CHROMATIC HOMOTOPY AND TELESCOPIC LOCALIZATION

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Contents

1. MU and $\mathcal{M}_{fg}$

A ring spectrum R is **complex oriented** if it is a equipped with a ring map $MU \to R$. Such a map provides the cohomology theory R^* with Chern classes for complex vector bundles satisfying a Whitney sum formula.

Given two line bundles $L_1, L_2 : X \to BU(1)$, there is a universal formula for the Chern class of their tensor product:

$$c_1(L_1 \otimes L_2) = F_R(c_1(L_1), c_1(L_2))$$

F_R is a power series in two variables with coefficients in R^*, and encodes the structure of a (1-dimensional commutative) **formal group law**. A formal group law is an abelian group structure on the formal R-scheme $\mathrm{Spf}(R[[x]])$. More concretely, this means that F_R satisfies group axioms, such as associativity: $F_R(x, F_R(y, z)) = F_R(F_R(x, y), z)$.

An important result of Quillen says that MU, the universal complex oriented ring spectrum, has the universal formal group law. In particular, $MU_* = L$, where L is the Lazard ring, defined by the universal property $\mathrm{Hom}(L, R) = FGL(R)$, where FGL is the set of formal group laws on R.

However, the connection between MU and formal group laws doesn't stop there. Recall we have the Adams-Novikov spectral sequence:

$$E_2 = \mathrm{Ext}_{MU_*MU}(MU_*, MU_*X) \implies \pi_*X$$

The E_2 term can be interpreted in terms of formal groups (which are formal group schemes Zariski-locally isomorphic to a formal group law). The Ext in the spectral sequence is taken in the category of comodules over the (graded) Hopf algebroid (MU_*, MU_*MU). However, this Hopf algebroid presents $\mathcal{M}_{fg}$, the moduli stack of formal groups. $\mathcal{M}_{fg}$ has a line bundle ω that is the Lie algebra of the universal formal group. Then the Adams-Novikov E_2 term can be reinterpreted as

$$E_2 = H^*(\mathcal{M}_{fg}; (MU_*X)_{\text{even/odd}} \otimes \omega^{\otimes *}) \implies \pi_*X$$

Where we treat the even and odd degree parts of MU_*X as a quasicoherent sheaf on $\mathcal{M}_{fg}$.

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Thus via MU, stable homotopy is tied to formal groups.

The study of formal groups simplifies a bit when localized at a prime. $(\mathcal{M}_{fg})_{(p)}$ has a simpler presentation as a graded Hopf algebroid (BP_*, BP_*BP), where BP_* is the ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, with $|v_i| = 2(p^i - 1)$. In fact this Hopf algebroid (as suggested by the notation) comes from a ring spectrum called BP. In fact, $MU_{(p)}$ decomposes into summands that are shifts of BP.

$(\mathcal{M}_{fg})_{(p)}$ is a well understood stack. We can draw a picture of its points $\mathrm{Spc}((\mathcal{M}_{fg})_{(p)})$

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & \infty \\ . & . & . & . & \cdots & . \end{array}$$

There is one for each natural number and a point ∞. One way to interpret each point is that it classified a formal group over an algebraically closed field up to isomorphism. The point n corresponds to a **height n formal group**. For example, when $p = 3$, a height n formal group is one such that if we choose coordinates so that the formal group is defined by a power series F, then $F(x, F(x, x)) = ux^{3^n} + \ldots$ where u is a unit and $\ldots$ indicates higher order terms.

The point 0 classifies a formal group in characteristic 0, and the rest of the points classify formal groups in characteristic p.

Another way to interpret the picture is that it classifies invariant prime ideals of BP_* in the Hopf algebroid (BP_*, BP_*BP). The point n corresponds to the ideal $(v_0, v_1, \ldots, v_{n-1})$ where $v_0 = p$.

The space also has a topology, where the open sets are the intervals from 0 to n. In particular, specialization increases height.

