Disclaimers

- implicit prime p in chromatic localization is same as in "p-group".
- specificity / generality, esp. genuine eqvt theory.

Main reference: "Descent & vanishing in chromatic alg. K-theory" by Clausen-Mathew-Naumann-Noel.

§0. Introduction

<u>Goal</u>: put everything together from last few talks $(+\varepsilon)$ to prove:

<u>Thm</u> A (Clausen-Mathew-Naumann-Noel). "vanishing"

Let R $\mathbb{E}_\infty$-ring & $n \geq 0$. If $L_{T(n)} A \simeq 0$, then

$$L_{T(n+1)} K(A) \simeq 0.$$

<u>Thm</u> B (CMNN) "descent".

Let R $\mathbb{E}_\infty$-ring & $n \geq 0$.

Suppose $L_{T(n)} (R^{tC_p}) \simeq 0$ trivial action. Then:

- for any R-linear stable ∞-category $\mathscr{C}$ w/ action of a finite p-group G, the following map

$$L_{T(n+1)} K(\mathscr{C}^{hG}) \longrightarrow \left(L_{T(n+1)} K(\mathscr{C}) \right)^{hG}$$

<u>Rmk</u> generalizes Thomason's Galois descent for $T(1)$ local K-theory.

<u>Rmk</u> ε true for non-p-groups.

Thm B does not hold in general for arbitrary finite G.

§1. Previously on....

<u>Notation</u> $L_n^{p.f} =$ localization w.r.t. $T(0) \oplus \cdots \oplus T(n)$.

<u>Thm</u> (Land-Mathew-Meier-Tamme). "weak" purity

A $\mathbb{E}_1$-ring spectrum & $n \geq 1$. Then

$$K(A) \longrightarrow K(L_n^{p.f} A)$$

is a $T(m)$ equivalence. for $1 \leq m \leq n$.

blueshift

<u>Thm</u> (CMNN)

Let A be an $\mathbb{E}_\infty$-ring. Then.

$$L_{T(n)} A^{tC_p} = 0 \Rightarrow L_{T(m)} A = 0 \quad m > n.$$

<u>Thm</u> (Kuhn; CM).

Let X be a $T(n)$-local spectrum w/ G-action.

Then $\quad X_{hG} \xrightarrow{\sim} X^{hG} \quad$ i.e. $X^{tG} \simeq *$.

§1. Recollections

Def. G a finite group. The <u>effective Burnside category</u> $\underline{\mathrm{Burn}}_G^{\mathrm{eff}}$:

the (nerve of the) $(2,1)$-category:

- objects: finite left G-sets S
- morphisms: spans of G-equivariant maps
$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ S & & T. \end{tikzcd}$$
  - composition: $\rightsquigarrow$ pullback.
- 2-morphisms: iso of spans.

Def. $\mathcal{C}$ an ∞-cat. The category of $\mathcal{C}$-valued <u>Mackey functors</u>:

$$\mathrm{Mack}_G(\mathcal{C}) := \mathrm{pShv}_\Sigma(\mathrm{Burn}_G^{\mathrm{eff,op}}, \mathcal{C}).$$

↳ functors which take $\sqcup \mapsto \times$.

$$\simeq \underbrace{\mathrm{pShv}_\Sigma(\mathrm{Burn}_G^{\mathrm{eff,op}}; \mathrm{Spc})}_{\text{nonabelian derived } \infty\text{-cat.}} \otimes \mathcal{C}.$$

↳ Lurie $\otimes$ in Pr^L

Rmk.
- Prove something about $\mathrm{Mack}_G(\mathcal{C})$ by reducing to
- CMNN use terminology "semiMackey functors".
- $M(G/H) = M^H$ categorical / "genuine" fixed points.
  $M(G) = M^{\{e\}} \circlearrowleft G$ "underlying object".

$$\begin{tikzcd} M_{hG} \arrow[r] \arrow[d, equal] & M^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \text{Tate}\ \square"] & \Phi^G M \arrow[d] \\ M_{hG} \arrow[r] & M^{hG} \arrow[r] & M^{tG} \end{tikzcd}$$

] isotropy separation sequence.

- pushforward along finite product-preserving functors.

Note $\mathcal{O}(G)$ = cat. of finite left G-sets & G-eqvt. maps.

There are two inclusions.

$$\mathcal{O}(G), \mathcal{O}(G)^{\mathrm{op}} \longrightarrow \mathrm{Burn}_G^{\mathrm{eff}}$$

$$S \to T \longmapsto \begin{cases} S = S \to T \\ T \leftarrow S = S. \end{cases}$$

If $\mathcal{C}$ presentably symmetric monoidal, then $\mathrm{Mack}_G(\mathcal{C})$ has $\otimes$ via Day convolution. using $\times$ on $\underline{\mathrm{Burn}}_G^{\mathrm{eff}}$.

