18.917 NOTES SPRING 2020

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1. Introduction: Nishida's Nilpotence Theorem

2. A quick introduction to ∞-categories

People who haven't seen ∞-categories before will want to look at the Kerodon website for a more detailed introduction. Much of what I say in the next few lectures is heavily influenced by Jacob Lurie's class notes, including in particular Lecture 5 of his 282y course and Lectures 2-4 of his iteration of 18.917 (available on MIT OCW).

3. Kan complexes, mapping spaces, and (co)limits

Definition 3.1. A Kan complex is a simplicial set $X_\bullet$ such that the maps

$$X_n \to \Lambda^n_i(X_\bullet)$$

are surjective for all $n \geqslant 0$ and all $0 \leqslant i \leqslant n$.

Kan complexes are special sorts of ∞-categories.

Example 3.2. If Z is a topological space, then $\mathrm{Sing}(Z)_\bullet$ is a Kan complex.

Definition 3.3. A *classical groupoid* is a classical category in which every morphism is an equivalence. A group G can be viewed as a one-object classical category BG, which is an example of a groupoid.

Example 3.4. Suppose that $\mathcal{C}$ is a classical groupoid. Then $N(\mathcal{C})$ is a Kan complex.

In class, we drew a picture indicating how the surjectivity of $X_2 \to \Lambda^2_0(X_\bullet)$ is related to the existence of inverses of morphisms.

For this reason, Kan complexes are sometimes called ∞-*groupoids*. We will also call them *homotopy types*, or just *spaces*, but they shouldn't be confused with topological spaces. As we will explore, they play a role in higher category theory analogous to the role played by sets in classical category theory.

Remark 3.5. Given a classical category $\mathcal{C}$, and two objects $c, c' \in \mathcal{C}$, there is a set of maps $\mathrm{Hom}_{\mathcal{C}}(c, c')$.

Given an infinity category $\mathcal{C}_\bullet$, and two objects $c, c' \in \mathcal{C}_0$, we can discuss the set of arrows in $\mathcal{C}_1$ that have domain c and codomain c'. However, in higher category theory the notation $\mathrm{Hom}_{\mathcal{C}}(c, c')$ is reserved for a somewhat more sophisticated object, which remembers for example that certain pairs of arrows are homotopic:

Definition 3.6. For each integer $n \geqslant 0$, the simplicial set Δ^n is the unique simplicial set such that, for any other simplicial set $X_\bullet$,

$$\mathrm{Hom}(\Delta^n, X_\bullet) \cong X_n.$$

One way to define Δ^n is to recall that a simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathbf{Sets}$, where Δ is the simplex category with objects $[0], [1], [2], \cdots$. In this language, the symbol Δ^n refers $Y([n])$, the Yoneda embedding applied to $[n]$.

Definition 3.7. Suppose that $\mathcal{C}_\bullet$ is an ∞-category, and that $c, c' \in \mathcal{C}_0$ are two objects in $\mathcal{C}_\bullet$. Then we define a simplicial set $\mathrm{Hom}_{\mathcal{C}}(c, c')_\bullet$ to have n-simplices the subset of $\mathcal{C}_{n+1}$ consisting of those $(n+1)$-simplices $\Delta^{n+1} \to \mathcal{C}$ such that the restriction to $\Delta^{\{0,1,\cdots,n\}}$ is the constant n-simplex at c, and the restriction to $\Delta^{\{n+1\}}$ is the constant 0-simplex at c'.

In class, we drew some pictures for small values of n.

Proposition 3.8. *If $c, c' \in \mathcal{C}_0$ are objects in an ∞-category, then $\mathrm{Hom}_{\mathcal{C}}(c, c')_\bullet$ is a Kan complex.*

Exercise A. Suppose $\mathcal{C}$ is an ∞-category, and that $f, g : c \to c'$ are two elements of $\mathcal{C}_1$. Then f and g are homotopic if and only if they are equivalent when considered as objects of the ∞-groupoid $\mathrm{Hom}_{\mathcal{C}}(c, c')_\bullet$.

