Blueshift & a converse for $\mathbb{E}_\infty$-ring spectra

Overview Our goal is to show that for $\mathbb{E}_1$-rings,

- $L_{T(i)} K(R) = L_{T(i)} K(L_{T(i) \oplus T(i-1)} R)$
- Descent results

} (1)

Last time Shachar showed that (1) follows from

$$L_{T(i)} K(L^f_{i-2} \mathbb{S}) = 0 \quad (2)$$

If in addition we know that for $\mathbb{E}_\infty$-ring R,

$$L_{T(i-1)} R = 0 \Rightarrow L_{T(i)} K(R) = 0, \quad (3)$$

taking $R = L^f_{i-2}\mathbb{S} \Rightarrow (2) \Rightarrow (1)$.

Today • Discuss Kuhn's blueshift result;

- Prove a main technical result that will help us prove (3) next time:

Thm [CMNN, 4.5] Let A be an $\mathbb{E}_\infty$-ring.

$$L_{T(i-1)} A^{tC_p} = 0 \Rightarrow L_{T(j)} A = 0, \ \forall j \geq i.$$

Rmk Fails for $\mathbb{E}_1$-rings, e.g. Morava k-theory:

$$L_{K(i-1)} k(i) = 0 \Rightarrow L_{K(i-1)} k(i)^{tC_p} = 0 \Rightarrow L_{T(i-1)} k(i)^{tC_p} = 0$$

But $L_{T(i)} k(i) \neq 0$

Blueshift & converse for E_∞-ring spectra

§1. Blueshift / Tate vanishing.

- Let G be a finite group. Denote by $\widetilde{EG}$ the cofiber of $EG_+ \to \mathbb{S}$. Thus we have a diagram of cofiber sequences

$$\begin{array}{ccccc}
EG_+ \otimes X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \otimes X \\
\downarrow & & \downarrow & & \downarrow \\
EG_+ \otimes F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge F(EG_+, X)
\end{array}$$

$$\xrightarrow{(-)^G} \quad \begin{array}{ccccc}
X_{hG} & \longrightarrow & X^G & \longrightarrow & \Phi^G X \\
\| \downarrow & & \downarrow & & \downarrow \\
X_{hG} & \xrightarrow{\text{norm}} & X^{hG} & \longrightarrow & X^{tG}
\end{array} \quad \text{(Tate construction)}$$

- Let L be a Bousfield localization functor and LSp the (∞-)category of L-local spectra.

- L is a right adjoint, so it commutes with $(-)^{hG}$.

- In LSp, $(-)_{hG}$ and $(-)^{tG}$ are the L-localizations of these constructions in Sp

Thm [Kuhn; Clausen-Mathew]

Let X be a $T(i)$-local spectrum with a G-action.
Then $X_{hG} \xrightarrow{N} X^{hG}$ is an equivalence in $L_{T(i)}Sp$, i.e. $X^{tG} \simeq *$.

Remark: We will only prove the case $G = C_p$.
For arbitrary G, one can first reduce to p-groups using transfer, and then induct along extensions of solvable groups. (See 2.7, 2.8 of [Kuhn]).

Key idea of Clausen & Mathew's proof:

Tate-vanishing <=> the transfer $\Sigma^\infty_+ BG \to \Sigma^\infty_+ *$ <= Kahn-Priddy
has a section after localization

Prop ([Clausen-Mathew], 2.3) TFAE:

(1). For $X \in LSp$ with a C_p-action, $X_{hG} \xrightarrow{N} X^{hG}$ is an equivalence in LSp and $X^{tG} = 0$.

(2). Condition (1) holds for $X = L\mathbb{S}$ with trivial action.

(3). The transfer $X_{hG} \to X$ admits a section after applying L, where G acts trivially on X.

Proof. $\underline{(1) \Rightarrow (2)}$: Clear.

$\underline{(2) \Rightarrow (1)}$: X is an $L\mathbb{S}$-module $\Rightarrow$ X^{tG} is a $(L\mathbb{S})^{tG}$-mod since $(-)^{tG}$ is lax-monoidal. Any module over 0 is 0.

$\underline{(2) \Leftrightarrow (3)}$. Consider the following diagram in $L\mathrm{Sp}$

$$\begin{array}{ccccc} (L\mathbb{S})_{hG} & \xrightarrow{N} & (L\mathbb{S})^{hG} & \xrightarrow{\text{can}} & (L\mathbb{S})^{tG} \\ & \searrow^{f} & \downarrow r & & \\ & & L\mathbb{S} & & \end{array}$$

Here r is induced by inclusion of the basepoint $* \hookrightarrow BG$ and f is the L-localization of the transfer

$$L(L\mathbb{S} \otimes \Sigma^\infty_+ BG) \simeq L\,\Sigma^\infty_+ BG \to L\mathbb{S}.$$

- Suppose that (2) holds, i.e. N is an equivalence. Since r has a section, so does f.

