matrix* of the graph to be a matrix B with rows indexed by vertices and columns indexed by edges with entries of 1 representing the source of a directed edge and -1 representing the sink. The Laplacian matrix of a graph can also be represented in terms of its incidence matrix as $L = BB^T$ independent of the orientation of the edges.

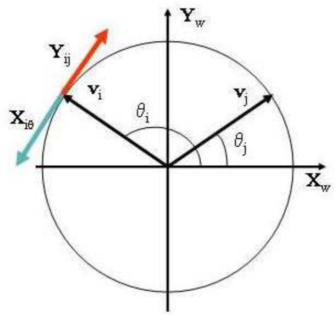


Fig. 1. The velocity vectors of agent i and its neighbor j are projected on the unit circle. $X_{i\theta}$ is the tangent vector to v_i, and depending on the relative orientations of v_i and v_j we could have $Y_{ij} = X_{i\theta}$ or $Y_{ij} = -X_{i\theta}$.

III. Distributed Control of planar nonholonomic vehicles

Consider a group of N agents on a plane. Each agent is capable of sensing some information from its neighbors as defined by:

$$\mathcal{N}_i \doteq \{j | i \sim j\} \subseteq \{1, \ldots, N\} \backslash \{i\}. \tag{1}$$

The neighborhood set of agent i, $\mathcal{N}_i$, is a set of agents that can share their heading (orientation) information with agent i. The size of the neighborhood depends on the characteristics of the communication device. We therefore assume that there is a predetermined radius R which determines the neighborhood relationship. The location of agent $i, (i = 1, \ldots, N)$ in the world coordinates is given by (x_i, y_i) and its velocity is $v_i = (\dot{x}_i, \dot{y}_i)^T$. The heading or orientation of agent i is θ_i and is given by: $\theta_i = \text{atan2}\big(\dot{y}_i, \dot{x}_i\big)$.

It is assumed that all agents move with constant unit speed. Thus, the kinematic model of each agent can be written as

$$\begin{aligned} \dot{x}_i &= \cos\theta_i \\ \dot{y}_i &= \sin\theta_i \\ \dot{\theta}_i &= \omega_i \qquad\qquad i = 1, \ldots, N \end{aligned} \tag{2}$$

The goal is to design the control input ω_i so that the group of mobile agents' headings reach agreement and velocity vectors are aligned. We therefore define the consensus state as follows:

Definition 3.1: The state where the headings of all agents are the same is called the consensus state.

In this paper, we say agents are "flocking" when their headings reach asymptotic agreement [7]. We consider the case where the neighboring relations among agents are represented by a fixed weighted graph.

Definition 3.2: The proximity graph $\mathbb{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ is a weighted graph consisting of:

- *a set of vertices $\mathcal{V}$ indexed by the set of mobile agents;*
- *a set of edges $\mathcal{E} = \{(v_i, v_j) \mid v_i, v_j \in \mathcal{V}$, and $i \sim j\}$;*
- *a set of weights $\mathcal{W}$, over the set of edges $\mathcal{E}$.*

In order to design the desired control law for agent i, let us view all the velocity vectors of neighbors of agent i in a unit circle as shown in Figure 1. Each velocity vector v_i can be written in terms of the heading angle θ_i (measured in a fixed inertial frame) as follows

$$v_i = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}, \ i = 1, \ldots, N. \tag{3}$$

As the velocity vector v_i changes, we can write the dynamics equation corresponding to the motion of agent i as $\dot{v}_i = \omega_i X_{i\theta}$ where vector $X_{i\theta}$ is tangent to v_i and is given by

$$X_{i\theta} = \begin{pmatrix} -\sin\theta_i \\ \cos\theta_i \end{pmatrix}, \ i = 1, \ldots, N.$$

Note: In the following the standard inner product is denoted by $< \cdot, \cdot >$, and the cross product by $\times$.

Let α_{ij} be the angle between two velocity vectors v_i and v_j, $\alpha_{ij} = |\theta_i - \theta_j|$. When v_i and v_j are neither equal nor opposite $(0 < \alpha_{ij} < \pi)$, we can define a unit vector Y_{ij} tangent to v_i such that it is pointing towards the velocity vector v_j. This unit-length vector is defined as:

$$Y_{ij} \doteq \frac{v_j^{\perp}}{|v_j^{\perp}|} = \frac{(v_i \times v_j) \times v_i}{\| (v_i \times v_j) \times v_i \|} = \frac{v_j - < v_i, v_j > v_i}{\sin\alpha_{ij}} \tag{4}$$

where $v_j^{\perp}$ is the component of v_j orthogonal to v_i. Now, we can prove the following theorem for the distributed control of the velocity vectors of a group of N agents.

