Disclaimers

- implicit prime $p$ in chromatic localization
  is same as $p$-group.
- specificity/generality, esp. genuine equiv theory

Main reference: "Descent & vanishing in chromatic alg. K-theory".
by Clausen- Madsen- Naumann- Noel.
§0. Introduction

Goal: put everything together from last few talks (+2) to prove.

**Thm A (Clausen-Mathew-Naumann-Noel)**  "vanishing."

Let $R$ be a ring of characteristic $p$, and $A$ be a module over $R$. If $\text{lim} A_i = 0$, then

$$L_{\text{lim}} K(A) \approx 0.$$ 

**Thm B (CMNN)**  "descent."

Let $R$ be a ring of characteristic $p$, and $A$ be a module over $R$. Suppose $L_{\text{lim}} (R^{\otimes p}) = 0$. Then:

- for any $R$-linear stable derivator $\mathcal{C}$ with action of a finite $p$-group $G$, the following map

$$L_{\text{lim}} K(\mathcal{C}^G) \rightarrow L_{\text{lim}} K(\mathcal{C})^G$$

**Remark** generalizes Thomason’s Galois descent for $\text{Th}$ local K-theory.

**Remark** true for non-$p$-groups.

Thm B does not hold in general for arbitrary finite $G$. 
§1. Previously on...

\textbf{Notation} \quad L_n^f = \text{localization w.r.t. } T(0) \oplus \cdots \oplus T(n).

\textbf{Thm (Land-Mathew-Meier-Tamme).} \quad \text{"weak" purity}

A E_
"verting spectrum \& n \geq 1. Then
\[ K(A) \overset{\sim}{\longrightarrow} K(L_n^f A). \]

is a $T(m)$ equivalence, for $1 \leq m \leq n$.

\textbf{Thm (CMNN).} \quad \text{blueshift}

Let $A$ be an $E_
"verting$. Then
\[ L_{T(n)} A^\wedge = 0 \Rightarrow L_{T(m)} A = 0 \quad m > n. \]

\textbf{Thm (Kuhn; CM).} \quad \text{blueshift}

Let $X$ be a $T(m)$-local spectrum w/ G-action.

Then \[ X^{hG} \overset{\sim}{\longrightarrow} X^{hG} \quad \text{i.e. } X^G \overset{\sim}{\longrightarrow} \ast. \]
§1. Recollections

Def. G a finite group. The effective Burnside category $\text{Burn}_G^e$ is the nerve of the $\text{(2,1)}$-category:

- objects: finite left $G$-sets $S$
- morphisms: spans of $G$-equivariant maps $S \rightarrow U \leftarrow T$
- composition: pullback.

> 2 morphisms: iso of spans.

Def. $C$ an $\omega$-cat. The category of $C$-valued Mackey functors:

$n_\text{Mack}_C(C) := \text{pSh}_{n_\text{Mack}}(\text{Burn}_G^e, \text{op}, C)$.

(functors which take $U \mapsto X$)

$= \text{pSh}_{n_\text{Mack}}(\text{Burn}_G^e, \text{Sp}_{\text{der}}, C)$, a monoidal derived $\omega$-cat.

Rank. Prove something about $n_\text{Mack}_C(C)$ by reducing to

- CMNN use terminology “semi-Mackey functors.”
- $M(G/H) = M^n$ categorized “genuine” fixed points.
- $M(G) = M^{\mu_1}_G$ “underlying object”:

\[
\begin{array}{c}
M_G \\ \downarrow \\
M^{\mu_1}_G \\ \downarrow \\
M^{\mu_0}_G
\end{array} \rightarrow
\begin{array}{c}
M^G \\ \downarrow \\
\Phi^0 M \\ \downarrow \\
M^G
\end{array}
\] isotropy separation sequence.

- pushforward along finite product-preserving functors.

Note. $\mathcal{O}(G) = \text{cat.}$ of finite left $G$-sets & $G$-equiv. maps.

There are two inclusions:

$\mathcal{O}(G), \mathcal{O}(G)^\text{op} \rightarrow \text{Burn}_G^e$.
If \( \mathcal{C} \) is presentable symmetric monoidal, then \( \text{Mackey}(\mathcal{C}) \) has \( \otimes \) via Day convolution using \( \otimes \) on \( \text{Burnet} \).