Definition 3.9. Given two simplicial sets $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$, we define a simplicial set

$$\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)_\bullet$$

with n-simplices

$$\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)_n = \mathrm{Hom}_{\mathbf{Simplicial\ Sets}}(\mathcal{C}_\bullet \times \Delta^n, \mathcal{D}_\bullet).$$

Note that the 0-simplices of $\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$ are just the simplicial set maps from $\mathcal{C}_\bullet$ to $\mathcal{D}_\bullet$.

Proposition 3.10. *Suppose that $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ are ∞-categories. Then $\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$ is an ∞-category, called the ∞-category of functors from $\mathcal{C}_\bullet$ to $\mathcal{D}_\bullet$. The objects of the ∞-category of functors are just the functors from $\mathcal{C}_\bullet$ to $\mathcal{D}_\bullet$, as defined last lecture.*

Definition 3.11. A natural transformation between two functors $F, G : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ is an arrow in the functor category $\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$. Two functors $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$ are *naturally isomorphic* if they are equivalent in the ∞-category $\underline{\mathrm{Hom}}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$.

Kan complexes form the objects of an ∞-category **Spaces** of *spaces*, or *homotopy types.* To be clear, **Spaces** is a simplicial set, so we might more properly denote it by $\mathbf{Spaces}_\bullet$. However, I will start dropping the $\bullet$ from the notation for ∞-categories: it is implicitly there, since ∞-categories are just special sorts of simplicial sets.

We won't define the ∞-category of Kan complexes simplex by simplex, as we did with the ∞-category of $\mathbb{F}_2$-module spectra last lecture. Instead, I'll just let you know some of its basic properties. First of all, if $\mathcal{C}$ and $\mathcal{D}$ are two ∞-groupoids, then the functor category $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ is also an ∞-groupoid (and not just an ∞-category). If $\mathcal{C}, \mathcal{D} \in \mathbf{Spaces}_0$ are Kan complexes, then inside **Spaces** there are equivalent objects $\mathrm{Hom}_{\mathbf{Spaces}}(\mathcal{C}, \mathcal{D})$ and $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$.

Example 3.12. Let C_2 denote the cyclic group of order 2. Then one example of a Kan complex is $N(BC_2)$. It turns out that this is equivalent to a different example, namely $\mathrm{Sing}(\mathbb{RP}^\infty)$.

Definition 3.13. Suppose $\mathcal{C}$ and $\mathcal{D}$ are two ∞-categories. An adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

consists of the following data:

- A functor $F : \mathcal{C} \to \mathcal{D}$, called the *left adjoint.*
- A functor $G : \mathcal{D} \to \mathcal{C}$, called the *right adjoint.*
- A *unit* natural transformation

  $$\epsilon : \mathrm{Id}_{\mathcal{C}} \to G \circ F.$$

  such that for all objects $c \in \mathcal{C}_0$ and $d \in \mathcal{D}_0$ the induced composite

  $$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(G(F(c)), G(d)) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(\epsilon, G(d))} \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

  is a homotopy equivalence of Kan complexes.

Proposition 3.14. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories. If F is the left adjoint of some adjunction (F, G, ϵ), then the right adjoint G is unique up to equivalence in the functor category $\underline{\mathrm{Hom}}(\mathcal{D}, \mathcal{C})$. Similarly, left adjoints are unique up to natural equivalence.*

Definition 3.15. Let $\mathcal{C}, \mathcal{D}$ denote ∞-categories. The category of $\mathcal{D}$-shaped diagrams in $\mathcal{C}$ is just the functor ∞-category $\underline{\mathrm{Hom}}\mathcal{D}, \mathcal{C}$. There is a *constant diagram* functor

$$\mathcal{C} \to \underline{\mathrm{Hom}}(\mathcal{D}, \mathcal{C}).$$

We say that $\mathcal{C}$ *admits $\mathcal{D}$-shaped colimits* if there is a left adjoint

$$\mathrm{colim} : \underline{\mathrm{Hom}}(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$$

to this constant diagram functor, and $\mathcal{C}$ *admits $\mathcal{D}$-shaped limits* if there is a right adjoint.