Theorem 3.3: Consider the system of N equations $\dot{v}_i = \omega_i X_{i\theta}$, $i = 1, \ldots, N$. If the proximity graph is fixed and connected, then by applying the control law

$$\omega_i = \sum_{j \in \mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\theta} > = \sum_{j \in \mathcal{N}_i} < v_j, X_{i\theta} > \tag{5}$$

all trajectories converge to the set of equilibrium points given by $\dot{\theta} = 0$. Furthermore, consensus state is locally asymptotically stable.

Proof: We observe that on the unit circle $Y_{ij} = X_{i\theta}$ or $Y_{ij} = -X_{i\theta}$, depending on the orientations of v_i and v_j. Hence we write the input (5) as

$$\omega_i = -\sum_{j \in \mathcal{N}_i} \sin(\theta_i - \theta_j). \tag{6}$$

Notice that the above input is exactly the one used in the Kuramoto model of coupled nonlinear oscillators [14], [18], [24].

Assume an arbitrary orientation for the edges of graph $\mathbb{G}$. Consider the $N \times d$ incidence matrix, B, of this oriented graph with N vertices and d edges. Then, we can write (6) as:

$$\dot{\theta} = \omega = -B \sin(B^T \theta) \tag{7}$$

where $\theta = [\theta_1, \ldots, \theta_N]^T$, and $\sin(B^T\theta) \in \mathbb{R}^d$ is a vector whose elements are $\sin(\theta_i - \theta_j)$. Equation (7) can be written in a more compact form of:

$$\dot{\theta} = \omega = -BW(\theta)B^T\theta = -L_w(\theta)\theta, \tag{8}$$

where $W(\theta) = \text{diag}\{\text{sinc}(\theta_i - \theta_j) \mid (i,j) \in \mathcal{E}\}$ is a diagonal matrix whose entries are the edge weights for $\mathbb{G}$. The ordering of the elements on the diagonal of $W(\theta)$ is consistent with the ordering of the edges in the incidence matrix B. The matrix $L_w(\theta) = BW(\theta)B^T$ can be thought of as the weighted Laplacian of $\mathbb{G}$, when $\text{sinc}(\theta_i - \theta_j) = \sin(\theta_i - \theta_j)/(\theta_i - \theta_j)$ is positive. For this to hold θ should belong to the open cube $(-\pi/2, \pi/2)^N$, where N is the number of vertices of the graph. In other words, over any compact subset of the cube $(-\pi/2, \pi/2)^N$, the dynamics can be represented by a state-dependent weighted Laplacian.

Now consider the Lyapunov function

$$U = \frac{1}{2}\sum_{j\sim i} \parallel v_i - v_j \parallel^2 = \frac{1}{2}[e^{j\theta}]^* L[e^{j\theta}] = \sum_{j\sim i} 1 - \cos(\theta_i - \theta_j) \tag{9}$$

where the sum is over all the neighboring pairs, denoted by $i \sim j$, L is the Laplacian of the graph, and $[e^{j\theta}]$ is the stack of velocity vectors in complex notation. The above sum represents the total misalignment energy between velocity vectors. Since we have $U = e - \mathbf{1}^T \cos(B^T\theta)$, and because of (8), the time derivative of U along the trajectories of the system becomes

$$\dot{U} = \nabla U \ \dot{\theta} = \theta^T L_w \dot{\theta} = -\dot{\theta}^T \ \dot{\theta} \leq 0$$

A simple application of LaSalle's invariance principle over the configuration space which is an N-Torus and therefore compact reveals that all trajectories starting in anywhere on the N-torus converge to the largest invariant sets in $E = \{\theta \mid \dot{U} = 0\}$. Note that this is a very rich set and contains many equilibria other than the consensus state, some of which could be stable. See Remark 3.4 for an instance of this situation.

In order to prove local stability of the consensus state we utilize a simple quadratic Lyapunov function $V = \frac{1}{2}\theta^T\theta$, and a compact set $\Omega_c = \{\theta \mid V \leq c\}$. This se which is characterized by the largest level set of V that is contained inside the cube $(-\pi/2, \pi/2)^N$ can be used to show that the synchronized state is the only equilibrium within the set $E = \{\theta \in \Omega_c \mid \dot{V} = 0\}$. This is true since $\dot{V} = -\theta^T L_w \theta \leq 0$. Thus, equilibrium points are the set of solutions of $L_w\theta = 0$. If graph $\mathbb{G}$ is connected, within Ω_c the null space of weighted Laplacian L_w is the span of the vector $\mathbf{1} \doteq [1, \ldots, 1]^T$. Thus, the solution is Null$\{L_w\}$, which is the set $S = \{\theta \mid \theta \in \text{span}\{\mathbf{1}\}\}$. This suggests that all agents reach the same heading as $t \to \infty$.