**Def** \( \text{A Mackey functor } F \) is

- \( \text{Boed} \) if \( \mathcal{V} \mathcal{G} \oslash \mathcal{S} \) finite,
  - \( F(S) \to F(S \times G)^{\mathcal{G}} \) is an equivalence.
  - \( F \) is RKE'd from full subcategory on \( \mathcal{G} \).
  - \( F^H \cong (F^{\mathcal{G}})^{\mathcal{H}} \) equivalence for all \( H \leq G \).
- \( \text{coBoed} \) if \( (M \otimes y(G))_{\mathcal{G}} \to M \) is an equivalence.
  - \( \text{you can embedding} \)
  - \( (F^H)_{\mathcal{H}} \to F^H \) equivalence for all \( H \leq G \).
  - \( F \) is KE'd from full subcategory on \( \mathcal{G} \).

**Prop 2.8** There is a \( \otimes \) Bousfield localization

\[ \text{Mackey}(\mathcal{C}) \to \text{Fun}(BG, \mathcal{C}). \]

**Fact** Mackeyification is given by \( (- \otimes y(G))_{\mathcal{G}} \).

Have "norm" natural transformation.
\[ (c)^{\text{coBoed}} \to (c)^{\text{Boed}} \]

For our purposes: take \( \mathcal{C} = \text{Sp}, \text{Spc}, \text{Cat}^{\text{perf}}(\mathcal{R}^\times) \).

\[ \text{since coends } \]
\[ \text{are computed via passage to } \]
\[ \text{up} \]
A useful construction: G-space.

**Def.** Let $EI$ be the $C_p$-Hurewicz functor in $Sp$, given by
\[(EI)^{hS} = *, \quad EI C_p = \varnothing.\]

So write $\widetilde{EI} = cofib\ (S^0 \to EI_+)$.

**Prop.** The Board-completion of a $C_p$-spectrum $X$ is
\[x^{\infty} = F(EI_+, X)\quad \text{internal mapping spectrum.}\]

**Note.** $M, N \in \mathcal{H}ack_{C_p}(Sp)$

\[
\begin{array}{ccc}
\text{Bun}^\text{eff}_{C_p} \times \text{Bun}^\text{eff}_{C_p} & \to & Sp \times Sp \\
\downarrow & & \downarrow \otimes \\
\text{Bun}_{C_p} & \to & M \otimes N.
\end{array}
\]

Q: Express $(M \otimes N)_{C_p}$ only in terms of $M_{C_p}, N_{C_p}$?

Idea: Force $\text{modf}(\cdot)^{C_p}$ to be symmetric monoidal by killing the $S^3$ piece first.

exactly picked out by $EI_+$.

i.e. smash with $cofib\ (EI_+ \to S^0) = \widetilde{EI}$

**Def.** Geometric fixed points functor $\widetilde{EI}_p$ for $C_p$

\[\text{Mack}_{C_p}(Sp) \to \text{Mack}_{C_p}(Sp) \to \text{Sp}.\]

*Rank Analogous formula for arbitrary $G$.\]
Pop \text{ is:}
- symmetric monoidal
- colimit-preserving.

There is a pullback square of G-spectra.

\[
\begin{array}{ccc}
X & \longrightarrow & E \mathbb{X} \otimes X \\
\downarrow & & \downarrow \\
F(E \mathbb{X}, X) & \longrightarrow & E \mathbb{X} \otimes F(E \mathbb{X}, X)
\end{array}
\]

which, upon taking \( (-)^{op} \), recovers the Tate square.

\[\text{Equivariant algebraic k-theory}\]

\[\text{Def. } \quad \text{Fun}(BG, \text{Cat}^{\text{perf}}) \xrightarrow{j^*} \text{Mack}_0(\text{Cat}^{\text{perf}}) \xrightarrow{K} \text{Mack}_0(\text{Sp})\]

I write \( K(G)(R) = K(\text{Perf}(R)^{\text{perf}}) \).

\[\text{Aside: Colimits in } \text{Cat}^{\text{perf}} \]

1. Diagram \( F \circ j \rightarrow \text{Cat}^{\text{perf}} \xrightarrow{\text{Ind}} \text{Pr}^{\text{perf}} \)

Since \( \text{Pr}^{\text{perf}} = (\text{Pr}^{\text{perf}})^{op} \), colim \( \text{Ind} \circ F = \lim_{j^{op}} \text{Ind} \circ F \)

