Descent for $L_{T(n)}K$ (Part I):

Thm: $L_{T(n+1)}K : \mathrm{Cat}^{\mathrm{Perf}}_{L_n^f \mathbb{S}} \longrightarrow \mathrm{Sp}_{T(n+1)}$ Preserves

G – fixed points for every p-group G

$$\boxed{L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} L_{T(n+1)}K(\mathcal{C})^{hG}}$$

Will be proven next Week.

Question: what Kind of functor is $L_{T(n)}K(\mathcal{C}^{h(-)})$?

or even $K(\mathcal{C}^{h(-)})$?

$H \leq L \leq G$ $\quad \mathrm{Res}^L_H : K(\mathcal{C}^{hL}) \to K(\mathcal{C}^{hH})$ $\quad$ Satisfies "Mackey formulas".

$\qquad\qquad \mathrm{Ind}^L_H : K(\mathcal{C}^{hH}) \longrightarrow K(\mathcal{C}^{hL})$

Def: $A^{\mathrm{eff}}(G) = \mathrm{span}(\mathrm{Fin}_G)$. $\quad \boxed{\begin{array}{l} \mathrm{Ob} = \{A\text{-finite}, G \curvearrowright A\} \\ \mathrm{Map}(A,B) = \{A \leftarrow C \rightarrow B\} \end{array}}$

$$\boxed{\mathrm{Mack}_G(\mathcal{C}) = \mathrm{Fun}^{\times}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathcal{C})},$$

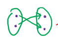
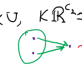
Example: $\mathrm{Mack}_G(\mathcal{S}) \simeq \mathcal{P}_{\Sigma}(A^{\mathrm{eff}}(G)) := \mathrm{CMon}_G$

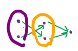
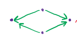
Example: $\mathrm{Mack}_G(\mathrm{Sp}) \simeq \mathrm{Sp}_G := \mathcal{S}_{G*}[(\wedge S^{\rho})^{-1}]$

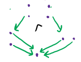
$\mathcal{C} \in \mathrm{Pr}^L \Rightarrow \mathrm{Mack}_G(\mathcal{C}) \simeq \mathrm{CMon}_G \otimes \mathcal{C}$ (Lurie's $\otimes$)

G-spectra: $\mathrm{Sp}_G = \mathrm{Mack}_G(\mathrm{Sp})$. $\quad X \in \mathrm{Sp}_G$, $\quad H \leq G$, $\quad X^H := X(G/H)$ "declared fixed points".

Example: $K\mathbb{R} \in \mathrm{Sp}_{C_2}$, $\quad K\mathbb{R}^{\{e\}} \simeq KU$, $\quad K\mathbb{R}^{C_2} \simeq KO$,

maps:  $\rightsquigarrow$ $\underset{KU}{[V]} \longmapsto \underset{KU}{[\bar{V}]}$,  $\rightsquigarrow$ $\underset{KO}{[V]} \longmapsto \underset{KU}{[V \otimes \mathbb{C}]}$,  $\rightsquigarrow$ $\underset{KU}{[V]} \mapsto \underset{KO}{[V_{\mathbb{R}}]}$

relations:  $\rightsquigarrow$ $\overline{(V \otimes \mathbb{C})} \simeq V \otimes \mathbb{C}$ $\quad$  $\rightsquigarrow$ $V_{\mathbb{R}} = \bar{V}_{\mathbb{R}}$. $\quad$  $\rightsquigarrow$ $(V \otimes \mathbb{C})_{\mathbb{R}} \simeq V \oplus V$

 $\rightsquigarrow$ $V_{\mathbb{R}} \otimes \mathbb{C} \simeq V \oplus \bar{V}$

Example: $K_G : (\mathrm{Cat}^{\mathrm{Perf}})^{BG} \longrightarrow \mathrm{Sp}_G$ $\quad \boxed{K_G(\mathcal{C})^H \simeq K(\mathcal{C}^{hH}).}$

$$L_{T(n)}K_G : (\mathrm{Cat}^{\mathrm{Perf}})^{BG} \xrightarrow{K_G} \mathrm{Sp}_G \xrightarrow{L_{T(n)}} (\mathrm{Sp}_{T(n)})_G$$