Alternatively, one can prove invariance of any compact subset of the open cube $(-\pi/2, \pi/2)^N$ without the use of a Lyapunov function. This can be easily seen by noting that any trajectory that approaches the boundary of the cube is pushed back to the interior of the cube by the dynamics (6). Specifically, suppose θ_i is the first to approach the boundary of the cube. The term $-sin(\theta_i - \theta_j)$ is therefore negative for each $j \in \mathcal{N}_i$, therefore θ_i can not push the boundary $\pi/2$. Similar argument can be made for the boundary $-\pi/2$. Once the invariance of the cube is established, LaSalle's invariance principle can be used in conjunction with properties of the Laplacian exactly as done above for the Lyapunov function to prove convergence of headings to the consensus state. ■

Remark 3.4: When the proximity graph $\mathbb{G}$ has the *ring topology* (i.e. all agents have exactly two neighbors), there are two sets of equilibrium: $\theta \in \text{span}\{\mathbf{1}\}$ and $B^T\theta \in \text{span}\{\mathbf{1}\}$ where the former corresponds to the set $\{\theta_i = \theta_j, \ \forall i \neq j\}$ and the latter corresponds to $\{\theta_i - \theta_j = \frac{2\pi}{N}, \ \forall i \neq j\}$. See [15] for details.

Remark 3.5: Local asymptotic stability of the consensus state can be established even when the proximity graph changes with time. As will be shown in Section VI, this holds as long as a weak notion of connectivity called *joint connectivity* [13] holds.

Remark 3.6: The geodesic control input (6) for a group of planar nonholonomic vehicles is basically the same controller that can stabilize the Kuramoto model of coupled nonlinear oscillators [14]. The term $\sin(\theta_i - \theta_j)$ in the angular velocity can be explained by noting that in the planar case the angular velocity is the rate of rotation about the axis $v_i \times v_j$ where v_i is given by (3). The norm of $v_i \times v_j$ is nothing but $\sin(\theta_i - \theta_j)$.

Remark 3.7: The geodesic controller (6) is the nonlinear version of the control law

$$\omega_i = -\sum_{j\in\mathcal{N}_i} \theta_i - \theta_j$$

proposed in [13], [19] as the continuous analogue of Viscek's model [29].

IV. Leader Following

One could envision in a social aggregation scenario such as flocking of birds, one of the flock-mates acts as the leader of the group and others follow the leader while staying in a formation. Similarly, here we consider the case that one additional agent, labelled 0, acts as the group's leader. Agent 0 moves with the constant unit velocity (same as others) and a fixed heading θ_0. Other agents in the group may or may not have the leader as a neighbor. Here we prove that the control law (6) results in a stable formation of the group while following the leader, so that in the end all agents reach the desired heading θ_0 (cf. [13] for more details on leader following).

Consider the input of each agent in the leaderless case that is given by (6). We can separate the leader from other agents and write:

$$\dot{\theta}_i = -\sum_{j\in\mathcal{N}_i} \sin(\theta_i - \theta_j) - c_i \sin(\theta_i - \theta_0), \tag{10}$$

where $c_i = 1$ if agent i and the leader are neighbors and $c_i = 0$ otherwise.

To show that all the headings become equal to θ_0, we consider the error term $e_i = \theta_i - \theta_0$. Since $\dot{e}_i = \dot{\theta}_i$, we can write (10) as

$$\dot{e}_i = -\sum_{j\in\mathcal{N}_i} \sin(e_i - e_j) - c_i \sin e_i.$$

Consider the error vector $e = [e_1, \ldots, e_N]^T$. Similar to calculations of section III, the error dynamics becomes:

$$\dot{e} = -BW(e)B^T e - W_l e = -L_w e - W_l e = -H_l e \tag{11}$$

where $W(e) = \text{diag}\{\text{sinc}(e_i - e_j) \mid (i,j) \in \mathcal{E}\} \in \mathbb{R}^{d\times d}$ and $W_l = \text{diag}\{c_i \ \text{sinc}(e_i) \mid (i,0) \in \mathcal{E}\} \in \mathbb{R}^{N\times N}$. Both $W(e)$ and W_l are weight matrices with positive entries, because $\text{sinc}(e_i - e_0) = \text{sinc}(\theta_i - \theta_0)$ is positive for $\theta_i \in (-\pi/2, \pi/2)$, and c_i is a nonnegative coefficient.