Proposition 3.16. *Suppose $\mathcal{C}$ is an ∞-category admitting $\mathcal{D}$-shaped colimits. Then for any diagram*

$$F : \mathcal{D} \to \mathcal{C}$$

and any object $c \in \mathcal{C}_0$, we have that

$$\mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_{\mathcal{D}} F(d), c) \simeq \lim_{\mathcal{D}} \mathrm{Hom}_{\mathcal{C}}(F(d), c).$$

If $\mathcal{C}$ admits $\mathcal{D}$-shaped limits, we similarly learn that

$$\mathrm{Hom}_{\mathcal{C}}(c, \lim_{\mathcal{D}} F(d),) \simeq \lim_{\mathcal{D}} \mathrm{Hom}_{\mathcal{C}}(c, F(d)).$$

On the right hand side of the $\simeq$ of the above Proposition, both the limits occur in the ∞-category **Spaces** of homotopy types, which admits all (small) (co)limits. So it will be helpful to get some idea of how to calculate limits and colimits in **Spaces**.

4. Some examples of (co)limits

Consider the following pair of basic examples:

$$N(\textbf{Topological spaces}) \xrightarrow{Sing_\bullet} \textbf{Spaces}$$

$$N(\textbf{Chain Complexes}) \longrightarrow \mathbb{F}_2\textbf{-module spectra},$$

where the categories on the left hand side are (nerves of) ordinary categories, and we have ∞-categories on the right hand side.

If you are taking this class, you should be familiar already with how to calculate limits and colimits in the classical categories on the left hand side. I'd like to describe how to calculate some limits and colimits on the right hand side, at least in a few examples.

Example 4.1. Let $\mathcal{D}$ denote the 3-object classical category depicted below, with two non-identity morphisms.

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

Suppose I'm given a $\mathcal{D}$-shaped diagram $F : \mathcal{D} \to$ **Topological spaces** of topological spaces

$$\begin{tikzcd} & Z_1 \arrow[d, "f"] \\ Z_2 \arrow[r, "g"] & Z_3. \end{tikzcd}$$

Let's denote the limit of this diagram by $\lim(F)$. Then $\lim(F)$ is a topological space with points consisting of all pairs $(z_1, z_2) \in Z_1 \times Z_2$ such that $f(z_1) = g(z_2)$.

On the other hand, we can consider the composite functor

$$N(\mathcal{D}) \xrightarrow{F} N(\textbf{Topological spaces}) \xrightarrow{\mathrm{Sing}} \textbf{Spaces}.$$

The limit of this functor is an object of the ∞-category **Spaces**, which is sometimes called the *homotopy limit* $\mathrm{holim}(F)$. To calculate it, consider the topological space Z' with points consisting of triples (z_1, z_2, γ), where $z_1 \in Z_1$, $z_2 \in Z_2$, and $\gamma : [0, 1] \to Z_3$ is a path beginning at $f(z_1)$ and ending at $g(z_2)$. Then $\mathrm{holim}(F) \simeq \mathrm{Sing}(Z')$.

Remark 4.2. One can think of the homotopy limit Z' as an expanded variant of the limit Z: instead of demanding that $f(z_1)$ and $g(z_2)$ be *equal*, we give a specified path γ witnessing that they are *equivalent*.

This sort of construction may be familiar to you from 18.906.

5. Cochains, symmetric products, and Steenrod operations

6. Adem relations & Hopf Invariant 1

7. Spectra, or the ∞-category of $\mathbb{S}$-modules

8. Milnor's Calculation of the dual Steenrod algebra

9. A Dyer–Lashof operation on the dual Steenrod algebra

10. $\mathbb{E}_n$-algebras and Mahowald's Theorem

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