On the K-theory of Pullbacks

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Last Time

Recall the universal characterisation of algebraic K-theory by Blumberg-Gepner-Tabuada (after Cisinski-Tabuada):

Theorem

Let $\mathscr{C}$ be a small, stable, idempotent-complete ∞-category, e.g. $\mathrm{Perf}(R)$ for an $\mathbb{E}_1$-ring R. Then there are natural equivalences of spectra

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathscr{S}^\omega), \mathcal{U}_{\mathrm{add}}(\mathscr{C})) \simeq K^{\mathrm{cn}}(\mathscr{C})$$
$$\mathrm{Map}(\mathcal{U}_{\mathrm{loc}}(\mathscr{S}^\omega), \mathcal{U}_{\mathrm{loc}}(\mathscr{C})) \simeq K(\mathscr{C})$$

where $\mathscr{S}^\omega$ is the small stable ∞-category of compact spectra.

These arise from the universal finitary additive and localising invariants $\mathcal{U}_{\mathrm{a/l}}\colon \mathbf{Cat}_\infty^{\mathrm{ex}} \to \mathcal{M}_{\mathrm{a/l}}$ valued in the (stable presentable) ∞-categories of non-commutative motives.

Last Time

Recall that a functor $F\colon \mathbf{Cat}_\infty^{\mathrm{ex}} \to \mathscr{D}$ valued in a stable presentable ∞-category $\mathscr{D}$ is called:

- *additive* if it inverts Morita equivalences and sends split exact sequences to cofibre sequences
- *localising* if it inverts Morita equivalences and sends all exact sequences to cofibre sequences
- *finitary* if it preserves filtered colimits

Recall further that 'Morita equivalences' in this context are functors $F\colon \mathscr{A} \to \mathscr{B}$ that become equivalences after idempotent completion; this is how the exact sequences are defined anyway.

The separation of the adjective *finitary* is needed to state in full completeness Land-Tamme's results.

A brief history lesson

Consider the classical K-theory of rings. A *Milnor square* is a pullback square of discrete rings:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \qquad (\square)$$

such that $B \to B'$ is surjective. In this case, we get a long exact sequence on K-theory

$$K_1(A) \to K_1(A') \oplus K_1(B) \to K_1(B') \to K_0(A) \to K_0(A') \oplus K_0(B) \to \cdots$$

extending infinitely into negative K-theory on the right (Milnor, Bass, Murthy). Recall that negative K-theory was defined by Bass in the so-called Fundamental Theorem of Algebraic K-theory: for a regular ring R, $K_n(R[t, t^{-1}]) \cong K_n(R) \oplus K_{n-1}(R)$.

A brief history lesson

It should be noted that both connective and non-connective K-theory commute with products. Quillen's Q-construction commutes with finite products. Further, a result of Nenashev (generalised by Grayson) shows that, for an exact category $\mathcal{E}$, $K_n(\mathcal{E})$ admits a presentation in terms of K_0 of a different exact category; Kasprowski and Winges used their results to show that the (non-connective) algebraic K-theory of exact categories commutes with arbitrary products (of exact categories – not rings).

The point being: the situation for fibre products differs greatly from that of (unfibred) products.

A brief history lesson

But this is all back in the 1960's, before Quillen's definition of higher algebraic K-theory, and long before ∞-categories came into the picture. Therefore, armed with our knowledge, we can conclude that the sequence extends infinitely to the left as well.

Alas, no we cannot. Swan in 1971 proved that there exists *no functor* $F_2\colon \mathbf{Ring} \to \mathbf{Ab}$ that could add on

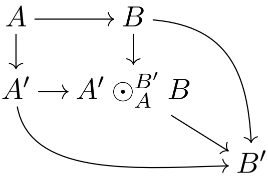
$$F_2(A') \oplus F_2(B) \to F_2(B') \to K_1(A) \to \cdots$$

to the left and remain exact. In particular, he showed that Milnor's K_2 didn't work.

We must generally remedy the failure of extending the long exact sequence by passing to ring spectra.

Theorem (Main Theorem, Land-Tamme)

To any pullback square of $\mathbb{E}_1$-ring spectra ($\square$), there is a natural $\mathbb{E}_1$-ring spectrum $A' \odot_A^{B'} B$ fitting into a diagram



Moreover, any localising invariant sends the inner square above to a pullback square in spectra, i.e. a long exact sequence extending infinitely both ways.