<u>Def</u> A Mackey functor F is

- <u>Borel</u> if $\forall\, G \curvearrowright S$ finite,

$$F(S) \to F(S \times G)^{hG} \quad \text{is an equivalence.}$$

$\iff$ $F|_{\mathcal{O}(G)^{op}}$ is RKE'd from full subcat on G.

$\iff$ $F^H \xrightarrow{\sim} (F^{\{e\}})^{hH}$ equivalence for all $H \leq G$.

- <u>coBorel</u> if $(M \otimes \underset{\text{yoneda embedding}}{y(G)})_{hG} \longrightarrow M$ is an equivalence.

$\iff$ $(F^{\{e\}})_{hH} \longrightarrow F^H$ equivalence for all $H \leq G$.

$\iff$ $F|_{\mathcal{O}(G)^{op}}$ is LKE'd from full subcategory on G.

<u>Prop</u> 2.8 There is a $\otimes$ Bousfield localization.

$$\mathrm{Mack}_G(\mathcal{C}) \longrightarrow \mathrm{Fun}(BG, \mathcal{C}).$$

"Borelification.

w/ fully faithful lax $\otimes$ right adjoint $(-)^{\mathrm{Bor}}$

<u>Fact</u>. coBorelification is given by $(- \otimes y(G))_{hG}$.

Have "norm" natural transformation.

$$(\ )^{\mathrm{coBor}} \longrightarrow (\ )^{\mathrm{Bor}}$$

For our purposes: take $\mathcal{C} = \underline{\mathrm{Sp}}, \underline{\mathrm{Spc}}, \underline{\mathrm{Cat}}_{\infty}^{\mathrm{perf}}$ $(\underline{\mathrm{Pr}}^L)$.

since colims are computed via passage to

A useful construction / G-space.

Def. Let $E\mathcal{I}$ be the C_p-Mackey functor in $\underline{Spc}$ given by

$$(E\mathcal{I})^{\{e\}} = * \qquad E\mathcal{I}^{C_p} = \phi.$$

& write $\widetilde{E\mathcal{I}} := \mathrm{cofib}(S^0 \to E\mathcal{I}_+)$.

Prop. The Borel-completion of a C_p-spectrum X is

$$X^{Bor} \simeq F(E\mathcal{I}_+, X) \qquad \text{internal mapping spectrum.}$$

Note $M, N \in \mathrm{Mack}_{C_p}(Sp)$

$$\begin{tikzcd}
\underline{Burn}^{eff,op}_{C_p} \times \underline{Burn}^{eff,op}_{C_p} \arrow[r] \arrow[d, "\times"'] & Sp \times Sp \arrow[r, "\otimes"] \arrow[dl, phantom, "\Downarrow LKE"] & Sp \\
\underline{Burn}^{eff,op}_{C_p} \arrow[urr, dashed, "M\otimes N"'] & &
\end{tikzcd}$$

$C_p \circlearrowleft$

$\{*\}\times C_p$

slice cat over

$C_p/C_p \simeq \{*\}\times\{*\}$

contribution of $M^{C_p} \otimes N^{hC_p}$.

...

Q Express $(M\otimes N)^{C_p}$ only in terms of M^{C_p}, N^{C_p}?

Idea Force/modify $(\ )^{C_p}$ to be symmetric monoidal by killing $\underbrace{(\ )^{\{e\}} \text{ piece}}_{\text{exactly picked out by } E\mathcal{I}_+}$ first.

i.e. smash w/ $\mathrm{cofib}(E\mathcal{I}_+ \to S^0) \simeq \widetilde{E\mathcal{I}}$

Def. <u>Geometric fixed points</u> functor Φ^{C_p} for C_p:

$$\mathrm{Mack}_{C_p}(Sp) \xrightarrow[\otimes \widetilde{E\mathcal{I}}]{} \mathrm{Mack}_{C_p}(Sp) \xrightarrow{(\ )^{C_p}} Sp.$$

Rmk Analogous formula for arbitrary G.

Prop. Φ^{C_p} is:
- symmetric monoidal
- colimit-preserving.

Then $\exists$ a pullback square of G-spectra.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{E}\mathcal{I} \otimes X \arrow[d] \\ F(E\mathcal{I}_+, X) \arrow[r] & \tilde{E}\mathcal{I} \otimes F(E\mathcal{I}_+, X) \end{tikzcd}$$

which, upon taking $(\ )^{C_p}$, recovers the Tate square

Equivariant algebraic K-theory.