2. Important Cohomology Theories and Theorems

Two important families of complex oriented cohomology theories are Morava E-theory and Morava K-theory.

The n^{th} Morava E-theory, denoted E_n, is an $\mathbb{E}_\infty$-ring spectrum that depends on a choice of perfect field k and formal group law on k of height n. However, none of the choices will matter for anything said here about it. Its coefficient ring is $(E_n)_* = W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$ where $|\beta| = 2$, and $W(k)$ denotes the Witt vectors of k, and its formal group law is the universal deformation of the formal group law on k, which was studied by Lubin and Tate.

One of its important properties is that $(E_n)_*(X) = 0$ if and only if $BP_*(X)$ is supported at height $\geq n+1$ on $\mathcal{M}_{fg}$. Thus it detects information from height 0 to height n.

The n^{th} Morava K-theory, denoted $K(n)$, is an $\mathbb{E}_1$-ring spectrum with coefficient group $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$. Once again, there are different versions of it, but the different versions will not be relevant here. $K(n)$ can be constructed from BP by iteratively taking cofibres by v_i for $i \neq n$ and inverting v_n. $K(\infty)$ is just $H\mathbb{F}_p$. An important property of $K(n)$ is that it is a **field**, since its homotopy groups are a graded field. This means that any module over $K(n)$ is free.

$K(n)$ detects information at height n. For example, $(E_n)_*(X) = 0 \iff (\oplus_0^n K(n))_* X = 0$.

Next, we turn to some fundamental results in chromatic homotopy theory. The first is the nilpotence theorem, due to Devinatz, Hopkins, and Smith.

Theorem 2.1 (Nilpotence Theorem v1)**.** *Let R be a ring spectrum, and $\alpha \in \pi_n(R)$ be an element sent to 0 in $MU_n(R)$. Then α is nilpotent.*

This says that MU is able to detect nilpotence of rings. An equivalent version of the theorem is

Theorem 2.2 (Nilpotence Theorem v2)**.** *Let $f : X \to Y$ be a map of finite spectra such that $f \otimes MU$ is 0. Then $f^{\otimes n} : X^{\otimes n} \to Y^{\otimes n}$ is null for $n \gg 0$.*

This formulation emphasizes the fact that MU detects nilpotence phenomena for finite spectra. When working p-locally, $MU_* f = 0$ iff $BP_* f = 0$ iff $K(n)_* f = 0$ for all n.

Definition 2.3. *A finite complex/spectrum X is **type** n if $K(i)_*(X) = 0$ for all $i < n$ and $K(n)_* X \neq 0$.*

Remark 2.3.1. Every nonzero p-local finite spectrum is type n for some n. This is because for $i >> 0$, the $K(i)$-based Atiyah-Hirzebruch spectral sequence degenerates for degree reasons for a fixed finite spectrum X, so $K(i)_*(X) = (H\mathbb{F}_p)_*(X) \otimes_{(H\mathbb{F}_p)_*} K(i)_* \neq 0$.

Remark 2.3.2. For a finite spectrum X, $K(m)_* X = 0 \implies K(m-1)_*(X) = 0$. This is essentially because its MU-homology is a *coherent* sheaf over $\mathcal{M}_{fg}$, so has closed support. This shows that for a type n spectrum X, $K(m)_*(X) \neq 0$ for $m \geq n$.

Definition 2.4. *Let C be a stable ∞-category. A **thick subcategory** $C' \to C$ is a stable subcategory closed under retracts.*

Example 2.4.1. Let $\mathrm{Sp}^{\omega}_{(p)}$ be the category of p-local finite spectra, and let $\mathrm{Sp}_{\geq n}$ be the category of type $\geq n$ spectra. Then $\mathrm{Sp}_{\geq n} \to \mathrm{Sp}^{\omega}_{(p)}$ is a thick subcategory.

It turns out these are all the examples. This is the content of the following result, which is a corollary of the nilpotence theorem, due to Hopkins and Smith.