In order to show that the error is asymptotically stable, consider the Lyapunov function $U = \frac{1}{2}e^T e$. The derivative of this along the trajectory of the error system can be written as $\dot{U} = -e^T H_l e$, where $H_l = L_w + W_l$. We will prove that H_l is positive definite, and the error will asymptotically decay to zero.

Note that both L_w and W_l are positive semi-definite matrices and so is H_l. Matrix L_w has property SC, because if we replace the nonzero elements of L_w with 1, we obtain the adjacency matrix of the neighboring graph that is strictly connected. matrix W_l is diagonal, thus adding it to L_w doesn't change the neighboring graph. Thus $H_l = L_w + W_l$ has property SC. A matrix is irreducible if and only if it has property SC. Thus, H_l is irreducible. See [12], Chapter 6, for more details on irreducible matrices.

We need to show that H_l is actually positive definite. To do this, we make the following observations:

- H_l is an irreducible matrix.
- L_w is diagonally dominant.
- For at least one of the rows of H_l the diagonal entry is strictly greater than the sum of off-diagonal entries (because W_l ia a diagonal matrix with nonnegative entries).

According to *Taussky theorem* [12] matrix H_l is an *irreducibly diagonally dominant* matrix and is invertible. Thus, H_l must be a positive definite matrix. As a result, $\dot{U} < 0$ and the error vector asymptotically decays to zero; consequently $\theta_i = \theta_0$ for every $i = 1, \ldots, N$, as $t \to \infty$.

V. Distributed Coordination of nonholonomic agents in 3D

Consider a group of N agents in the 3 dimensional space. Our goal in this section is to design a control law for each agent such that it guarantees they reach the consensus state.

Each agent is capable of communicating some information with its neighbors, defined by (1). The neighborhood set of agent i, $\mathcal{N}_i$, is a set of agents that can share their headings and attitudes (orientation) information with agent i. As before, it is assumed that there is a predetermined sphere with radius R which determines the neighborhood relationship. The location of agent i in the fixed world coordinates is given by (x_i, y_i, z_i) and its velocity is $v_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i)^T$. The orientation of the velocity vector of agent i can be characterized by specifying two angles θ_i (heading) and ϕ_i (attitude) relative to the world frame (see Figure 2.a), and they are defined as:

$$\theta_i = \text{atan2}\big(\dot{y}_i, \dot{x}_i\big),\ 0 \le \theta_i \le 2\pi \tag{12}$$

$$\phi_i = \text{atan2}\Big(\sqrt{\dot{x}_i^2 + \dot{y}_i^2}, \dot{z}_i\Big),\ 0 < \phi_i < \pi\ . \tag{13}$$

Without loss of generality, it is assumed that all agents move with a constant unit speed. The velocity of agent i in 3 dimensions is given by:

$$v_i = \begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i \sin\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\phi_i \end{pmatrix}$$

Hence, all velocity vectors are on a unit sphere $\mathbf{S}^2 \doteq \{p = (x, y, z) \in \mathbb{R}^3 \ :\ \parallel p \parallel = 1\}$ (see Figure 2.b). We represent each vector v_i as a point on this unit sphere. As the direction of the velocity vector of agent i changes, the corresponding point v_i will move along a curve on the sphere. The tangent vector to this curve at $v_i \in \mathbf{S}^2$ can be uniquely represented as a vector $\dot{v}_i \in \mathbb{R}^3$ such that $\dot{v}_i \perp v_i$ and $\dot{v}_i \in T_i\mathbf{S}^2$ where $T_i\mathbf{S}^2$ is the tangent plane at v_i. A basis for the tangent space $T_i\mathbf{S}^2$ can be obtained by differentiating v_i, and thus $\dot{v}_i$ can be written as

$$\dot{v}_i = U_{i\theta} X_{i\theta} + U_{i\phi} X_{i\phi}\ \ \in T_i\mathbf{S}^2$$

where $\mathbf{B}_i = \{X_{i\theta}, X_{i\phi}\}$ is an *orthonormal basis* for the tangent plane $T_i\mathbf{S}^2$, and

$$X_{i\theta} = \begin{pmatrix} -\sin\theta_i \\ \cos\theta_i \\ 0 \end{pmatrix},\ X_{i\phi} = \begin{pmatrix} \cos\theta_i \cos\phi_i \\ \sin\theta_i \cos\phi_i \\ -\sin\phi_i \end{pmatrix}\ .$$