Def. $\mathrm{Fun}(BG, \mathrm{Cat}_\infty^{\mathrm{perf}}) \xrightarrow{(\ )^{\mathrm{Bur}}} \mathrm{Mack}_G(\mathrm{Cat}_\infty^{\mathrm{perf}}) \xrightarrow{K} \mathrm{Mack}_G(\mathrm{Sp})$ $(\simeq \mathrm{Sp}_G)$

& write $K_G(R) := K(\mathrm{Perf}(R)^{\mathrm{Bur}})$.
$\circlearrowleft$ trivial G-action.

Aside: Colimits in $\mathrm{Cat}_\infty^{\mathrm{perf}}$

1. Diagram $F: \mathcal{J} \to \mathrm{Cat}_\infty^{\mathrm{perf}} \xrightarrow{\mathrm{Ind}} \mathrm{Pr}^L$.

   Since $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{op}$ $\quad \operatorname{colim}_{\mathcal{J}} \mathrm{Ind} \circ F \simeq \lim_{\mathcal{J}^{op}} \mathrm{Ind} \circ F$

2. $F \circ \mathrm{Ind}$ classifies a bicartesian fibration $\begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{J}. \end{matrix}$

3. $\lim_{\mathcal{J}^{op}} (\mathrm{Ind} \circ f) \simeq \{\text{cocartesian sections of } \pi\}$

   $\Vert$ when $\mathcal{J}$ an ∞-groupoid. $(*)$

   $\operatorname{colim}_{\mathcal{J}^{op}} (\mathrm{Ind} \circ f) \simeq \{\text{cartesian sections of } \pi\}$

4. Take compact objects $(-)^\omega$.

Ex Let G act trivially on $\mathrm{Perf}(R)$ & consider $\mathrm{Perf}(R)^{\mathrm{coBor}}$

$$\begin{array}{ccl} (\mathrm{Perf}(R)^{\mathrm{coBor}})^{H} & \simeq & \mathrm{Perf}(R)_{hH} = (\mathrm{Mod}(R)_{hH})^{\omega} \qquad * . \\ & \simeq & \mathrm{Perf}(R[H]). \\ \downarrow & & \downarrow \\ (\mathrm{Perf}(R)^{\mathrm{coBor}})^{hH} & \simeq & \mathrm{Fun}(BH, \mathrm{Perf}(R)) \end{array}$$

Nilpotence

Def. A C_p-spectrum $X \in \mathrm{Mack}_{C_p}(Sp)$ is $\underline{\mathcal{I}\text{-nilpotent}}$ if it belongs to the thick subcategory / thick $\otimes$-ideal gen by $\{G_+\}$ i.e. the freely induced spectra

Note $\mathcal{I}$ nilpotent $\Longrightarrow$ Borel but $\underline{\text{not}}$ $\Longleftarrow$.

Rmk "Borel" not preserved by pushforward along arbitrary exact functors,

i.e. $X^G \xrightarrow{?} \lim_{BG} X^{\{e\}}$. $\leftarrow$ infinite limit

but $\mathcal{I}$-nilpotent is. $\leftarrow$ $\underline{\text{finite}}$ condition.

Note R ring M R-module. Then R $\mathcal{I}$-nilpotent

$\Rightarrow$ M $\mathcal{I}$-nilpotent

Aside: "Borel" $\Longleftrightarrow$ complete w.r.t. an alg. object in Sp_G.

$\mathcal{I}$-nilpotent $\Longleftrightarrow$ sseq computing "completion" converges $\underline{\text{quickly}}$.

Thm Let A a homotopy ring C_p-spectrum, i.e. $\mathrm{HoRing}(\mathrm{Mack}_{C_p}(Sp))$

Then A is $\mathcal{I}$ nilpotent $\Longleftrightarrow$ $\Phi^{C_p} A \simeq *$.

§2 Outline.

$$\begin{tikzcd}
 & & \text{Thm A \& B} \\
 & {[\text{LMMT}]} \arrow[dr, Rightarrow, bend left] & \\
 & \text{Thm C} \arrow[r, Rightarrow] & \text{Cor 4.9.} \arrow[u, Rightarrow] \\
\text{nilpotence 3.5 \& 3.6} \arrow[r, Rightarrow] & \text{Lemma 4.1} & \\
 & & \text{Lemma 4.7} \arrow[uu, Rightarrow] \\
 & & \text{Hesselholt-Nikolaus \& Lemma 4.3.} \arrow[u, Rightarrow]
\end{tikzcd}$$

descent & vanishing

descent for $R = L_n^{p.f} \mathbb{S}^0$.
& purity

inductive vanishing.
relates descent & vanishing

Assembly map
computations.