Theorem 2.5 (Thick subcategory Theorem)**.** *Let $C \subset \mathrm{Sp}^{\omega}_{(p)}$ be a nonzero thick subcategory. Then $C = \mathrm{Sp}_{\geq n}$ for some n.*

It is true that $\mathrm{Sp}_{\geq n}$ are distinct as n varies, but showing this requires a bit more work.

Definition 2.6. *Let X be a finite complex/spectrum. A v_n**-self map** $v_n : \Sigma^d X \to X$ is a map that*

(1) induces 0 on $K(m)_*$ for $m \neq n$.
(2) induces an isomorphism on $K(n)_*$.

The use of v_n as a name for the v_n-self map is slightly misleading: a more appropriate name is v_n^k, because when they exist, they can be chosen to induce multiplication by a power of v_n on $K(n)_*$.

Using a construction due to Smith, Hopkins and Smith proved the following result:

Theorem 2.7 (Periodicity Theorem)**.** *Every type n spectrum admits a v_n-self map.*

From this theorem, it is easy to see why $\mathrm{Sp}_{\geq n}$ are distinct. For example, the sphere $\mathbb{S}$ is a type 0 but not type 1 spectrum. Given a type n but not type $n+1$ spectrum, we can take the cofibre of a v_n-self map to obtain a type $n+1$ but not type $n+2$-spectrum, thereby inductively distinguishing the categories $\mathrm{Sp}_{\geq n}$.

v_n-self maps are well behaved. After replacing one with a sufficiently large power, we can assume

- the v_n-self map induces multiplication by v_n^i on $K(n)_*$.

- the v_n-self map is central in $\mathrm{End}_*(X) = \pi_* X \otimes DX$.

Given a map of finite type n-spectra $f : X \to Y$ equipped with a v_n-self map, we can replace the v_n-self maps by an iterate to make the diagram below commute:

$$\begin{array}{ccc} \Sigma^d X & \xrightarrow{f} & \Sigma^d Y \\ \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle v_n} \\ X & \xrightarrow{f} & Y \end{array}$$

In this sense, v_n-self maps are almost functorial. Note that if we take f above to be the identity, we see that v_n-self maps are also unique up to taking iterations.

3. Chromatic Localizations

The moduli stack of formal groups is filtered by the open substacks of formal groups of height $\leq n$. Chromatic localizations are a way to turn this algebraic filtration into a topological one, and their study was pioneered by Doug Ravenel. To talk about them, we will briefly review Bousfield localizations of the category Sp.

Given a spectrum X, there is an adjunction

$$L_X : \mathrm{Sp} \leftrightarrows \mathrm{Sp}_X : i$$

such that

- L_X inverts X**-equivalences**: that is morphisms f such that $f \otimes X$ is an equivalence.
- L_X kills (sends to 0) the X**-acyclic objects**, i.e those objects Y such that $Y \otimes X = 0$.
- i is fully faithful, so Sp_X is a reflective subcategory of Sp.
- The essential image of i consists of X**-local spectra**, that is objects Z such that there are no nonzero maps from X-acyclic objects to Z.

The composite $i \circ L_X$ will often be shortened to L_X. The unit of the adjunction gives a natural map $Y \to L_X Y$, characterized by the fact that it is an X-equivalence to an X-local object.

The construction L_X doesn't depend on all of X but rather on the **Bousfield class**, that is $\langle X \rangle = \{X\text{-acyclic objects}\}$.

We can often break up a Bousfield localization into smaller pieces, and glue them back together.

Lemma 3.1. *Suppose L_E preserves F-acyclic objects. Then*

$$\begin{array}{ccc} L_{E\oplus F} & \longrightarrow & L_F X \\ \downarrow & \lrcorner & \downarrow \\ L_E X & \longrightarrow & L_E L_F X \end{array}$$

is a pullback square.

Proof. Let $P = L_E X \times_{L_E L_F X} L_F X$.