Opening remarks

Every Milnor square is, in particular, a pullback square in $\mathbb{E}_1$-rings. Since we know that non-connective K-theory is a localising invariant, this solves the old question completely – although it would be nice to know when $A' \odot_A^{B'} B \to B'$ is a K_n-equivalence for $n \geq 2$ (as when we start with discrete rings, this map is a K_n-equivalence for all $n \leq 1$).

There have also been a number of other results in the intervening years between 1971 and 2018 that answered part of the question; the other main theorems in Land-Tamme subsume them completely. We'll mention them in due time.

A few more words on the Main Theorem

The object $A' \odot_A^{B'} B$ is actually not that mysterious; its underlying spectrum is the same as $A' \otimes_A B$ ($\otimes$ being the smash product of spectra). Moreover, if all the $\mathbb{E}_1$-rings in the diagram ($\square$) are $\mathbb{E}_k$-rings, then the tensor $A' \otimes_A B$ carries a natural $\mathbb{E}_{k-1}$-ring structure. Therefore if we start with $\mathbb{E}_\infty$-rings or even $\mathbb{E}_2$-rings, we actually know what the ring structure on $A' \odot_A^{B'} B$ is.

Unfortunately, that was a big old lie. The ring structure on $A' \odot_A^{B'} B$ is *not the same* as the natural one. I will work out Land-Tamme's explicit example proving this.

Proving the Main Theorem

Consider a cospan of ∞-categories $\mathscr{A} \xrightarrow{p} \mathscr{C} \xleftarrow{q} \mathscr{B}$. Then we can form the *oriented pullback* or *comma category* which goes by many names:

- Land-Tamme: $\mathscr{A} \overset{\rightarrow}{\times}_{\mathscr{C}} \mathscr{B}$. We will use this below because p, q will never change.
- Category Theorists: Sometimes this is written $(p \downarrow q)$, but I prefer (p/q) which says to me 'p over q' which defines the orientation.

In any case, it's formed as the pullback in simplicial sets

$$\begin{array}{ccc} (p/q) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathscr{C}) \\ \big\downarrow & & \big\downarrow {\scriptstyle 0\times 1} \\ \mathscr{A} \times \mathscr{B} & \xrightarrow[p\times q]{} & \mathscr{C} \times \mathscr{C} \end{array}$$

Since the righthand map is a categorical fibration, (p/q) is a quasicategory and this pullback square is still a pullback in $\mathbf{Cat}_\infty$.

Comma categories

We define the comma category via this pullback square to make sure all the higher morphisms work out properly, but the objects and 1-morphisms are easy enough to define:

- The objects of (p/q) consist of a triple $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $f \colon p(A) \to q(B)$ in $\mathscr{C}$.
- A morphism $(A, B, f) \to (A', B', f')$ is $\alpha \colon A \to A'$, $\beta \colon B \to B'$ such that the following square commutes:

$$\begin{array}{ccc} p(A) & \xrightarrow{p(\alpha)} & p(A') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ q(B) & \xrightarrow[q(\beta)]{} & q(B') \end{array}$$

Comma categories continued

If the original cospan $\mathscr{A} \xrightarrow{p} \mathscr{C} \xleftarrow{q} \mathscr{B}$ was a cospan of exact functors between stable ∞-categories (i.e. in $\mathbf{Cat}_\infty^{\mathrm{ex}}$), then (p/q) is also a stable ∞-category and the projections are exact functors.

We have some natural fully faithful inclusions in the stable (or even pointed) case: $j_1 \colon \mathscr{A} \to (p/q)$ given by $A \mapsto (A, 0, 0)$ and $j_2 \colon \mathscr{B} \to (p/q)$ given by $B \mapsto (0, B, 0)$. In fact, for any $(X, Y, f) \in (p/q)$, there is a canonical cofibre sequence

$$(X, 0, 0) \to (X, Y, f) \to (0, Y, 0)$$

The structure of the comma category

Again in the $\mathbf{Cat}_\infty^{\text{ex}}$ situation, we have the following lemma.