§3. The first reduction.

Thm C (CMNN) Let $\mathcal{C}$ $L_n^{p.f}$-local stable ∞-cat. & G finite p-group.

Then $L_{T(m)} K(\mathcal{C}) \simeq 0$ $\quad m \geq n+2$ & for any action of G on $\mathcal{C}$, have.

$$L_{T(n+1)} K(\mathcal{C}^{hG}) \xrightarrow{\sim} L_{T(n)} K(\mathcal{C})^{hG}.$$

Cor. 4.9 Let A an $\mathbb{E}_1$-ring. Then the map $A \to L_{T(n)\oplus T(n-1)} A$ induces an equivalence.

$$L_{T(n)} K(A) \xrightarrow{\sim} L_{T(n)} K(L_{T(n)\oplus T(n-1)} A).$$

(Thm $C \Rightarrow 4.9$): purity $\rightsquigarrow$ WLOG assume A is $L_n^{p.f}$-local

Have a pullback square of nonconnective K-theory:

$$\begin{array}{ccc} K(A) & \longrightarrow & K(L_{T(n)\oplus T(n-1)} A) \\ \downarrow & & \downarrow \\ K(L_{n-2}^{p.f} A) & \longrightarrow & K(L_{n-2}^{p.f} L_{T(n)\oplus T(n-1)} A) \end{array}$$

b/c both vertical homotopy fibers are given by

$$K(\underbrace{\text{thick subcat}}_{\text{of } \mathrm{Perf}(A)} \text{ on } A \otimes \underbrace{F}_{\text{finite type } n-1 \text{ cplx}}).$$

Thm $C \Rightarrow$ $L_{T(n)}$(bottom row) $\simeq$ $(0 \simeq 0)$ $\square$

§4. The Inductive Lemma; reduction to it

"inductive"

Prop (CMNN 4.7) Let R be an $\mathbb{E}_\infty$-ring & $n \geq 1$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $L_{T(n)} R \simeq 0$ & $L_{T(n)} K(R^{tC_p}) \simeq 0$ for trivial C_p-action.

(2) $L_{T(n)} \Phi^{C_p} K_{C_p}(R) \simeq 0$ $\quad (\ )_{hC_p} \to (\ )^{C_p} \to \cancel{\Phi^{C_p}}$ "coBorel"

(3) $L_{T(m)} K(R) \simeq 0$ $\quad m \geq n+1$.

Q why/how does this imply "descent"?

"nilpotence"

Prop (CMNN 4.1, 3.6).
Let R be an $\mathbb{E}_2$-ring & $m \geq 0$. Then TFAE:

(2) $L_{T(m)} K_{C_p}(R) \otimes F$ is $\mathcal{I}$-nilpotent for any finite type m complex F.

(1). $L_{T(m)} \Phi^{C_p} K_{C_p}(R) \simeq 0$.

(3). $L_{T(m)} K_{C_p}(R) \simeq 0$ is Borel-complete. explain?

Claim ("inductive" & "nilpotence") $\Rightarrow$ Thm C.

Idea Take $R = L_n^{p,f} \mathbb{S}^0$: Suffices to show

1. $L_{T(n+1)} R \simeq 0$. ✓ by defn

2. $L_{T(n+1)} K(R^{tC_p}) \simeq 0$.

By induction on n:

$n=0$: $R = \mathbb{S}[\frac{1}{p}] \Rightarrow \mathbb{S}[\frac{1}{p}]^{tC_p} \simeq 0$.

$n>0$: blueshift $\Rightarrow$ R^{tC_p} is $L_{n-1}^{p,f}$-local

$\therefore$ $K(R^{tC_p})$ is a module over $\underbrace{K(L_{n-1}^{p,f} \mathbb{S}^0)}$

so does this $\Leftarrow$ vanishes $T(n+1)$-locally

Preliminary observation.

If $\mathcal{C}$ presentable stable ∞-cat & $X \in \mathrm{Fun}(BH, \mathcal{C})$, then (finite group)
$X_{hH} \otimes T(m)$ belongs to the thick subcat gen by $X \otimes T(m)$.

Notation: $Y \in \langle Z \rangle$ for Y belongs to the thick subcat gen by Z.

Pf. Choose $T(m)$ to have (htpy) ring structure.   T(n)-local Tate vanishing $\Downarrow$

Consider $T(m)^{hG} \simeq T(m) \otimes EH_+ \simeq \operatorname{colim}_{\ell} (T(m) \otimes sk_\ell EH_+)^{hH}$

S^0 cpct $\rightsquigarrow$ unit for $T(m)^{hH}$ factors through
$T(m) \otimes sk_\ell EH_+$ some ℓ.   induced/free G-action/cells.