- P is $E \oplus F$ local. Indeed, if Z is $E \oplus F$ acyclic, $P^Z = 0 \otimes_0 0 = 0$.
- $X \to P$ is an $E \oplus F$ equivalence. To see it is an E-equivalence, after tensoring with E it becomes

$$X \otimes E \xrightarrow{\sim} X \otimes E \times_{E \otimes L_F X} E \otimes L_F X$$

To see it is an F-equivalence, by the hypothesis on L_E, we learn that the natural transformation $Y \to L_E Y$ is an F-equivalence. Thus after tensoring with F, we get

$$X \otimes F \xrightarrow{\sim} X \otimes F \times_{X \otimes F} X \otimes F$$

□

If X is a type n spectrum, We can invert a v_n-self map to get $X[v_n^{-1}]$, which is called the **telescope** of X and denoted $T(n)$. By the almost uniqueness of v_n-self maps, $T(n)$ only depends on X. Essentially by the thick subcategory theorem, $\langle T(n) \rangle$ only depends on n.

There are two flavors of chromatic localizations that are studied. The first are the telescopic and finite localizations $L_{T(n)}$ and $L_n^f := L_{\oplus_0^n T(i)}$. The second are the $K(n)$ and E_n localizations $L_{K(n)}$ and $L_n := L_{E(n)} = L_{\oplus_0^n K(n)}$. The hope is that we can understand stable homotopy via the towers of localizations

$$X \to \dots \to L_n X \to L_{n-1} X \to \dots L_1 X \to L_0 X$$

(and similarly for L_n^f in place of L_n).

The two flavors of localizations are related to each other. If $Y \otimes T(n) = 0$, then $X \otimes T(n) \otimes K(n) = 0$, but $T(n) \otimes K(n)$ is a nonzero sum of copies of $K(n)$, so $X \otimes K(n) = 0$. Thus we get factorizations of the natural maps

$$X \to L_n^f X \to L_n X \;\;,\;\; X \to L_{T(n)} X \to L_{K(n)} X$$

An important property of $L_n X$ is that it is colimit preserving:

Theorem 3.2 (Smashing Theorem). $L_n X = L_n \mathbb{S} \otimes X$

The same is true of L_n^f, but it is easier to prove, as will now be explained.

Lemma 3.3. *The L_n^f-acyclic spectra coincide with* $\mathrm{Ind}(\mathrm{Sp}_{\geq n+1})$*: that is they are filtered colimits of type $\geq n+1$ spectra.*

Proof. It is easy to see that $\mathrm{Ind}(\mathrm{Sp}_{\geq n+1})$ consists of $T(n)$-acyclic spectra; we will show the reverse inclusion. First let $n = 0$, and suppose X is $T(0)$-acyclic. Then there is a cofibre sequence

$$X \to p^{-1} X = X \otimes T(0) \to X \otimes \mathbb{S}/p^{\infty}$$

where $\mathbb{S}/p^{\infty}$ is the colimit of $\mathbb{S}/p^n$ over all n. Since $X \otimes T(0)$ vanishes, we learn that $X = \Sigma^{-1} X \otimes \mathbb{S}/p^{\infty}$. X is a filtered colimit of finite spectra, and after tensoring with $\mathbb{S}/p^n$, this becomes a filtered colimit of type 1 spectra.

Now we can induct on n. For example, let $n = 1$, and assume that in addition, X is $T(1)$-acyclic. Then there is a cofibre sequence

$$X \otimes \mathbb{S}/p^n \to X \otimes v_1^{-1} \mathbb{S}/p^n = X \otimes T(1) \to X \otimes \mathbb{S}/p^n, v_1^{\infty}$$

, so since $X \otimes T(1) = 0$, we learn that $X = \Sigma^{-2} X \otimes \mathbb{S}/p^{\infty}, v_1^{\infty}$, which is in $\mathrm{Ind}(\mathrm{Sp}_{\geq 2})$. □

Remark 3.3.1. The argument in the above lemma shows that there is a cofibre sequence

$$\Sigma^{-1-n} \mathbb{S}/v_0^{\infty}, \dots v_n^{\infty} \to \mathbb{S} \to L_n^f \mathbb{S}$$