Lemma (1.5, Land-Tamme)

Suppose that $p\colon \mathscr{A} \to \mathscr{C}$ admits a right adjoint r. Then the functor $j_1\colon \mathscr{A} \to (p/q)$ also admits a right adjoint, given by

$$(X, Y, f) \mapsto \operatorname{fib}(X \xrightarrow{\eta_X} rp(X) \xrightarrow{r(f)} rq(Y))$$

where $\eta\colon 1_{\mathscr{A}} \Rightarrow rp$ is the unit of the adjunction.

Land-Tamme offers us the ability to prove this lemma ourselves using the definition of the mapping spaces in the comma category. I was personally confused so I decided this is worth looking at.

Sketch of Lemma 1.5

To see it explicitly, recall $j_1(A) = (A, 0, 0)$. Then we have the defining pullback

$$\begin{array}{ccc} \mathrm{Map}_{(p/q)}(j_1(A), (X, Y, f)) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathscr{C})}(0, f) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathscr{A}}(A, X) \times \mathrm{Map}_{\mathscr{B}}(0, Y) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(p(A), p(X)) \times \mathrm{Map}_{\mathscr{C}}(q(0), q(Y)) \end{array}$$

The categories $\mathrm{Map}_{\mathscr{B}}(0, Y)$ and $\mathrm{Map}_{\mathscr{C}}(q(0), q(Y))$ are contractible, so of no consequence. Looking in the upper-right corner, we see the data consists of

$$\begin{array}{ccc} p(A) & \xrightarrow{0} & q(0) = 0 \\ \downarrow & & \downarrow \\ p(X) & \xrightarrow[f]{} & q(Y) \end{array}$$

Sketch of Lemma 1.5 continued

Starting from $g\colon A \to X$, a map

$$p(A) \xrightarrow{p(g)} p(X) \xrightarrow{f} q(Y)$$

is the same via the (p, r) adjunction as

$$A \xrightarrow{\eta_A} rp(A) \xrightarrow{rp(g)} rp(X) \xrightarrow{r(f)} rq(Y)$$

and by functoriality of the unit we can rearrange this to

$$A \xrightarrow{g} X \xrightarrow{\eta_X} rp(X) \xrightarrow{r(f)} rq(Y)$$

And we retain the property that the composition of the latter two maps is zero, so we get a factorisation through the fibre:

$$A \to \mathrm{fib}(X \xrightarrow{\eta_X} rp(X) \xrightarrow{r(f)} rq(Y))$$

Back to rings

For an $\mathbb{E}_1$-ring A, let $\mathrm{Mod}(A)$ denote the (stable presentable) ∞-category of right A-modules. Let $\mathrm{Perf}(A)$ be what it was last week, so that $\mathrm{Ind}(\mathrm{Perf}(A)) = \mathrm{Mod}(A)$. Since these are presentable, attaining a right adjoint as in the above Lemma will be easier. Starting from a square ($\square$) of $\mathbb{E}_1$-rings, we obtain a square of module categories

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathrm{Mod}(A) & \longrightarrow & \mathrm{Mod}(B) \\ \downarrow & & \downarrow^{q} \\ \mathrm{Mod}(A') & \underset{p}{\longrightarrow} & \mathrm{Mod}(B') \end{array}$$

where all functors here are 'extension of scalars'. We obtain a natural map $i\colon \mathrm{Mod}(A) \to (p/q)$ which admits a right adjoint and satisfies a nice explicit formula.

Explicit formulas

The existence of a right adjoint is simply because i preserves colimits and everything in sight is presentable. The formula follows from the same sort of general diagram chase we performed above. Call it $s\colon (p/q) \to \mathrm{Mod}(A)$. Then

$$s(M,N,f) = M \times_{N\otimes_B B'} N \quad \text{via} \quad \begin{array}{ccc} & & N \\ & & \downarrow \\ M \longrightarrow M \otimes_{A'} B' & \underset{f}{\longrightarrow} & N \otimes_B B' \end{array}$$

There is more: since ($\square$) was a pullback square, we can prove that the unit map $X \to si(X)$ is an equivalence for any $X \in \mathrm{Mod}(A)$:

$$si(X) = s(X\otimes_A A', X\otimes_A B, \cong) = X\otimes_A A' \times_{X\otimes_A B\otimes_B B'} X \otimes_A B$$

This fibre product is taken over an object isomorphic to $X \otimes_A B'$, whence the pullback of the tensors is isomorphic to $X \otimes_A A$. There is a (little) bit more work to do to show that the unit map realises the equivalence $X \simeq X \otimes_A A$.