$\Rightarrow$ $T(m)$ is a retract of

$\Rightarrow$ $T(m) \in \langle T(m) \otimes H_+ \rangle$.

$\Rightarrow$ $X \otimes T(m) \in \langle X \otimes T(m) \otimes H_+ \rangle$.

$\Rightarrow$ $X_{hH} \otimes T(m) \in \langle (X \otimes T(m) \otimes H_+)_{hH} = X \otimes T(m) \rangle$

As a consequence we have: $E\mathcal{I}_+ \otimes T(m) = (G_+)_{hG} \otimes T(m)$
is $\mathcal{I}$-nilpotent

Pf of lemma: (1) $\Rightarrow$ (2). Write $M = L_{T(m)} K_{C_p}(R)$.
$\therefore$ $M \otimes F \simeq M \otimes T(m)$ has trivial Φ^{C_p}
$\simeq E\mathcal{F}_+ \otimes M \otimes T(m)$ is $\mathcal{I}$-nilpotent by

(2) $\Rightarrow$ (3) Can write $M \simeq \varinjlim_{F \text{ finite type } m} M \otimes F$. (Hovey-Sadofsky).

(2) $\Rightarrow$ (1) $0 \simeq L_{T(m)} \Phi^{C_p}(M \otimes F) \simeq L_{T(m)} \Phi^{C_p}(M) \otimes F$.
$\Rightarrow$ $L_{T(m)} \Phi^{C_p}(M) \simeq 0$.

(3) $\Rightarrow$ (2) Choose a finite type m complex F w/ ring structure.
$\therefore$ $M \otimes F$ has a $T(m)$-module structure.

$E\mathcal{I}_+ \otimes M \otimes F$ is $T(m) \otimes E\mathcal{I}_+$ module

$\Rightarrow$ belongs to thick subcategory gen by $E\mathcal{I}_+ \otimes T(m)$

$\Rightarrow$ is $\mathcal{I}$-nilpotent

Cor. Let $R \in L_{T(m)} Sp^{C_p}$ an algebra object which is Borel complete. Then any R-module $M \in L_{T(m)} Sp^{C_p}$ is also Borel complete.

Pf. Φ^{C_p} is symmetric monoidal.

Cor. (3) $L_{T(m)} K_{C_p}(R)$ is Borel-complete

IMPLIES:

(4) For any R-linear stable ∞-category $\mathcal{C}$ w/ action of a finite p-group G & every additive invariant E which takes values in $T(m)$-local spectra,

$$E(\mathcal{C}^{hG}) \xrightarrow{\sim} E(\mathcal{C})^{hG}.$$

Pf Sketch: $E_G(\mathcal{C})$ is a module over $L_{T(m)} K_{C_p}(R)$.

since E $T(m)$-local!

§5. Pf of the inductive proposition.

Prop. (CMNN). Let R be an $\mathbb{E}_\infty$-ring & $n \geq 1$.

(1) $L_{T(n)} R = 0$ & $L_{T(n)} K(R^{tC_p}) = 0$

(2) $L_{T(n)} \Phi^{C_p}(K_{C_p}(R)) = 0$.

(3) $L_{T(j)} K(R) = 0 \quad \forall j \geq n+1$.

Note: have map of $\mathbb{E}_\infty$-rings $\Phi^{C_p}(K_{C_p}(R)) \to K(R)^{tC_p}$.

$\therefore$ (2) $\Rightarrow$ (3) by blueshift

Pf of (1) $\Rightarrow$ (2) proceeds via:

Step 1: Reduction to case: R connective

- $L_n^{p.f}(H\mathbb{Z}^{tC_p}) = 0$.
- $(\ )^{tC_p}$ preserves exact sequences.
- $(\ )^{tC_p}$ commutes w/ colimits on coconnective spectra.

$$(\tau_{\geq 0} R)^{tC_p} \to R^{tC_p} \to \cancel{(\tau_{<0} R)^{tC_p}}$$

is an $L_n^{p.f}$ equivalence

hypotheses of (1) for $R \Rightarrow \tau_{\geq 0} R$.

& conclusion of (2) for $\tau_{\geq 0} R \Rightarrow$ for R.

Step 2: Reduction to $K(\mathrm{Perf}(R)^{coBor})$.