2. \( F \circ \text{Ind} \) classifies a bicartesian \( E \)

3. \( \lim_{j^{op}} (\text{Ind} \circ F) = \{ \text{equivariant sections of } \pi^3 \} \)

4. \( \text{Take compact objects } (-)^{wc} \).
Ex: Let $G$ act trivially on $\text{Perf}(R)$ & consider $\text{Perf}(R)^{\text{Bar}}$.

$$(\text{Perf}(R)^{\text{Bar}})^{H} \cong \text{Perf}(R)^{H} \Rightarrow (\text{Mod}(R)^{H})^{0} \cong \text{Perf}(R[H])$$

$$\downarrow$$

$$(\text{Perf}(R)^{\text{Bar}})^{H} \cong \text{Fun}(BH, \text{Perf}(R))$$

Nilpotence

Def. A $C_p$-spectrum $X \in \text{Mack}_C(Sp)$ is $I$-nilpotent if it belongs to the thick subcategory $I_{\text{trivk}}$ $I$-ideal gen by $I_{\text{trivk}}$.

i.e. the freely induced spectra.

Note $I$-nilpotent $\Rightarrow$ Borel but not $\Leftarrow$.

Remark: "Borel" not preserved by pushforward along arbitrary exact functors, i.e. $X^G \not\Rightarrow \lim_{\to} X^G$.

$\Leftarrow$ infinite limit

but $I$-nilpotent $\Rightarrow$ finite condition.

Note: $R$ ring $M$ $R$-module. Then $R$ $I$-nilpotent $\Rightarrow M$ $I$-nilpotent

Aside: "Borel" $\Leftrightarrow$ complete wrt an alg object in $Sp$.

$I$-nilpotent $\Rightarrow$ ssseq computing "completion"

converges quickly.

Then, let $A$ a homotopy ring $C_p$-spectrum, i.e. $\text{HoRays}(\text{Mack}_C(Sp))$,

Then $A$ is $I$ nilpotent $\Rightarrow \quad \square^p A \Rightarrow \ast$.
§2 Outline

Thm A & B

[LMNT]
nilpotence

Thm C \Rightarrow \text{Cor 4.9.}

Lemma 4.1

Lemma 4.7

Hesselholt-Madsen &

Lemma 4.8,

inductive vanishing
relates descent & vanishing

assembly map
computations

descent & vanishing

descent for $R = \mathbb{L}^{0}\mathcal{S}^0$

& purity
§3. The first reduction.

**Thm C (CMNN)** Let $C \in L^f_n$-local stable oo-cat. $G$ finite $p$-group.

Then $L(T_n) K(C) = 0$ for any action of $G$ on $C$.

For any $m \geq n+2$ have $L(T_n) K(C)^G \cong L(T_n) K(C)^G$.

**Cor. 4.9** Let $A$ be an $E_\infty$ ring. Then the map $A \to L(T_n \otimes_{T_n^{-1}}) A$ induces an equivalence.

$L(T_n) K(A) \cong L(T_n) K(L(T_n \otimes_{T_n^{-1}}) A)$.

**(Thm C ⇒ 4.9):** purity wlog assume $A$ is $L^f_n$-local have a pullback square of nonconnective K-theory

$$
\begin{array}{ccc}
K(A) & \to & K(L(T_n \otimes_{T_n^{-1}}) A) \\
\downarrow & & \downarrow \\
K(L^f_{n+2}(A)) & \to & K(L^f_{n+2}(L(T_n \otimes_{T_n^{-1}}) A))
\end{array}
$$

Both vertical homotopy fibers are given by

$K$ (trunc. subset on $A \otimes F$)

of $\text{Perf}(A)$ limit type $n-1$ cpx.

**Thm C ⇒ $L(T_n)$ (bottom row) = (0 = 0)**
§4. The Inductive Lemma: reduction to it

Prop (CMNN 4.3) Let \( R \) be an \( E_{\infty} \)-ring \( n \geq 1 \). Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

1. \( L_{\text{dim}} R = 0 \) \& \( L_{\text{dim}} K(R^{\text{top}}) = 0 \) for trivial \( G \)-action.

2. \( L_{\text{dim}} \bar{R}^{\text{top}} K_{G}(R) \cong 0 \) \( (\bar{R}^{\text{top}} \overset{G}{\longrightarrow} \bar{R}^{\text{top}}) \) "co Bord."