The interesting map we want

Since the unit of the adjunction is an equivalence, the left adjoint is fully faithful. Thus upon restricting to perfect modules, we obtain a fully faithful functor (so long as we start from a pullback square)

$$i\colon \operatorname{Perf}(A) \to \operatorname{Perf}(A') \overset{\to}{\times}_{\operatorname{Perf}(B')} \operatorname{Perf}(B)$$

where we have reintroduced this $\overset{\to}{\times}$ notation since all functor names have lost their meaning.

We are now ready to start proving the main theorem. The cofibre of i (shortly called $\mathscr{Q}$) is the correct category to take the place of $\operatorname{Perf}(B')$ in the long exact sequence on K-theory, but we were promised an $\mathbb{E}_1$-ring. The main theorem is proven by identifying

$$\mathscr{Q} \simeq \operatorname{Perf}(A' \odot_A^{B'} B)$$

Key Higher-Topical-Theoretic Lemma

How can we tell when an ∞-category is secretly $\mathrm{Perf}(R)$ for some $\mathbb{E}_1$-ring R? The Schwede-Shipley recognition theorem has the following form in ∞-category theory.

The category $\mathrm{Alg}_{\mathbb{E}_0}(\mathbf{Cat}_\infty^{\mathrm{perf}})$ of $\mathbb{E}_0$-algebras in the ∞-category $\mathbf{Cat}_\infty^{\mathrm{perf}}$ is equivalent (by HTT.2.1.3.10) to $\mathscr{C} \in \mathbf{Cat}_\infty^{\mathrm{perf}}$ and an object $C \in \mathscr{C}$.

Lemma (1.10, Land-Tamme & HTT.7.1.2, Lurie)

The association $R \mapsto (\mathrm{Perf}(R), R)$ extends to a fully faithful functor

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathbf{Cat}_\infty^{\mathrm{perf}})$$

whose essential image consists of $(\mathscr{C}, C)$ for which C generates $\mathscr{C}$. A quasi-inverse is given by $(\mathscr{C}, C) \mapsto \mathrm{End}_{\mathscr{C}}(C)$.

Thus: if $\mathscr{Q}$ admits a generator, we get our ring.

Recognising $\mathscr{Q}$

Definition

A category $\mathscr{C} \in \mathbf{Cat}_\infty^{\mathrm{perf}}$ is *generated* by a set of objects S if the smallest stable, idempotent complete full subcategory of $\mathscr{C}$ containing S is all of $\mathscr{C}$. This means taking finite direct sums, retracts of those, then close everything up under suspensions and cones (with perhaps more retracts after that).

Our first observation: take a cospan $\mathscr{A} \xrightarrow{p} \mathscr{C} \xleftarrow{q} \mathscr{B}$ in $\mathbf{Cat}_\infty^{\mathrm{perf}}$. Suppose that S generates $\mathscr{A}$ and T generates $\mathscr{B}$. Then based on the existence of the cofibre sequences

$$(X, 0, 0) \to (X, Y, f) \to (0, Y, 0)$$

we see that $j_1(S) \cup j_2(T)$ generates the comma category (p/q). Thus we know $\{A', B\}$ generates $\mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B)$.

Recognising $\mathscr{Q}$

It turns out that *either* of these objects generates the cofibre $\mathscr{Q}$, but this requires some more work. Consider the (not commutative!) diagram:

$$\begin{array}{ccc} \mathrm{Perf}(A) & \xrightarrow{\quad i_2 \quad} & \mathrm{Perf}(B) \\ {\scriptstyle i_1}\downarrow & \overset{\tau}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle j_2} \\ \mathrm{Perf}(A') & \xrightarrow[\Omega j_1]{} \mathrm{Perf}(A') \overset{\rightarrow}{\times}_{\mathrm{Perf}(B')} & \mathrm{Perf}(B) \end{array}$$

where i_1, i_2 are the extension by scalar functors and j_1, j_2 the inclusions. The natural transformation $\tau\colon \Omega j_1 i_1 \Rightarrow j_2 i_2$ comes from the canonical cofibre sequence: for $X \in \mathrm{Perf}(A)$,

$$(\Omega X \otimes_A A', 0, 0) \overset{\tau_X}{\to} (0, X \otimes_A B, 0) \\ \to (X \otimes_A A', X \otimes_A B, \cong) \to (X \otimes_A A', 0, 0)$$