- Suppose that (3) holds, i.e. f has a section.

  $\underline{\text{Fact}}$: $(L\mathbb{S})^{tG}$ is a ring and $\underline{\text{can}}$ a ring map.

Hence to show that $(L\mathbb{S})^{tG} = 0$, it suffices to show that $N: \pi_0 L((L\mathbb{S})_{hG}) \to \pi_0(L\mathbb{S}^{hG})$ hits a unit u, since u is then mapped to $0 \in \pi_0 L((L\mathbb{S})^{tG})$.

Since f has a section, there exists $x \in \pi_0((L\mathbb{S})_{hG})$ such that $f = r \circ N$ maps x to $1 \in \pi_0(L\mathbb{S})$.

• <u>Claim</u>. $u = Nx$ is a unit in $\pi_0((L\$)^{hG})$.

$L\$$ represents a commutative multiplicative cohomology theory. We want to show that if $u \in L\$^0(BG)$ restricts to a unit in $L\$^0(*)$, then cupping with u is an iso on $L\$^*(BG)$.

Let $K \subset BG$ be any finite pointed subcomplex. The Atiyah-Hirzebruch sseq. of $L\$^*(K)$ is multiplicative. Since $F^m(L\$^*(K)) = 0$ for $m > \dim(K)$, we deduce that $F^1(L\$^*(K)) = \ker(L\$^*(K) \to L\$^*(*))$ is nilpotent. Hence u restricts to a unit under $L\$^*(BG) \to L\$^*(K)$ and $u : L\$^*(K) \to L\$^*(K)$ is an iso.

Using the Five Lemma on the Milnor sequence

$$
\begin{array}{ccccccccc}
0 \to & \lim_K^1 L\$^{*-1}(K) & \to & L\$^*(BG) & \to & \lim_K L\$^*(K) & \to 0 \\
 & u \downarrow \cong & & u \downarrow & & u \downarrow \cong & \\
0 \to & \lim_K^1 L\$^{*-1}(K) & \to & L\$^*(BG) & \to & \lim_K L\$^*(K) & \to 0
\end{array}
$$

we conclude that $u : L\$^*(BG) \to L\$^*(BG)$ is an iso. □

- Now we only need to find a section for the transfer after $T(n)$-localization.

Thm [Kahn-Priddy]

The transfer $\Sigma^\infty_+ BC_p \to \Sigma^\infty_+ *$ admits a section σ after applying $\Omega^{\infty+1}$.

Recall that $L_{T(n)} \simeq \Phi \circ \Omega^\infty$, where Φ is the Bousfield-Kuhn functor. Since Σ^{-1} commutes with $L_{T(n)}$, $\Sigma \circ \Phi \circ \sigma$ provides a section.

This concludes the proof of Tate-vanishing. $\square$.

§2. Fix a height $i+1$ Morava E-theory $\bar{E}_{i+1}$.

$\pi_*(\bar{E}_{i+1}) = W(k)[[v_1, \cdots, v_i]][\beta^{\pm 1}]$

$BP_* \to \bar{E}_*$

$v_j \mapsto v_j \beta^?$, $j \le i$

$v_{i+1} \mapsto \beta^?$

Thm [CMNN, Lemma 4.5]

Let A be an $\mathbb{E}_\infty$-ring with trivial C_p-action.
If $L_{T(i)}(A^{tC_p}) = 0$, then $L_{T(j)} A = 0$, $\forall j \ge i+1$.

$$L_{T(i)} A = 0 \Rightarrow L_{T(i)} A^{tC_p} = 0$$

[Hahn] $\Searrow$ $\quad$ $\Swarrow$ Thm

$$L_{T(j)} A = 0, \; j \ge i+1$$

Main idea

(0). Reduce to $K(i+1)$-local $\mathbb{E}_\infty$-$\bar{E}_{i+1}$-algebras.

(1). Relate A and A^{tC_p} via $A^{tC_p} = A \otimes_{\bar{E}_{i+1}} \bar{E}_{i+1}^{tC_p}$.

(2). Algebraic massage + [Hahn]

Proof. Recall from Ishan's talk that for ring R,

$$R \otimes T(i) = 0 \iff R \otimes K(i) = 0,$$

so it suffices to prove the result for $L_{K(i)}$.