3. \( L_{\text{dim}} K(R) = 0 \) \( m \geq n+1 \).

Q Why? How does this imply "descent"?

Prop (CMNN 4.1, 3.6).

Let \( R \) be an \( E_{\infty} \)-ring \( m \geq 0 \). Then TFAE:

1. \( L_{\text{dim}} K_{G}(R) \otimes F \) is \( I \)-nilpotent for any finite type \( m \)

complex \( F \).

2. \( L_{\text{dim}} \bar{R}^{\text{top}} K_{G}(R) = \mathbb{Q} \).

3. \( L_{\text{dim}} K_{G}(R) = 0 \) is Bord-complete. explain?

Claim ("Inductive" \& "nilpotence") \( \Rightarrow \). Thm C.

Idea: Take \( R = L_{0}^{\mathcal{S}^{0}} \). Suffices to show

1. \( L_{\text{dim}} R = 0 \). \( \checkmark \) by defn.

2. \( L_{\text{dim}} K(R^{\text{top}}) = 0 \).

By induction on \( n \):

\( n = 0 \) : \( R = \mathcal{S}^{0} \Rightarrow \mathcal{S}^{1} \overset{\mathcal{S}^{0}}{\longrightarrow} \mathcal{S}^{1} \).

\( n > 0 \) : Blueshift \( \Rightarrow R^{\text{top}} \) is \( L_{n-1} \)-local.

\( \Rightarrow \) \( K(R^{\text{top}}) \) is a module over \( K(L_{n-1}^{\mathcal{S}^{0}}) \).

so does \( \mathcal{S}^{0} \) vanish \( L_{n+1} \)-locally.
Preliminary observation.

If \( C \) is a presentable stable \( \infty \)-category and \( X \in \text{Fun}(BH, C) \), then \( X_{HH} \otimes \text{L}m \) belongs to the thick subcategory generated by \( X \otimes \text{L}m \).

Notation. \( Y \in \langle Z \rangle \) for \( Y \) belongs to the thick subcategory generated by \( Z \).

Proof. Choose \( \text{L}m \) to have (mpy) ring structure. Consider

\[
\text{L}(m) = \text{L}(m) \otimes EH_+ \cong \lim_{\Delta^{op}} (\text{L}(m) \otimes \text{sk}_E EH_+)^{HH}
\]

So \( \text{L}(m) \) is a retract of \( \text{L}(m) \otimes EH_+ \), hence belongs to \( \langle \text{L}(m) \otimes EH_+ \rangle \).

\[
\Rightarrow \text{L}(m) \in \langle \text{L}(m) \otimes H_+ \rangle
\]

and

\[
\Rightarrow \text{L}(m) \otimes \text{sk}_E EH_+ \in \langle \text{L}(m) \otimes H_+ \rangle.
\]

\[
\Rightarrow \quad X_{HH} \otimes \text{L}(m) \in \langle (X \otimes \text{L}(m) \otimes H_+)^{HH} = X \otimes \text{L}(m) \rangle.
\]

As a consequence, we have: \( E \text{L}(m) \otimes \text{L}(m) = (G_+)^{HH} \otimes \text{L}(m) \) is \( \mathbb{L} \)-nilpotent.

Proof of lemma: (1) \( \Rightarrow \) (2). Write \( M = \text{L}(m) \otimes \text{sk}_E (R) \).

(2) \( \Rightarrow \) (3). Can write \( M = \varprojlim_{\text{finite type } m} M \otimes F \).

(3) \( \Rightarrow \) (1) Choose a finite type \( n \)-complex \( F \) with ring structure.

\[
M \otimes F \text{ has a } \text{L}(m) \text{-module structure.}
\]
$E_{1+} \otimes M \otimes F \cong \mathcal{T}(m) \otimes E_{1+}$ module

$\Rightarrow$ belongs to thick subcategory gen by $E_{1+} \otimes \mathcal{T}(m)$

$\Rightarrow$ is $T$-nilpotent

Cor. Let $R \in \mathcal{L}(\mathcal{T}(m))Sp$ an algebra object which is $Bord$ complete

Then any $R$-module $M \in \mathcal{L}(\mathcal{T}(m))Sp$ is also $Bord$ complete

$\mathcal{Pf}$. $\overline{\mathcal{E}}^C$ is symmetric monoidal.