The control inputs $U_{i\theta}$ and $U_{i\phi}$ are related to $\dot{\theta}_i$ and $\dot{\phi}_i$:

$$U_{i\theta} = \dot{\theta}_i \sin\phi_i, \qquad U_{i\phi} = \dot{\phi}_i\ . \tag{14}$$

When points v_i and v_j are neither equal nor opposite, a vector $Y_{ij} \in T_i\mathbf{S}^2$ called the *geodesic versor* can be defined to show the geodesic direction from v_i to v_j (see Figure 2.b). The unit length geodesic versor is defined by equation (4). The difference from the 2-dimensional case is that on the sphere the angle α_{ij} is the radian distance between points v_i and v_j over the great circle path.

Now, we can prove the following theorem for the geodesic control of the velocity vectors of a group of N agents, which is a generalization of Theorem 2 in [3] to an arbitrary number of agents and connected topologies.

Theorem 5.1: Consider the system of N equations $\dot{v}_i = U_{i\theta} X_{i\theta} + U_{i\phi} X_{i\phi},\ i = 1, \ldots, N$. If the proximity graph of the agents is fixed and connected, then by applying the control laws

$$U_{i\theta} = \sum_{j \in \mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\theta} >= \sum_{j \in \mathcal{N}_i} < v_j, X_{i\theta} > \tag{15}$$

$$U_{i\phi} = \sum_{j \in \mathcal{N}_i} \sin\alpha_{ij} < Y_{ij}, X_{i\phi} >= \sum_{j \in \mathcal{N}_i} < v_j, X_{i\phi} > \tag{16}$$

all trajectories converge to the equilibria given by $U_{i\theta} = 0$ and $U_{i\phi} = 0$, for $i = 1, \ldots, N$. Furthermore, the consensus state is locally asymptotically stable.

Proof: Convergence to equilibria can be established using the Lyapunov function

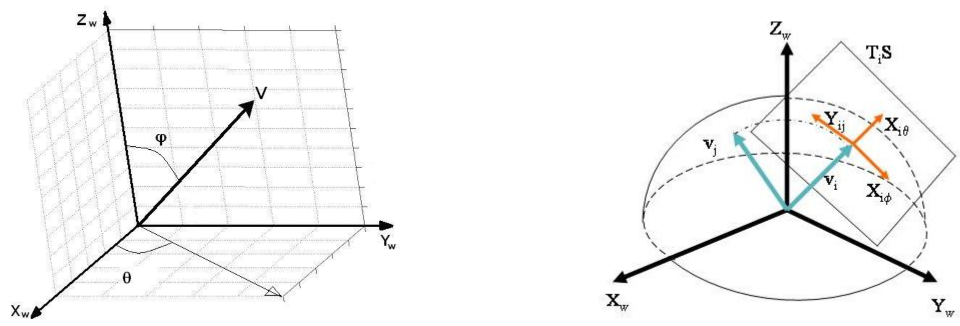
$$V = \frac{1}{2} \sum_{j \sim i} \parallel v_i - v_j \parallel^2 = \sum_{j \sim i} 1 - < v_i, v_j >\ . \tag{17}$$

which is a measure of discrepancy among the velocity vectors. The time derivative of V becomes

$$\dot{V} = -\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} < \dot{v}_i, v_j > = -\sum_{i=1}^{N} (U_{i\theta}^2 + U_{i\phi}^2) \le 0.$$

Similar to the 2D case, the configuration space (which is now an N copies of a sphere) is compact and therefore LaSalle's invariance principle can be used to establish convergence of all trajectories to invariant sets, including the synchronized state where all θ_i s are the same and all ϕ_is are the same.

To prove local asymptotic stability of the consensus state for the system of N agents with the control laws given in



(a) (b)

Fig. 2. a) The heading of each agent is determined by two angles θ_i (heading) and ϕ_i (attitude) relative to the world frame. b) The velocity vectors of agents in 3D are projected onto a unit sphere. $X_{i\theta}$ and $X_{i\phi}$ form an orthonormal basis of the tangent plane. The geodesic versor Y_{ij} points towards the geodesic direction from v_i to v_j.

Theorem (5.1), we need to write (15) and (16) in terms of the heading and attitude angles.