Recognising $\mathscr{Q}$

Since we aren't going to be using our original cospan notation, we are free to let

$$\mathrm{Perf}(A) \xrightarrow{i} \mathrm{Perf}(A') \overset{\rightarrow}{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B) \xrightarrow{p} \mathscr{Q}$$

denote the cofibre sequence. Since we know that the middle is generated by $\{j_1(A'), j_2(B)\}$, we know that $\mathscr{Q}$ is generated by $\{pj_1(A'), pj_2(B)\}$.

Lemma (1.11, Land-Tamme)

The whiskered natural transformation $p \star \tau \colon p\Omega j_1 i_1 \Rightarrow pj_2 i_2$ is an equivalence. Thus $p\Omega j_1(A') \simeq pj_2(B)$, so $\mathscr{Q}$ is generated by one element.

Look at the cone of τ_X: it's in the image of i, thus it becomes zero in the cofibre $\mathscr{Q}$, and equivalences are detected exactly by having a zero cone. We can therefore define $A' \odot_A^{B'} B := \mathrm{End}_{\mathscr{Q}}(pj_2(B))$.

Recognised, but so what?

One thing that we've not yet proved is that $\mathscr{Q} \simeq \mathrm{Perf}(A' \odot_A^{B'} B)$ is the appropriate replacement for $\mathrm{Perf}(A)$. This relies on the exact sequence

$$\mathrm{Perf}(A) \xrightarrow{i} \mathrm{Perf}(A') \overset{\rightarrow}{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B) \xrightarrow{p} \mathrm{Perf}(A' \odot_A^{B'} B)$$

and the commutative (up to the equivalence $p \star \tau$) diagram

$$\begin{array}{ccc} \mathrm{Perf}(A) & \xrightarrow{i_2} & \mathrm{Perf}(B) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle pj_2} \\ \mathrm{Perf}(A') & \xrightarrow[p\Omega j_1]{} & \mathrm{Perf}(A' \odot_A^{B'} B) \end{array}$$

We need to show that for any localising invariant E, we get a cofibre sequence in spectra (or whatever value category)

$$E(\mathrm{Perf}(A)) \xrightarrow{(i_1, i_2)} E(\mathrm{Perf}(A')) \oplus E(\mathrm{Perf}(B)) \xrightarrow{pj_1 - p\Omega j_2} E(\mathrm{Perf}(A' \odot_A^{B'} B))$$

The finale

We have the (dramatically) split exact sequence

$$\mathrm{Perf}(A') \xrightarrow[j_1]{} \mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B) \longrightarrow \mathrm{Perf}(B)$$

So for any additive invariant E, we have

$$E(\mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B)) \cong E(\mathrm{Perf}(A')) \oplus E(\mathrm{Perf}(B))$$

But we also the non-split exact sequence, which gives us:

$$E(\mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B)) \cong E(\mathrm{Perf}(A)) \oplus E(\mathrm{Perf}(A' \odot_A^{B'} B))$$

for any localising invariant.

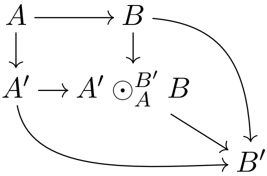
The finale 2

Putting this all together, we can conclude that the sequence

$$E(\mathrm{Perf}(A)) \xrightarrow{(i_1, i_2)} E(\mathrm{Perf}(A')) \oplus E(\mathrm{Perf}(B)) \xrightarrow{pj_1 - p\Omega j_2} E(\mathrm{Perf}(A' \odot_A^{B'} B))$$

is a cofibre sequence once we remember that $E(\Omega) = -1$ for any additive invariant, so replace the righthand map by $pj_1 + pj_2$.