Cor. (3) $\mathcal{L}(\mathcal{T}(m))K^C(R)$ is Bord-complete

IMPLIES

(4) For any $R$-linear stable $\omega$-category $E$ with action of

a finite $p$-group $G$ & every additive invariant $E$

which takes values in $\mathcal{T}(m)$-local spectra,

$E(C^G) \Rightarrow E(C)^K$

$\mathcal{Pf}$ Sketch: $E_G(C)$ is a module over $\mathcal{L}(\mathcal{T}(m))K^C(R)$.

since $E$ $\mathcal{T}(m)$-local!
§5. Pf of the inductive proposition

Prop. (CMNN). Let $R$ be an $E_\infty$-ring \& $n \geq 1$.

1. $L_{T(\eta)} R = 0$ \& $L_{T_{\eta1}} K(R)^{t_{\eta}} = 0.$

2. $L_{T(\eta)} \overline{\text{CP}}(K_{CP}(R)) = 0.$

3. $L_{T_{\eta j}} K(R) = 0 \; \forall \; j \geq n+1.$

\textbf{Note}: have map of $E_\infty$-rings $\overline{\text{CP}}(K_{CP}(R)) \to K(R)^{t_{\eta}}$.

\therefore \; (2) \implies (3) \; \text{by blueshift}

Pf of (1) \Rightarrow (2) proceeds via:

\textbf{STEP 1}: Reduction to case: $R$ connective

\[
\left\{ \begin{array}{l}
L_{n_{n_0}}^2 (H \mathbb{Z}^{t_{\eta}}) = 0.
\end{array} \right.
\]

\text{ Phenomena exact sequences.}

\text{ Commutes with commutes on connective spectra.}

\[
(T_{\eta \omega} R)^{t_{\eta}} \xrightarrow{T_{\eta \omega} R} R^{t_{\eta}} \xrightarrow{T_{\eta \omega} R} \]

\text{Hypothesis of (2) for $R = T_{\eta \omega} R.$}

\& conclusion of (2) for $T_{\eta \omega} R \Rightarrow$ for $R.$

\textbf{STEP 2}: Reduction to $K(\text{Perf}(R)^{n_{\eta \omega} n_{\eta \omega}}).$

- Have functor $\text{Perf}(R)^{n_{\eta \omega} n_{\eta \omega}} \to \text{Perf}(R)^{\eta \omega} \to \text{Perf}(\eta \omega) \to \eta \omega$

\text{underlying objects}

- $\text{Perf}(R) \Rightarrow \text{Perf}(R) \Rightarrow 0$

$\text{(C)}^{\eta \omega}$

- $\text{Perf}(\text{Perf}(R)) \Rightarrow \text{Fun}(\eta \omega, \text{Perf}(R)) \Rightarrow \text{Fun}(\eta \omega, \text{Perf}(R)) \Rightarrow \text{Perf}(R)$

\text{R}_{\text{module}} \Rightarrow \text{R}_{\text{module}} \Rightarrow \text{R}_{\text{module}}$

\textbf{Take K-mean}.
get cofiber sequence of Cp-spectra

\[ \text{cofiber sequence of } \text{Cp-spectra} \]

\[ K(\text{Perf}(\mathbb{R})^{\text{cobar}}) \to K(\text{Perf}(\mathbb{R})^{\text{cobar}}) \to K(\text{Perf}(\mathbb{R})^{\text{cobar}}/\text{Perf}(\mathbb{R})^{\text{cobar}}). \]

\[ \text{module over } K(R^{\text{cobar}}). \]

\[ \Rightarrow \text{trivial } \text{LT}_{\text{R}}(\mathbb{R}). \]

\[ \vdash \text{have equivalent } \text{LT}_{\text{R}}(\mathbb{R}). \]

\[ \vdash \text{suffices to show } \text{LT}_{\text{R}}(\mathbb{R}) \cong K(\text{Perf}(\mathbb{R})^{\text{cobar}}) = 0 \]

**STEP 3:** Reduction to K(discrete rings) & TC(−).

Since R assumed connective, apply DGM to get pullback square of Cp-spectra.

\[ \begin{array}{ccc}
K(\text{Perf}(\mathbb{R})^{\text{cobar}}) & \longrightarrow & TC(\text{Perf}(\mathbb{R})^{\text{cobar}}). \\
\downarrow & & \downarrow \\
K(\text{Perf}(\text{GR}(\mathbb{R})^{\text{cobar}})) & \longrightarrow & TC(\text{Perf}(\text{GR}(\mathbb{R})^{\text{cobar}})).
\end{array} \]

\[ \vdash \text{suffices to show } \Phi_{\text{cobar}} \text{ of } LL, \text{UR} = 0. \]

(We will assume R connective & \text{LT}_{\text{R}}(\mathbb{R}) = 0.