Since L_n^f kills a category that is generated by compact objects, it preserves filtered colimits. It also preserves finite colimits, so L_n^f preserves all colimits. The only colimits preserving endomorphisms of Sp are given by tensoring, so we learn

Corollary 3.4. *$L_n^f X = L_n^f \mathbb{S} \otimes X$.*

The corollary above is one way to see that L_m^f preserves $\oplus_{m+1}^n T(i)$-acyclic objects. Thus we learn from Lemma 3.1:

Corollary 3.5. *There is a pullback diagram*

$$\begin{array}{ccc} L_n^f X & \longrightarrow & L_{\oplus_{m+1}^n T(i)} X \\ \downarrow & \lrcorner & \downarrow \\ L_m^f X & \longrightarrow & L_m^f L_{\oplus_{m+1}^n T(i)} X \end{array}$$

Note that the same is true with $K(n)$ replacing $T(n)$ and L_n replacing L_n^f by the smashing theorem. These pullback squares allow one to reduce the study of L_n^f to the study of $L_{T(n)}$.

The exact relation between $L_{T(n)}$ and $L_{K(n)}$ is not known. It was conjectured by Ravenel that there is no difference between the two.

Conjecture 3.6 (Telescope conjecture)**.** *The map $L_{T(n)}X \to L_{K(n)}X$ is an equivalence.*

This conjecture is known to be true for $n = 1, 0$, and many believe it to be false otherwise. Nevertheless, so long as we are concerned with rings or finite spectra, the nilpotence theorem implies that $T(n)$ and $K(n)$ behave similarly.

Lemma 3.7. *If R is a ring spectrum, $R \otimes T(n) = 0 \iff R \otimes K(n) = 0$.*

Proof. Let V_n be a type n spectrum that is an $\mathbb{E}_1$-ring. For example, one can start with any type n spectrum X and replace it with its endomorphism ring $X \otimes DX$. Let v_n be a central v_n-self map, so that $T(n) = V_n[v_n^{-1}]$ is a ring. Then

$$\begin{aligned} & R \otimes T(n) = 0 \\ \iff & \text{the unit of } R \otimes T(n) \text{ is nilpotent} \\ \iff & \text{the unit of } R \otimes T(n) \otimes K(m) \text{ is nilpotent for all } m \\ \iff & \text{the unit of } R \otimes T(n) \otimes K(n) \text{ is nilpotent} \\ \iff & R \otimes T(n) \otimes K(n) = 0 \\ \iff & R \otimes K(n) = 0 \end{aligned}$$

Where in the second step, we use the nilpotence theorem, and in the last step we use the fact that $T(n) \otimes K(n)$ is a free $K(n)$-module. □

4. Telescopic Localization and the Bousfield Kuhn functor

Now we will look the telescopic localization functors more in depth and see their to unstable homotopy theory. We will set $n \geq 1$, and let V_n denote a type n *space* with a v_n-self map $v_n : \Sigma^d V_n \to V_n$.

Definition 4.1. *For a space/spectrum X, the v_n-**periodic homotopy groups** with coefficients in V_n, denoted $v_n^{-1}\pi_*(X; V_n)$ are defined as $v_n^{-1}\pi_*(\mathrm{Map}_*(V_n, X))$.*

It isn't hard to see that $v_n^{-1}\pi_*(X; V_n)$ is the homotopy groups of a d-periodic spectrum called $\Phi_{V_n}(X)$, given by the formula $\mathrm{colim}_k \Sigma^{\infty - kd} \mathrm{Map}_*(V_n, X)$ where the colimit is taken with respect to the maps $\mathrm{Map}_*(V_n, X) \to \mathrm{Map}_*(\Sigma^d V_n, X)$ induced by v_n.

Note that if X is a spectrum, then $\Phi_{V_n}(X)$ is the spectrum $X \otimes DV[v_n^{-1}] = X \otimes T(n)$.