Using (14) we obtain expression for $\dot{\theta}_i$ and $\dot{\phi}_i$:

$$\dot{\theta}_i = -\sum_{j\in\mathcal{N}_i} \frac{\sin\phi_j}{\sin\phi_i}\sin(\theta_i-\theta_j) \tag{18}$$

$$\dot{\phi}_i = -\sum_{j\in\mathcal{N}_i}\sin(\phi_i-\phi_j) - \sum_{j\in\mathcal{N}_i}\sin\phi_j\cos\phi_i(1-\cos(\theta_i-\theta_j)) \;. \tag{19}$$

To prove local stability We now linearize equation (19), (18) around the synchronized state $\phi_i = \phi_j$, $\theta_i = \theta_j$. Let $\tilde{\theta}_i$ and $\tilde{\phi}_i$ be the deviations of θ_i and ϕ_i from the synchronized state. The linearized dynamics can be written as

$$\begin{aligned}\dot{\tilde{\theta}}_i &= -\sum_{j\in\mathcal{N}_i}(\tilde{\theta}_i-\tilde{\theta}_j)\\ \dot{\tilde{\phi}}_i &= -\sum_{j\in\mathcal{N}_i}(\tilde{\phi}_i-\tilde{\phi}_j)\end{aligned} \tag{20}$$

Now consider the quadratic Lyapunov function

$$V \doteq \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}\tilde{\phi}^T\tilde{\phi}. \tag{21}$$

Then by using (20) we can show that $\dot{V}$ is nonpositive:

$$\dot{V} \;=\; \tilde{\theta}^T\dot{\tilde{\theta}} + \tilde{\phi}^T\dot{\tilde{\phi}} = -\tilde{\theta}^T L\tilde{\theta} - \tilde{\phi}^T L\tilde{\phi} \le 0$$

where $L = BB^T$ is the graph Laplacian. Consider the compact set $\Omega_c = \{(\tilde{\theta},\tilde{\phi}) \mid V \le c\}$. By LaSalle's invariance principle any trajectory starting in Ω_c converges to the largest invariant set, S, contained in $E = \{(\tilde{\theta},\ \tilde{\phi}) \mid \dot{V} = 0\}$. The invariant set of this system is when $L\theta = 0$ and $L\phi = 0$, or when $\tilde{\theta},\ \tilde{\phi} \in$ Null$\{L\}$. Since the graph is assumed to be connected, $\tilde{\phi},\tilde{\theta} \in$ span **1** is the only invariant set. As a result, the synchronized state is locally asymptotically stable. ■

This analysis shows that geodesic controllers (15) and (16) will render the consensus state asymptotically stable locally.

VI. Stability Analysis for Switching Graphs in 2 dimensions

So far, the underlying assumption has been that the graph $\mathbb{G}$, representing the neighborhood relationship, is fixed and connected. In practice, the motion of individual agents will result in change in topology. This change in topology could be taken into account by using smooth "bump functions" [19], or by resorting to nonsmooth analysis [27]. To avoid complications that occur because of discontinuous change in the set of nearest neighbors, we will assume that there is always a minimum time, called a *dwell time*, over which the graph does not change. This simplifying assumption will avoid infinite switches over a finite period of time, and can be relaxed by using nonsmooth analysis [8].

Let $\mathcal{P}$ denote a set that indexes the class of all simple graphs defined on N vertices; so if $p \in \mathcal{P}$ then $\mathbb{G}_p$ is the corresponding proximity graph on N vertices. Let $\sigma(t)$ be a switching signal whose value at time t is the index of the graph representing the proximity graph of agent i. In the sequel we make two simplifying assumptions: First we assume the switching signal $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant signal with consecutive switching times separated by a non vanishing dwell time, μ_D. This means that the time between two consecutive switchings $t_{k+1} - t_k \ge \mu_D$, all $k \ge 0$. By 'piecewise constant' we mean a signal that exhibits a finite number of discontinuities in any finite time interval and that is constant between consecutive discontinuities. The set of all switching signals σ with this property is denoted by $\mathcal{S}_{dwell}$ [11]. This restriction on the admissible switching signals is required in order to obtain asymptotic stability for a switching nonlinear system and extend LaSalle's invariance principle [1], [11]. Second, we assume that the switching is such that some weak notion of connectivity "persists" over finite time intervals. Each agent i would use control laws similar to (5) (which is now hybrid, since the set of neighbors $\mathcal{N}_i$ changes discontinuously). With the dwell time assumption, the control inputs would be of the following form:

$$\omega_i(t) = \sum_{j\in\mathcal{N}_i(t_k)} < v_j(t), X_{i\theta}(t) > \;,\; t \in [t_k, t_k+\mu_D) \tag{22}$$

The hybrid controller may result in change of the proximity graph as the switching occurs. As mentioned earlier, following [13], we need to define a weaker notion of connectivity for a collection of graphs with a switching signal σ with $\sigma(t_k) = p_k$.