Recall: isotropy separation w fiber sequence.

\[ \begin{array}{ccc}
K(\text{Perf}(\mathbb{A})^{\text{cobar}})_{h \mathbb{P}} & \longrightarrow & K(\text{Perf}(\mathbb{A})^{\text{cobar}})_{h \mathbb{P}} \to \Phi_{\text{cobar}} \text{KiPerf}(\mathbb{A})^{\text{cobar}}. \\
\text{by def} \downarrow & & \text{by def} \\
K(\text{Perf}(\mathbb{A})^{\text{cobar}}, \mathbb{P}) & \to & \Phi_{\text{cobar}} \text{KiPerf}(\mathbb{A})^{\text{cobar}}.
\end{array} \]

**STEP 4:** Next section.
8.6 Assembly maps

HENJ

1.  

2. 

3. 

4. 

5. 

6. 

7. 

8. 

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Note: The images and components are from real-life models but are simplified for educational purposes. The figures and objects are not to scale. Small parts are not shown to scale. The images depict assembly processes, but the actual assembly process may vary. May contain rude.
Prop. Let $R$ a discrete ring. Then the assembly map

$$K(R) \otimes B(p) \rightarrow K(R[p])$$

is a $T(n)$-local equivalence for all $n \geq 1$.

**Note:** Nilpotence theorem + $e$ (LMMT) $\Rightarrow$ for a ring spectrum $A$, $L_{T(n)} A = 0 \iff L_{K(n)} A = 0$.

Since $KU_p$ is height 1 Moore E-theory & $\langle E \rangle = \langle E_6 \otimes K(n) \rangle$:

$$KU_p \otimes R = 0 \Rightarrow L_{K(n)} R = 0.$$  

**Pf:** $\frac{fib}{fib} \left( K(R) \rightarrow K(R[\frac{1}{p}]) \right) \rightarrow K(\{p\text{-torsion}\}) \otimes KU_p = HZ_p \quad \text{(by duality)}$

\[ \text{Th(surgery)} = 0. \]

$\therefore$ is an $\otimes$ after $p$-completion.

WLOG assume $\sqrt{p} \in R$.

$$K(R) \rightarrow K(R[\frac{1}{p}]) \quad p \text{ roots of unity}.$$  

Since $\otimes$ is multiplication by $p^{-1}$, can WLOG assume that $R$ contains $p^m$ roots of unity.

Want:  

$$(KU \otimes K(R))_{KU} \rightarrow KU \otimes K(R[p])$$

is an equivalence after $p$-completion.

Since assembly map comes from a Mackey functor,

$$KU \otimes (K(R)_{KU} \rightarrow K(R[p]) \rightarrow K(R)_{KU})$$

admits a section

retraction as $(KU \otimes K(R))_{p^\infty}$-modules.

Done if we can show both sides of $(\star)$ are free $(KU \otimes K(R))_{p^\infty}$-modules of rank $p$.  


> $(\text{KU} \otimes \text{K}(R)_{\text{ac}p})_{\ell}^\wedge$ : last talk
> $(\text{KU} \otimes \text{K}(\mathbb{Cp})_{\ell}^\wedge$ : $R(Cp) = R_{\ell}^{p}$ since we
> assumed $R$ contained
> $p$th roots of unity.

**Def.** The topological Hochschild homology (THH) of an $E_1$-ring $R$

$$
\text{THH}(R) = S' \otimes R \quad \text{(in the cat of $E_1$-rings)}

= \colim_{\Lambda^+} \left( R \Rightarrow R \otimes R \Rightarrow R \otimes R \rightarrow \cdots \right)

\cong_R C_2 \cong_R C_3
$$

A cyclotomic spectrum is a spectrum $\nu$ such that $\nu \otimes S'$ is a

$\infty$-category of $S'$-equivariant maps $\nu \rightarrow \nu_{\text{cycl}}$, all primes $p$.