There's one last thing to prove: the existence of



which holds so long as we can find a map $\mathrm{Perf}(A' \odot_A^{B'} B) \to \mathrm{Perf}(B')$ making the $\mathrm{Perf}(-)$ version of this diagram commute. This uses the 'fully faithful' part of the recognition theorem.

The finale 3

How can we get a map from $\mathrm{Perf}(A' \odot_A^{B'} B) \to \mathrm{Perf}(B')$? It suffices to give a map $c\colon \mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B) \to \mathrm{Perf}(B')$ such that precomposition with $i\colon \mathrm{Perf}(A) \to \mathrm{Perf}(A') \overrightarrow{\times}_{\mathrm{Perf}(B')} \mathrm{Perf}(B)$ gives zero. Specifically, that map is

$$c(X, Y, f) = \mathrm{cone}(f\colon X \otimes_{A'} B' \to Y \otimes_B B')$$

Since $i(A)$ has an isomorphism in its 'morphism' spot, $ci = 0$. The induced map $\bar{c}\colon \mathrm{Perf}(A' \odot_A^{B'} B) \to \mathrm{Perf}(B')$ does the job after a couple of naturality checks. □

One ring to rule them all

First, we want to understand the ring $A' \odot_A^{B'} B'$ a little more. We present these results without proof.

The underlying spectrum of $A' \odot_A^{B'} B$ is $A' \otimes_A B$. The ring maps $\Omega p j_1$ and $p j_2$ are the obvious inclusions into the tensor product. The ring map $A' \odot_A^{B'} B \to B'$ is given by the map $A' \otimes_A B \to B'$ induced by the original ring maps $A' \to B'$, $B \to B'$, and the multiplication in B'.

Corollary (1.4, Land-Tamme)

If the multiplication map $A' \otimes_A B \to B'$ is an equivalence (starting from our usual ($\square$) of $\mathbb{E}_1$-rings), then the square

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(B') \end{array}$$

is cartesian for any localising invariant E.

So when is that map an equivalence?

From classical ring theory, we have lots of examples when $A' \otimes_A B' \to B$ is an equivalence.

A map $A \to A'$ is called *Tor-unital* if its kernel I is Tor-unital, i.e. $\mathrm{Tor}_i^{\mathbb{Z} \ltimes I}(\mathbb{Z}, \mathbb{Z}) = 0$ for all $i > 0$. In this case, $A' \otimes_A A' \to A'$ is an equivalence, thus so is $A' \otimes_A B \to B'$ and we obtain excision for the original square. This recovers older results of Suslin.

However, since we are working in ring *spectra*, we can make an explicit connection between the vanishing of these Tor groups and homotopy theory. Specifically, about the *connectedness* of the map $A \to A'$ (more on this soon).

More on the ring structure

In the situation that we don't get excision on our original square, then it falls to us to understand the structure of the new ring $A' \odot_A^{B'} B$. The first thing to notice is that it carries with it a lot of bimodule structure.

Let I denote the fibre of $B \to B'$. Then the B-bimodule structure of $A' \odot_A^{B'} B$ is defined by a cofibre sequence

$$I \otimes_A B \to B \to A' \odot_A^{B'} B$$

Let J denote the fibre of $A' \to B'$. Then the A'-bimodule structure of $A' \odot_A^{B'} B$ is defined by a cofibre sequence

$$A' \otimes_A J \to A' \to A' \odot_A^{B'} B$$

An example

Armed with this, let's see how the constructed ring is *not* the same as the tensor product in the case that we begin with $\mathbb{E}_k$-rings for $k \geq 2$.

Let k be a discrete unital ring, and let $\alpha \in k$. Then we have the following pullback square of $\mathbb{E}_1$-rings:

$$\begin{array}{ccc} k & \longrightarrow & k[y] \\ \downarrow & & \downarrow \\ k[x] & \longrightarrow & k[x,y]/(yx-\alpha) \end{array}$$

where all maps are the obvious/canonical ones. From the main theorem, we get an $\mathbb{E}_1$-ring $k[x] \odot_k^{k[x,y]/(yx-\alpha)} k[y]$ whose underlying spectrum is $k[x] \otimes_k k[y]$. In the case that k is a commutative ring, then $k[x] \otimes_k k[y] = k[x,y]$. But this is not the appropriate ring structure!