- have functor $\mathrm{Perf}(R)^{coBor} \hookrightarrow \mathrm{Perf}(R)^{Bor} \to \mathrm{Perf}(-)^{Bor}/(-)^{coBor}$

underlying objects: $\mathrm{Perf}(R) = \mathrm{Perf}(R) \to 0$

$(\ )^{C_p}$: $\underbrace{\mathrm{Perf}(R[C_p])}_{R_{hC_p}\text{-linear}} \hookrightarrow \underbrace{\mathrm{Fun}(BC_p, \mathrm{Perf}(R))}_{R^{hC_p}\text{-linear}} \to \underbrace{\mathrm{Fun}(BC_p, \mathrm{Perf}(R))/\mathrm{Perf}(RH)}_{R^{tC_p}\text{-linear}}$

R^{hC_p}-linear $\Rightarrow$ R^{tC_p}-linear

$\Downarrow$ Take K-theory.

get cofiber sequence of C_p-spectra

$$K(\mathrm{Perf}(R)^{\mathrm{coBor}}) \to K(\mathrm{Perf}(R)^{\mathrm{Bor}}) \to K(\underbrace{\mathrm{Perf}(R)^{\mathrm{Bor}}/\mathrm{Perf}(R)^{\mathrm{coBor}}}_{\text{module over } K(R^{tC_p})}).$$

$\Rightarrow$ trivial $L_{T(n)}\Phi$.

$\therefore$ have equivalent $L_{T(n)}\Phi^{C_p}$.

$\therefore$ suffices to show $L_{T(n)}\Phi^{C_p} K(\mathrm{Perf}(R)^{\mathrm{coBor}}) = 0$

STEP 3: Reduction to K(discrete rings) & TC(−).

Since R assumed connective, apply DGM. to get pullback square of C_p-spectra.

$$\begin{tikzcd}
K(\mathrm{Perf}(R)^{\mathrm{coBor}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TC(\mathrm{Perf}(R)^{\mathrm{coBor}}). \arrow[d] \\
K(\mathrm{Perf}(\pi_0 R)^{\mathrm{coBor}}). \arrow[r] & TC(\mathrm{Perf}(\pi_0 R)^{\mathrm{coBor}}).
\end{tikzcd}$$

$\therefore$ suffices to show Φ^{C_p} of LL, UR $= 0$.

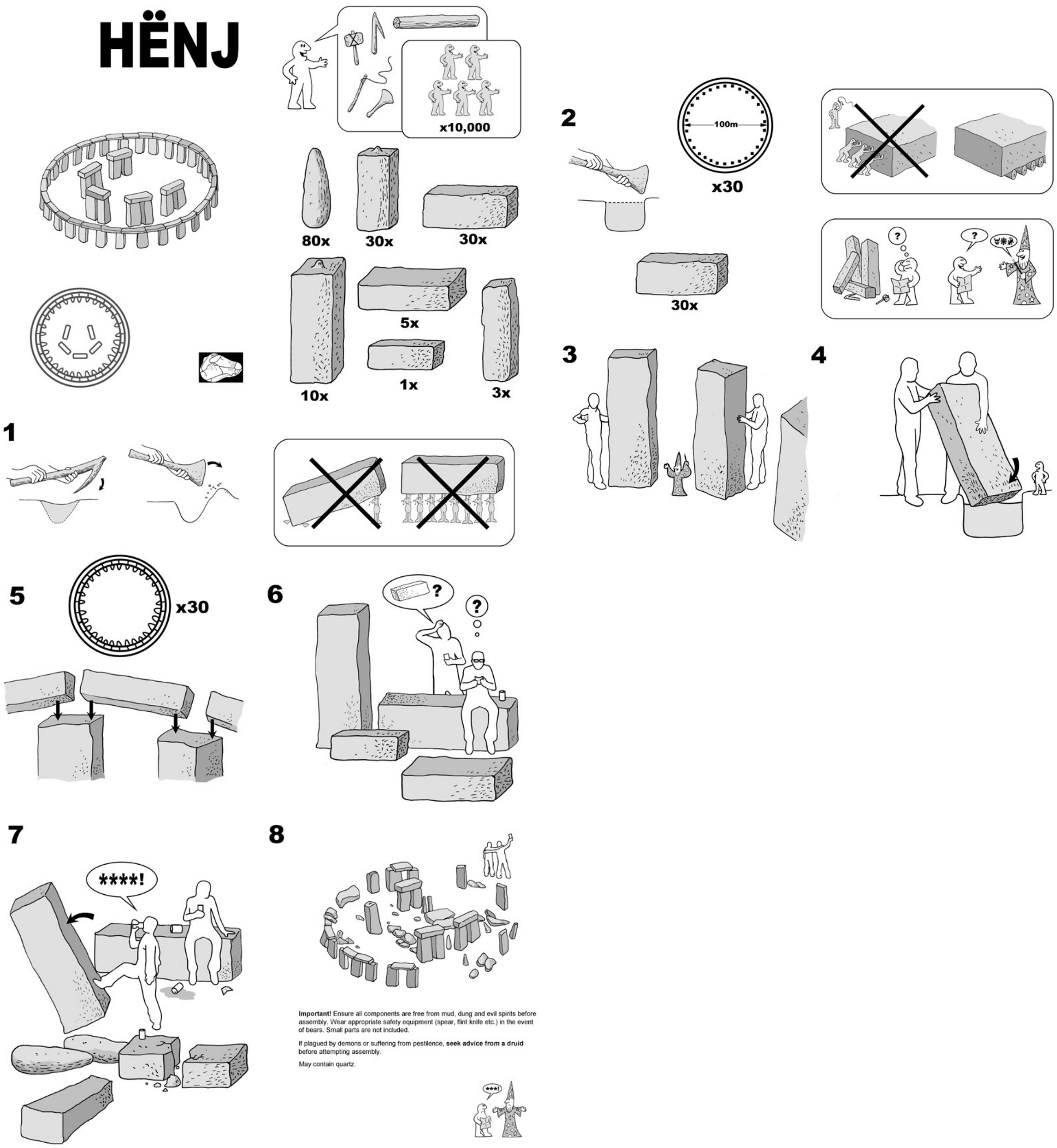
w/ assumption R connective & $L_{T(n)}R = 0$