Analogy: Sp_G is like $\mathrm{Shv}(X)$, with $\mathrm{Sp}^{BG} \simeq \mathrm{Shv}(U)$,

$U \subseteq X$ open, $\quad \mathrm{Sp} \simeq \mathrm{Shv}(Z)$, $\quad Z \subseteq X$ closed.

functors:

$\mathrm{Mack}_G(\mathcal{C}) \xrightarrow{ev_G} \mathcal{C}^{BG}$	$\mathrm{Shv}(X) \xrightarrow{j^*} \mathrm{Shv}(U)$
$\mathcal{C}^{BG} \xrightarrow{(-)^b} \mathrm{Mack}_G(\mathcal{C})$	$\mathrm{Shv}(U) \xrightarrow{j_*} \mathrm{Shv}(X)$
$\mathcal{C}^{BG} \xrightarrow{(-)_b} \mathrm{Mack}_G(\mathcal{C})$	$\mathrm{Shv}(U) \xrightarrow{j_!} \mathrm{Shv}(X)$
$\mathcal{C} \longrightarrow \mathrm{Mack}_G(\mathcal{C})$	$\mathrm{Shv}(Z) \xrightarrow{i_*} \mathrm{Shv}(X)$
geometric f.P. $\longrightarrow$ $\mathrm{Mack}_G(\mathcal{C}) \xrightarrow{\Phi^G} \mathcal{C}$	$\mathrm{Shv}(X) \xrightarrow{i^*} \mathrm{Shv}(Z)$
$(-)_b \xrightarrow{Nm} (-)^b$	$j_! \longrightarrow j_*$

Example: $R \in \mathrm{Alg}(\mathrm{Sp})$, $\quad (\mathrm{Perf}(R)_b)^H \simeq \mathrm{Perf}(R[H])$

$\qquad (\mathrm{Perf}(R)^b)^H \simeq \mathrm{Fun}(BH, \mathrm{Perf}(R))$

$Nm : \mathrm{Perf}(R[H]) \longrightarrow \mathrm{Fun}(BH, \mathrm{Perf}(R))$, $\quad \boxed{Nm(M)_y = M \otimes_{R[H]} R[P_{x,y}(BH)]}$

(for "large" categories: $Nm : \mathcal{C}_{hH} \xrightarrow{\sim} \mathcal{C}^{hH}$)

The case of C_p: In this case, Sp_{C_p} is glued from Sp and Sp^{BC_p}.

"$X = U \sqcup Z$".

$(\ )^{tC_p} : \mathcal{C}^{BC_p} \to \mathcal{C}$	$i^* j_* : \mathrm{Shv}(U) \to \mathrm{Shv}(Z)$
"Isotropy Sep." $\begin{array}{ccc} E^{C_p} & \longrightarrow & E^{hC_p} \\ \downarrow & \ulcorner & \downarrow \\ \Phi^{C_p}(E) & \longrightarrow & E^{tC_p} \end{array}$	$\begin{array}{ccc} \mathcal{F} & \longrightarrow & j_* \mathcal{F}_U \\ \downarrow & \ulcorner & \downarrow \\ i_* \mathcal{F}_Z & \longrightarrow & i_* i^* j_* \mathcal{F}_U \end{array}$
$E_{hC_p} \to E^{C_p} \to \Phi^{C_p}(E)$	$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F}$

T(n)-locally: $\mathrm{Mack}_{C_p}(\mathrm{Sp}_{T(n)}) \simeq \mathrm{Sp}_{C_p} \otimes \mathrm{Sp}_{T(n)}$

"Tate Vanishing" ( ): $\quad \boxed{Nm : E_b \xrightarrow{\sim} E^b}$

$\Downarrow$

$$\boxed{\mathrm{Mack}_{C_p}(\mathrm{Sp}_{T(n)}) \xrightarrow[\sim]{(\Phi^{\{e\}}, \Phi^{C_p})} \mathrm{Sp}_{T(n)}^{BC_p} \times \mathrm{Sp}_{T(n)}}$$

Cor: $L_{T(n)}K_{C_p} \simeq L_{T(n)}K(\mathcal{C})^b \iff \Phi^{C_p}(L_{T(n)}K_{C_p}(\mathcal{C})) = 0$