Definition 6.1: The *union* of a collection of simple graphs $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$, each with vertex set $\mathcal{V}$, is a simple graph with vertex set $\mathcal{V}$ and edge set equaling the union of the edge sets of all the graphs in the collection.

Definition 6.2: A collection of graphs is called *jointly connected*, if the union of its members is a connected graph.

It is natural to say that the N agents under consideration are *linked together* across a time interval $[t, \tau]$ if the collection of graphs $\{\mathbb{G}_{\sigma(t)}, \mathbb{G}_{\sigma(t+1)}, \ldots, \mathbb{G}_{\sigma(\tau)}\}$ encountered along the interval, is jointly connected.

In trying to extend the Theorem (3.3) to graphs with the above mentioned switching regime, we need the following lemma, which was proven in [13].

Lemma 6.3: If $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$ is a jointly connected collection of graphs with Laplacians $L_{p_1}, L_{p_2}, \ldots, L_{p_m}$, then

$$\bigcap_{i=1}^{m} \text{kernel } L_{p_i} = \text{span } \{\mathbf{1}\}. \tag{23}$$

The above lemma states that the intersection of the null space of the Laplacians of a set of jointly connected graphs is only the vector of ones. In other words, even though the graphs might be disconnected, and as a result their Laplacian have a larger kernel, the intersection is only the vector of ones.

The goal here is to use the above lemma in the context of a LaSalle-like invariance principle for switched systems with dwell time constraint on switching.

We can now state the following theorem:

Theorem 6.4: Let $\mu_D > 0$ and the initial heading vector θ_0 be fixed. Let $\sigma : [0, \infty) \to \mathcal{P}$ be a piecewise constant persistent switching signal corresponding to all graphs over N vertices whose switching times $t_1, t_2, \ldots$ satisfy $(t_{k+1} - t_k) \geq \mu_D, k \geq 1$. If there exists an infinite sequence of nonempty, bounded, time-intervals with the property that across each such interval the N-agent group is linked together, then by applying geodesic control law (22) all trajectories converge to equilibria of $\omega_i(t) = 0$. Furthermore, the consensus state is locally asymptotically stable.

Proof: Convergence to the equilibria of $\omega_i(t) = 0$ can be established by using a Lyapunov function similar to (9). In order to show that the consensus state is locally asymptotically stable, consider the following positive definite function

$$V(\theta) \doteq \frac{1}{2}\theta(t)^T\theta(t) \tag{24}$$

defined over $\Omega = (-\pi/2, \pi/2)^N$. Using the same notation, we now have

$$\dot{V}(\theta) = \theta(t)^T\dot{\theta}(t) = -\theta(t)^T L_{\sigma(t)}\theta(t) \leq 0.$$

Since for all $\theta \in \Omega$ and each $p \in \mathcal{P}$ we have $\dot{V}(\theta) \leq 0$, following [1] we say that $V(\theta)$ in (24) is a *weak common Lyapunov function* [1] for the switching system. Let $l > 0$ and let $\Omega_l = \{\theta \in \Omega \mid V(\theta) \leq l\}$ be a compact sublevel set of V. Define

$$Z \doteq \{\theta \in \Omega \mid \dot{V}(\theta) = 0\} = \{\theta \in \Omega \mid L_{\sigma(t)}\theta(t) = 0\}.$$

Theorem 1 in [1] states that all the trajectories starting in Ω_l with switching signal $\sigma \in \mathcal{S}_{dwell}$ converge to the union of the invariant sets $M \subset Z \cap \Omega_l$. For each $p \in \mathcal{P}$, the set span$\{\mathbf{1}\}$ is a subset of the set $Z \cap \Omega_l$, since at switching times that the graph is disconnected, the kernel of the weighted Laplacian matrix contains the span of $\mathbf{1}$, but could have other directions as well. However, given the restriction on the switching signal which requires an infinite sequence of nonempty, uniformly bounded time-intervals, with the property that across each such interval the collection of underlying graphs $\{\mathbb{G}_{\sigma(t_k)}, \mathbb{G}_{\sigma(t_{k+1})}, \ldots, \mathbb{G}_{\sigma(t_{k+m})}\}$ is jointly connected, Lemma 6.3 suggests that the null space of the Laplacians $\{L_{p_1}, L_{p_2}, \ldots, L_{p_m}\}$ of these jointly connected graphs (i.e. the invariant set M) is exactly span$\{\mathbf{1}\}$. In other words, the only persistent direction in the kernel of teh Laplacian, and as a result, the only invariant set inside $Z \cap \Omega_l$ is $\{\mathbf{1}\}$. Thus, the consensus state is locally asymptotically stable. ■

We should re-emphasize that the results are local, in the sense that the convergence is guaranteed for a subset of all possible initial orientations.