Recall: Isotropy separation $\rightsquigarrow$ fiber sequence.

$$\begin{tikzcd}
K(\mathrm{Perf}(A)^{\mathrm{coBor}})_{hC_p} \arrow[r] \arrow[d, equal, "\text{only depends on underlying object}"'] & K(\mathrm{Perf}(A)^{\mathrm{coBor}})^{C_p} \arrow[r] \arrow[d, phantom, "\cong\ \text{by defn}"] & \Phi^{C_p} K(\mathrm{Perf}(A)^{\mathrm{coBor}}) \\
 & K(\mathrm{Perf}(A)^{\mathrm{coBor}}, C_p). \arrow[d, phantom, "\cong"] & \\
K(A)_{hC_p} \simeq K(A) \otimes BC_{p+} \arrow[dr, "\text{``assembly map''}"'] & K(\mathrm{Perf}(A)_{hC_p}). \arrow[d, phantom, "\cong"] & \\
 & K(A[C_p]). \arrow[uuur, "\text{cofiber.}"'] &
\end{tikzcd}$$

STEP 4: Next section.

§6. Assembly maps



Prop. Let R a discrete ring. Then the assembly map

$$K(R)\otimes BC_{p+} \longrightarrow K(R[C_p])$$

is a $T(n)$-local equivalence for all $n\geq 1$.

Note: Nilpotence theorem + ε (LMMT) $\Rightarrow$ for a ring spectrum A,

$$L_{T(n)}A=0 \iff L_{K(n)}A=0.$$

Since $KU_p^\wedge$ is height 1 Morava E-theory & $\langle E_1\rangle = \langle E_0\oplus K(1)\rangle$.

$$KU_p^\wedge \otimes R = 0 \Rightarrow L_{K(1)}R=0.$$

Pf:

$$\mathrm{fib}\left(K(R) \longrightarrow K(R[\tfrac{1}{p}])\right)^\wedge_p \overset{\text{by dévissage}}{\overset{!}{\simeq}} K(\{p\text{-torsion}\})^\wedge_p \circlearrowleft \underbrace{K(\mathbb{F}_p)^\wedge_p}_{T(n)\text{-locally } =0} = H\mathbb{Z}_p^\wedge$$

$\therefore$ is an $\simeq$ after p-completion.

$\therefore$ WLOG assume $1/p\in R$.

$$K(R) \rightleftarrows K(R[\zeta_p]) \quad p \text{ roots of unity.}$$

Since $\circlearrowleft$ is multiplication by $p-1$, can WLOG assume that R contains p^n roots of unity.

Want: $(KU\otimes K(R))_{hC_p} \longrightarrow KU\otimes K(R[C_p])$

is an equivalence after p-completion.

Since assembly map comes from a Mackey functor.

$$KU\otimes\left(K(R)_{hC_p} \longrightarrow K(R[C_p]) \longrightarrow K(R)^{hC_p}\right)^\wedge_p$$

section.

admits a section

retraction as $(KU\otimes K(R))^\wedge_p$-modules.