Definition 4.2. *If $f : X \to Y$ is a map of spaces or spectra, f is a v_n**-periodic equivalence** if $\Phi_{V_n} X \to \Phi_{V_n} Y$ is an equivalence.*

Essentially by the thick subcategory theorem, the notion of V_n-periodic equivalence only depends on n.

Lemma 4.3. *Let $n \geq 1$, and X be a spectrum. Then*

(1) $\Phi_{V_n} X = \Phi_{V_n} \Omega^\infty X$.
(2) The map $\tau_{\geq k} X \to X$ is a v_n-periodic equivalence.
(3) The map $X \to X_p^\wedge$ is a v_n-periodic equivalence.

Proof. (1): This follows from the adjunction between Σ^∞ and Ω^∞.

(2): The fibre is bounded above, and $v_n^{-1}\pi_*(Y; V_n) = 0$ whenever Y is bounded above because $|v_n| > 0$, and the homotopy groups of the mapping space are bounded above.

(3): The fibre $F \to X \to X_p^\wedge$ is $\mathbb{S}/p$-acyclic. This means that it is killed by tensoring with $\mathbb{S}/p^m$ for all m. But for any type n spectrum, some power of p acts by 0, so the same is true for $T(n)$. Thus $F \otimes \mathbb{S}/p^k \otimes T(n) = F \otimes (T(n) \oplus \Sigma T(n)) = 0$ for $k \gg 0$, so F is $T(n)$-acyclic. □

The above lemma, along with the fact that $\Phi_{V_n} X = X \otimes T(n)$ implies that for $n \geq 1$, $L_{T(n)} X$ only depends on $\Omega^\infty X$ as an $\mathbb{E}_\infty$-space. A wonderful insight of Bousfield and Kuhn is that in fact it only depends on $\Omega^\infty X$ as a space!

To see this, we start by thinking about the construction taking a pair of a type n space and v_n-self map (V_n, v_n) to the functor $\Phi_{V_n} : S_* \to \mathrm{Sp}$. If we replace v_n by an iterate, it is easy to see that it doesn't change Φ_{V_n}, so since v_n-self maps are unique, the data of v_n is not important in the construciton of Φ_{V_n}.

Secondly, if we replace V_n by ΣV_n, Φ_{V_n} changes to $\Phi_{\Sigma V_n} = \Sigma^{-1} \Phi_{V_n}$. Thus Φ_{V_n} only depends on the spectrum $\Sigma^\infty V_n$.

These observations can be souped up to construct a functor

$$\mathrm{Sp}_{\geq n} \to \mathrm{Fun}(S_*, \mathrm{Sp})$$

that sends a type n spectrum V to Φ_V.

Definition 4.4. *The **Bousfield-Kuhn functor** Φ is a functor $S_* \to \mathrm{Sp}$ given by $\Phi := \lim_{V \to \mathbb{S}} \Phi_V$.*

Another way to describe it is that you right Kan extend the functor $\mathrm{Sp}_{\geq n} \to \mathrm{Fun}(S_*, \mathrm{Sp})$ along the inclusion to Sp, and evaluate on $\mathbb{S}$. An important property of Φ is that it realizes the factorization of $L_{T(n)}$ through Ω^∞ as a space:

Proposition 4.5. *$\Phi \Omega^\infty X = L_{T(n)} X$.*

Proof. We have from the definition and our previous observations $\Phi \Omega^\infty X = \lim_{V \to \mathbb{S}} \Phi_V X = \lim_{V \to \mathrm{Sp}} DV[v_n^{-1}] \otimes X$.

Each term in the limit is $T(n)$-local, so it agrees with $L_{T(n)}(DV[v^{-1}] \otimes X) = L_{T(n)}(DV \otimes X) = L_{T(n)}(X^V)$.