We will first show that $L_{K(i)} A^{tC_p} = 0 \Rightarrow L_{K(i+1)} A = 0$

This is equivalent to $L_{K(i+1)}(\bar{E}_{i+1} \otimes A) = 0$, since $K(i+1) \otimes \bar{E}_{(i+1)} \neq 0$ splits as a wedge of $\Sigma^? K(i+1)$'s

Hence it suffices to show: if A is a $K(i+1)$-local $\mathbb{E}_\infty$-$\bar{E}_{i+1}$-algebra with $L_{K(i)} A^{tC_p} = 0$, then $A = 0$.

Lemma $A^{tC_p} \simeq A \otimes_{E_{i+1}} E_{i+1}^{tC_p}$

proof. We will construct equivalences such that

$$\begin{array}{ccccc} A \otimes_{E_{i+1}} (E_{i+1})_{hC_p} & \xrightarrow{N} & A \otimes_{E_{i+1}} E_{i+1}^{hC_p} & \longrightarrow & A \otimes_{E_{i+1}} E_{i+1}^{tC_p} \\ \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \\ A_{hC_p} & \xrightarrow{N} & A^{hC_p} & \longrightarrow & A^{tC_p} \end{array}$$

commutes.

Then the cofibers are equivalent.

(i). ψ_1 comes from the definition of $(-)_{hC_p}$:

$$A \otimes_{E_{i+1}} (E_{i+1})_{hC_p} = A \otimes_{E_{i+1}} E_{i+1} \otimes_{\mathbb{S}[C_p]} \mathbb{S} \simeq A \otimes_{\mathbb{S}[C_p]} \mathbb{S} = A_{hC_p}.$$

(ii). (Lemma 4.2) $E_{i+1}^{hC_p} \simeq F(BC_{p+}, E_{i+1})$

The SES $C_p \to S^1 \xrightarrow{\cdot p} S^1$ induces a fiber sequence

$$\begin{array}{ccccc} S^1 & \longrightarrow & BC_p & \longrightarrow & ES^1 \\ & & \downarrow & & \downarrow \\ & & \mathbb{CP}^\infty \simeq BS^1 & \longrightarrow & BS^1 \end{array}$$

This classifies an S^1-bundle over $\mathbb{CP}^\infty$ sitting inside a disk (line)-bundle $\mathcal{L}^{\otimes p}$.

Taking the Thom space of $\mathcal{L}^{\otimes p}$ yields cofiber sequence

$$BC_p \hookrightarrow E(\mathcal{L}^{\otimes p}) \simeq \mathbb{CP}^\infty \to \mathrm{Thom}(\mathcal{L}^{\otimes p}).$$

This induces a LES in $\bar{E}_{i+1}^*$-cohomology

$$\bar{E}_{i+1}^*[[x]] \longrightarrow \bar{E}_{i+1}^*[[x]] \to \bar{E}_{i+1}^*(BC_p).$$

$\cdot [p](x)$ ← non-zero divisor

(cupping with the Euler class)

Hence $\bar{E}_{i+1}^*(BC_p) = \bar{E}_{i+1}^*[[x]]/[p](x)$. This is a free module of rank p^{i+1} over $\bar{E}_{i+1}^*$ by Weierstrass Preparation Theorem, so $\bar{E}_{i+1}^{hC_p} \simeq \bigoplus_{p^i} \Sigma^? \bar{E}_{i+1}$.

Since $\bar{E}_{i+1}$ is $T(i+1)$-local, Kuhn's Tate-vanishing implies that $L_{T(i+1)}(\bar{E}_{i+1})_{hC_p} \xrightarrow{\simeq} \bar{E}_i^{hC_p}$, so the LHS is a free $\bar{E}_{i+1}$-module on a dual basis. Working in $T(i+1)$-local $\bar{E}_{i+1}$-modules, we have

$$\mathrm{Map}_{L_{T(i+1)}\mathrm{Mod}_{\bar{E}_{i+1}}}(L_{T(i+1)}(\bar{E}_{i+1})_{hC_p}, A) \simeq L_{T(i+1)}(\bar{E}_{i+1}^{hC_p} \otimes_{\bar{E}_{i+1}} A) \simeq \bar{E}_{i+1}^{hC_p} \otimes_{\bar{E}_{i+1}} A$$

$$\simeq \mathrm{Map}_{\mathrm{Mod}_{\bar{E}_{i+1}}}((\bar{E}_{i+1})_{hC_p}, A)$$

$$\simeq \mathrm{Map}(\mathbb{S}_{hC_p}, A) = A^{hC_p}. \qquad \square$$

Next, we will apply some algebraic massage.