Done if we can show both sides of $(*)$ are free $(KU\otimes K(R))^\wedge_p$-modules of rank p.

- $(KU \otimes K(R)_{hC_p})^\wedge_p$: last talk
- $(KU \otimes K(R[C_p]))^\wedge_p$: $R[C_p] \simeq R^{\times p}$ since we assumed R contained p^{th} roots of unity! ∎

Def. The <u>topological Hochschild homology</u> (THH) of an $\mathbb{E}_\infty$ring R is

$$THH(R) = S^1 \otimes R \quad \text{(tensor in cat of } \mathbb{E}_\infty\text{-rings)}.$$

$$\simeq \operatorname*{colim}_{\Lambda^{op}} \left(R \rightrightarrows \underset{\circlearrowleft C_2}{R \otimes_{\mathbb{S}} R} \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \underset{\circlearrowleft C_3}{R \otimes_{\mathbb{S}} R} \cdots \right).$$

A cyclotomic spectrum is a spectrum w S^1 action $Y : BS^1 \to Sp$ + data of S^1-equivt. maps $\underbrace{\varphi_\ell : Y \to Y^{tC_\ell}}_{\text{"Frobenius"}}$ all primes ℓ.

Facts
- THH(R) has cyclotomic structure
- If Y connective cyclotomic, there's a canonical map:

$$\prod_{\ell \text{ prime}} (Y^{tC_\ell})^{hS^1} \longleftarrow Y^{tS^1}$$

which is an equivalence on profinite completion.

Def <u>Topological cyclic homology</u> of a cyclotomic spectrum Y is:

$$TC(Y) = Eq\left(Y^{hS^1} \underset{\prod \varphi_\ell}{\overset{can}{\rightrightarrows}} Y^{tS^1} \right).$$

Prop (Hesselholt-Nikolaus). Let R a connective $\mathbb{E}_1$-spectrum. The p-completion of the cofiber of the assembly map

$$TC(R) \otimes BC_{p+} \to TC(R[C_p])$$

can be identified with $\underbrace{\Sigma THH(R;\mathbb{Z}_p) \otimes BC_p}_{\simeq 0}$.

In particular if $L_{T(n)} R = 0$

Sketch: Fact: $THH(\mathbb{S}[G]) \simeq \mathbb{S} \otimes LBG_+$ free loop space.

Have an S^1-equivariant decomposition.

$$LBG_+ = \coprod_{x \in \text{conjugacy classes}} BC_G(x)_+$$

centralizer

where S^1 acts on RHS via:

$$B\left(\mathbb{Z} \times C_G(x) \longrightarrow C_G(x). \right)$$
$$(n, y) \longmapsto (x^n \cdot y).$$

Get S^1-eqvt cofiber sequence of spaces:

$$BC_p \longrightarrow LBG_+ \longrightarrow \underbrace{BC_{p+}^{res}}_{\text{residual } S^1 \simeq S^1/C_p\text{-action.}} \otimes C_p.$$

$\rightsquigarrow$ Get cofiber sequence of spectra w/ S^1-action

$$THH(R) \otimes BC_{p+}^{triv} \longrightarrow THH(R[C_p]) \longrightarrow THH(R) \otimes BC_{p+}^{res} \otimes C_p.$$

Lemma: Let Y spectrum w/ S^1 action which is bounded below w/ the residual S^1/C_p-action.

& let S^1 act on $Y \otimes BC_{p+}^{res}$

Then $$(Y \otimes BC_{p+}^{res})^{tC_p} \simeq 0.$$

Why? $\triangleright$ Reduce to case Y concentrated in a single degree.

$$\Rightarrow \quad Y \otimes BC_{p+}^{res} \simeq Y_{hC_p}$$

Tate orbit lemma $\Rightarrow$ $(Y_{hC_p})^{tC_p} \simeq 0.$

Note. Consider. $THH(R; \mathbb{Z}_p)$. Then.

$$\left(THH(R) \otimes BC_{p+}^{res}\right)^{tS^1} \simeq \prod_{\ell} \left(\left(THH(R) \otimes BC_p^{res}\right)^{tC_\ell}\right)^{hS^1}$$

after profinite completion

$$\begin{array}{c}\wr\!\wr \\ 0.\end{array}$$

$$\therefore \quad TC(\quad) \simeq \left(THH(R) \otimes BC_{p+}^{res}\right)^{hS^1} \otimes C_p.$$

$$(\ )^{tS^1} = 0 \qquad \simeq \left(THH(R) \otimes BC_{p+}^{res}\right)_{hS^1} \otimes C_p.$$

$$\simeq \Sigma\, THH(R)_{hS^1} \otimes C_p.$$