Putting this together, we have

$$\Phi \Omega^\infty X = \lim_{V \to \mathbb{S}} L_{T(n)} X^V$$

Now I claim that $T(n)$-locally $\mathbb{S}$ is a filtered colimit of type n spectra. This claim completes the proof, because it identifies $\lim_{V\to\mathbb{S}} L_{T(n)}X^V$ with $L_{T(n)}X^{\mathbb{S}} = L_{T(n)}X$.

To see the claim we recall that we had a cofibre sequence

$$\Sigma^{-n}\mathbb{S}/p^\infty, \dots, v_{n-1}^\infty \to \mathbb{S} \to L_{n-1}^f\mathbb{S}$$

$L_n^f\mathbb{S}$ is $T(n)$-acyclic since $T(n)$ is a filtered colimit of type n spectra, which L_n^f kills. Thus if we apply $L_{T(n)}$ to the cofibre sequence above, we obtain a formula for $L_{T(n)}\mathbb{S}$ as a filtered colimit of ($T(n)$-localizations of) type n spectra. □

Some other important facts about the Bousfield-Kuhn functor are:

- It inverts v_n-periodic equivalences, and takes values in $T(n)$-local spectra. This is indicated by the factorization below, where $S_*^{v_n}$ is the localization of pointed spaces at the v_n-periodic equivalences. The factored map is also denoted Φ.

$$\begin{array}{ccc} S_* & \xrightarrow{\Phi} & \mathrm{Sp} \\ \downarrow & & \downarrow \\ S_*^{v_n} & \xrightarrow{\Phi} & \mathrm{Sp}_{T(n)} \end{array}$$

- $S_*^{v_n} \to \mathrm{Sp}_{T(n)}$ preserves limits.

A consequence of the factorization in the proposition above is:

Corollary 4.6. *Let $f : X \to Y$ be a map of spectra. If $\Sigma^\infty\Omega^\infty f$ is a $T(n)$-equivalence, then so is f.*

Proof. By assumption, $L_{T(n)}\Sigma^\infty\Omega^\infty f$ is an equivalence. But this is equal to $\Phi\Omega^\infty\Sigma^\infty\Omega^\infty f$, and by the triangle identity for the adjunction between Σ^∞ and Ω^∞, the map $\Phi\Omega^\infty f$ is a retract of $\Phi\Omega^\infty\Sigma^\infty\Omega^\infty f$. Thus $\Phi\Omega^\infty f = L_{T(n)}f$ is also an equivalence. □

The functor $\Sigma^\infty\Omega^\infty$ doesn't *preserve* $T(n)$-local equivalences in general.

Example 4.6.1. $H\mathbb{Z}$ is $T(n)$-acyclic, but $\Sigma^\infty\Omega^\infty H\mathbb{Z}$ is a sum of spheres, so is not.

Nevertheless, for sufficiently connected maps, $\Sigma^\infty\Omega^\infty$ does preserve $T(n)$-local equivalences. Here is a version of that statement for the finite localizations.

Proposition 4.7. *Let $n \geq 1$. There is an $m \geq 2$ such that:*

(1) If F is an m-connected pointed space such that $v_i^{-1}\pi_*(F; V_i) = 0$ for $0 \leq i \leq n$, then F is L_n^f-acyclic.

(2) If $f : X \to Y$ is an m-connected map that is a v_i-periodic equivalence for $0 \leq i \leq n$, then $\Sigma^\infty f$ is an L_n^f equivalence.

(3) $\Sigma^\infty\Omega^\infty$ preserbes m-connected L_n^f equivalences.

Proof. (1): Omitted. This relies on results of Bousfield on unstable localization.

(2): The fibre F satisfies the hypotheses of (1). Then f can be identified with $\mathrm{colim}_Y F \to \mathrm{colim}_Y *$, which is a $T(n)$ equivalence since F is L_n^f-acyclic.

(3): Apply Ω^∞ and (2).

□

Remark 4.7.1. In fact in the above proposition, m can be taken to be $n + 1$. This is a consequence of ambidexterity of the $T(n)$-local category, which was proven by Carmeli, Schlank, and Yanovski.