- $\pi_0(E_{i+1}^{tC_p}) = \pi_*(E_{i+1}^{hC_p})[x^{-1}] = E_{i+1}^*[[x]][x^{-1}]/[p](x)$ is flat over $\pi_*(E_{i+1})$, so all higher Tor vanishes and $\pi_0(A^{tC_p}) \cong \pi_0(A) \underset{\pi_0(E_{i+1})}{\otimes} \pi_0(E_{i+1}^{tC_p})$

- Mod out by the ideal $(p, v_1, \ldots, v_{i-1})$ and invert x on both sides.

  Observe that $\pi_0(E_{i+1}^{C_p})/(p, v_1, \ldots, v_{i-1})$ is faithfuly flat over the field $\pi_0(E_{i+1})/(p, v_1, \ldots, v_{i-1})[v_i^{-1}] = k((v_{i-1}))$. This is because

$$[p](x) = v_{i+1}x^{p^{i+1}} + F(x)\,x^{p^{i+1}+1} \mod (p, v_1, \ldots, v_i).$$

In $\pi_0(E_{i+1}^{tC_p})/(p, v_1, \ldots v_{i-1}) = E_{i+1}^0[[x]][x^{-1}]/[p](x)/(p, v_1, \ldots, v_{i-1})$,

$$[p](x) = 0 \iff v_i \cdot G(x) = (v_{i+1} + F(x)\cdot x)\cdot x^{p^{i+1}}.$$

Since x and v_{i+1} are both invertible, so is v_i.

- Hence the induced map

$$\pi_0(A)/(p, v_1, \ldots, v_{i-1})[v_i^{-1}] \xrightarrow{\Phi} \pi_0(A^{tC_p})/(p, v_1, \ldots, v_{i-1})[v_i^{-1}]$$

is faithfuly flat.

Now we use the following result:

Lemma ([Hahn], 2.2)

If R is a $K(i)$-acyclic E_∞-E_{i+1}-algebra, then some power of v_i is in the ideal $(p, v_1, \cdots, v_{i-1}) \subset \pi_0(R)$.

Since A^{tC_p} is $K(i)$-acyclic by assumption, the target of Φ vanishes and so does the source.

- Finally, we want to show that
$(*) = \pi_0(A)/(p, v_1, \cdots, v_{i-1})[v_i^{-1}] = 0 \Rightarrow A = 0$

(i). For $i=0$, $(*) = \pi_0(A) \otimes \mathbb{Q} = 0$, so A is $K(1)$-acyclic by the May Nilpotence Conjecture [Mathew-Naumann-Noel]. Since A is $K(1)$-local by assumption, $A = 0$.

(ii). For $i \geq 1$, we use the following blackbox:

Thm ([Hahn], 1.2)

Let A be a $K(i+1)$-local E_∞-E_{i+1}-algebra. If in $\pi_0(R)$ some power of v_i is in the ideal $(p, v_1, \cdots v_{i-1})$, then so is 1.

It follows that $\pi_0(A)/(p, v_1, \cdots, v_{i-1}) = 0$, so $\pi_*(A)/(p, v_1, \cdots, v_{i-1}) = 0$. Since A is $K(i+1)$-local, $A = 0$.

- This concludes the proof for $L_{T(i)} A^{+Cp} = 0 \Rightarrow L_{T(i+1)} A = 0$
To see that $L_{T(j)} A = 0$, $\forall j \geq i+1$, we use the following:

Thm ([Hahn], 1.1)

Let A be a $K(n)$-acyclic E_∞-ring spectrum.
Then A is $K(n+1)$-acyclic

Pf. Note that A is $K(n+1)$-acyclic $\Leftrightarrow$ $A \wedge E_{n+1}$ is.
By Lemma 2.2, some power of v_n is in the ideal
$I = (p, v_1, \cdots, v_{n-1}) \subset \pi_0(A \wedge E_{n+1})$. The same is true
for $I \subset \pi_0(L_{K(n+1)}(A \wedge E_{n+1}))$, so $1 \in I$ by Thm 1.2.

As a result, $L_{K(n+1)}(A \wedge E_{n+1})$ is acyclic w.r.t. any
type $n+1$ Moore spectrum and thus its telescope.
It follows that $L_{K(n+1)}(A \wedge E_{n+1})$ is $K(n+1)$-acyclic,
and so is $A \wedge E_{n+1}$. □

□