Remark 6.5: In the case of changing topology, given that conditions of Theorem 6.4 hold, and by using Lemma 6.3, we can show that leader following is also achieved.

VII. Simulations

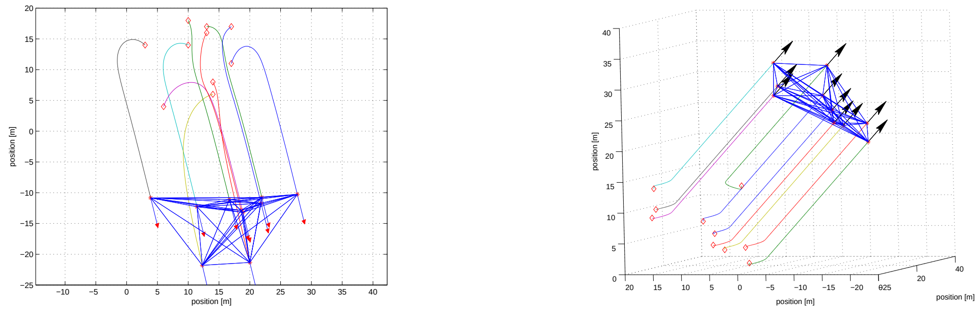
In this section we numerically show that the distributed control law (5), for the planar case, and the geodesic control laws (15) and (16), for the three dimensional case, can force a group of agents to reach the consensus state. Figures 3.a and 3.b show the leaderless flocking of 10 agents in 2 and 3 dimensions, respectively. The initial position and heading of all agents are generated randomly within a pre-specified area. The neighboring radius is chosen large enough so that agents form a connected graph at time $t = 0$. The arrows on each agent show the directions of the velocity vectors.

Simulations show that agents *smoothly* adjust their headings and after a reasonable amount of time they converge to a formation, and their relative distances stabilizes.

Figure 4.a shows the effect of the presence of a leader in the group. In the simulations, one of the agents is randomly chosen to be the leader of the group, and its heading is constant. Without knowing which one of them is the leader, all other agent adjust their headings to follow him so that the formation remains stable. Even if the leader's motion has dynamics, as long as the group remains connected, all agents follow the leader, as it is shown in Figure 4.b.

VIII. Conclusions and Future Work

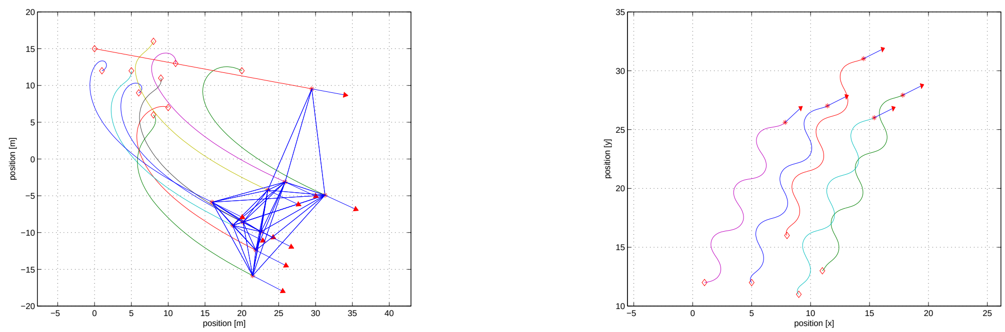
We provided a coordination scheme which resulted in flocking of a collection of kinematic agents. The control law was based on nearest neighbor sensing. It was shown that reaching the consensus state is possible despite possible changes in the topology of the proximity graph representing the neighborhood relationship.



(a) Flocking in 2D

(b) Flocking in 3D

Fig. 3. Geodesic control laws result in flocking of 10 agents in a) 2D and in b) 3D.



(a)

(b)

Fig. 4. a) In the presence of a leader in the group, all agents are forced to align their velocities with that of the leader. b) Agents follow a leader with dynamics. In this case, the leader (agent in red) travels a sinusoidal path.

A generalization of the current analysis would be to develop results similar to [26], [27] for dynamic models, by using artificial potential functions similar to [21]. An important question that we need to answer is how to enforce the connectivity condition of the proximity graph. A potential starting point would be to use results of [31] in topology control.

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