**Facts.** THH($R$) has cyclotomic structure — "Frobenius".

> If $Y$ is a connective cyclotomic $E_1$-ring, then's a canonical map:

$$
\text{THH}(Y) = \text{THH}(Y_{\text{cycl}}) \xrightarrow{\text{cycl}} Y_{\text{cycl}}
$$

which is an equivalence on profinite completion.

**Def.** Topological cyclotomic homology of a cyclotomic spectrum $Y$

$$
\text{TC}(Y) = \text{Eq} \left( Y_{\text{cycl}} \xrightarrow{\text{cycl}} Y_{\text{cycl}} \right)
$$

**Prop.** (Hesselholt-Nikolaus). Let $R$ a connective $E_1$-spectrum.

The $p$-completion of the counit of the assembly map

$$
\text{TC}(R) \otimes \text{BC}(p) \rightarrow \text{TC}(\mathbb{Cp})
$$

can be identified with $\Sigma \text{THH}(R; \mathbb{Cp}) \otimes \text{BC}(p)$.

In particular if $\text{THH}(R) = 0$ then $\Sigma \text{THH}(R; \mathbb{Cp}) \otimes \text{BC}(p) = 0$. 

Sketch: Fact: \( \text{THH}(\mathcal{S}[G]) = \mathcal{S} \otimes \text{LBG}_+ \)

Have an \( \mathcal{S} \)-equivariant decomposition.

\[ \text{LBG}_+ = \coprod_{x \in \text{centre}(G)} \text{BC}_G(x) \]

When \( \mathcal{S} \) acts on RHs via:

\[ B \left( \mathbb{Z} \times \mathcal{C}_G(x) \longrightarrow \mathcal{C}_G(x) \right), \]

\[ (n, y) \longmapsto (n^* y). \]

Get \( \mathcal{S} \)-equivariant cofiber sequence of spaces:

\[ \text{BC}_p \longrightarrow \text{LBG}_+ \longrightarrow \text{BC}_p^{\mathcal{S}^+} \otimes \mathcal{C}_p. \]

Get cofiber sequence of spectra w/ \( \mathcal{S} \)-action

\[ \text{THH}(\mathbb{R}) \otimes \text{BC}_p^{\mathcal{S}^+} \longrightarrow \text{THH}(\mathbb{R} \mathcal{C}_p) \longrightarrow \text{THH}(\mathbb{R}) \otimes \text{BC}_p^{\mathcal{S}^+} \otimes \mathcal{C}_p. \]

Lema: Let \( Y \) spectrum w/ \( \mathcal{S} \)-action which is bounded below.

Let \( \mathcal{S} \) act on \( Y \otimes \text{BC}_p^{\mathcal{S}^+} \) w/ the residual \( \mathcal{S} \)-action.

Then \( (Y \otimes \text{BC}_p^{\mathcal{S}^+})^{t\mathcal{C}_p} = 0 \).

Why? Reduce to case \( Y \) concentrated in a single degree.

\[ \Rightarrow Y \otimes \text{BC}_p^{\mathcal{S}^+} = Y_{t\mathcal{C}_p}. \]

Take orbit lemma \( (Y_{t\mathcal{C}_p})^{t\mathcal{C}_p} = 0 \).

Note: Consider \( \text{THH}(\mathbb{R}; \mathbb{Z}_p) \). Then.

\[ (\text{THH}(\mathbb{R}) \otimes \text{BC}_p^{\mathcal{S}^+})^{t\mathcal{S}^+} \cong \prod_{i} (\text{THH}(\mathbb{R}) \otimes \text{BC}_p^{\mathcal{S}^+})^{t\mathcal{C}_p} \]

after performing completion. 12

0.
\[ T_{\text{Z}}(\sigma) \cong (\text{THH}(R) \otimes BC_{\mathbb{P}^1}^{\mathbb{Q}})_{\text{hs}} \otimes \mathbb{C} \]

\[ \cong (\text{THH}(R) \otimes BC_{\mathbb{P}^1}^{\mathbb{Q}})_{\text{hs}} \otimes \mathbb{C} \]

\[ \cong \sum \text{THH}(R)_{\text{hs}} \otimes \mathbb{C} \]