So what is the ring structure?

Proposition (4.1, Land-Tamme)

The $\mathbb{E}_1$-ring $k[x] \odot_k^{k[x,y]/(yx-\alpha)} k[y]$ is discrete and isomorphic to

$$k\langle x, y\rangle/(yx - \alpha)$$

the quotient of the *non-commutative* polynomial algebra.

First, the ring maps $k[x] \to k[x] \otimes_k k[y]$ and $k[y] \to k[x] \otimes_k k[y]$ endow $k[x] \odot_k^{k[x,y]/(yx-\alpha)} k[y]$ with the 'expected' left $k[x]$- and right $k[y]$-module structure. We just have to figure out the right $k[x]$- and left $k[y]$-module structure. In fact, determining one will figure out the other, so we focus on the left $k[y]$-module structure, and specifically how to left multiply $x \otimes 1$ by $1 \otimes y$.

The interesting $k[y]$-module structure

Recall that this comes from a cofibre sequence

$$I \otimes_k k[y] \to k[y] \to k[x] \otimes_k k[y]$$

where I is the fibre of $k[y] \to k[x,y]/(yx-\alpha)$. We rotate this sequence and use an explicit model of $\Sigma(I \otimes_k k[y])$ and have the following cofibre sequence in left $k[y]$-modules:

$$\begin{bmatrix} 0 \\ \downarrow \\ k[y] \end{bmatrix} \to \begin{bmatrix} 0 \\ \downarrow \\ k[x] \otimes_k k[y] \end{bmatrix} \xrightarrow{j} \begin{bmatrix} k[y] \otimes_k k[y] \\ \downarrow \\ k[x,y]/(yx-\alpha) \otimes_k k[y] \end{bmatrix}$$

where $j \colon k[x] \to k[x,y]/(yx-\alpha)$ is the ring map from our original pullback square. This map is injective and left $k[y]$-linear (after tensoring), so we can compute the left $k[y]$-module structure on the middle module via its image under j.

The interesting $k[y]$-module structure

Because we can move x, y past the tensor in the image of j, we conclude for any $m \geq 1$:

$$j((1 \otimes y) \cdot (x^m \otimes 1)) = (yx^m) \otimes 1 = j(\alpha \cdot x^{m-1} \otimes 1)$$

and this concludes the argument. □

Funnily enough, for any localising invariant E, the E-theory of $k\langle x, y\rangle/(yx - \alpha)$ is independent of α:

$$E(k\langle x, y\rangle/(yx - \alpha)) \cong E(k) \oplus NE(k) \oplus NE(k)$$

where $NE(k) := \mathrm{cof}(E(k) \to E(k[x]))$ is 'nil E-theory'.

An example of the example

Let $\alpha = 1$. Then $k[x, y]/(yx - 1) = k[x, x^{-1}]$ is the ring of Laurent polynomials and $k\langle x, y\rangle/(yx - 1)$ is called the Toeplitz ring T_k. These fit in an extension

$$0 \to M(k) \to T_k \to k[x, x^{-1}] \to 0$$

where $M(k) = \operatorname{colim}_n M_n(k)$ is the ring of finite matrices. The inclusion is the map $e_{ij} \mapsto x^{i-1}(1 - xy)y^{j-1}$. As a filtered colimit of unital rings, $M(k)$ isn't itself unital, but is Tor-unital, whence for any localising invariant E we obtain a cofibre sequence

$$E(M(k)) \to E(T_k) \to E(k[x, x^{-1}])$$

and by Morita theory we have that $\mathrm{Perf}(k) \simeq \mathrm{Perf}(M(k))$ so we can replace this first term by $E(k)$. This let's use do computations!

An actual computation

Consider the element $e = 1 - xy$ in the Toeplitz ring T_k. As the notation suggests, it's an idempotent element (left as an exercise). We can use this element to achieve a swindle of sorts. The cyclic module eT_k is a k-T_k-bimodule, and the map $\mathrm{Perf}(k) \to \mathrm{Perf}(T_k)$ induced by $e_{ij} \mapsto x^{i-1}(1-xy)y^{j-1}$ once we interpret the Morita equivalence is exactly $(-) \otimes_k eT_k$.

Now, as k-T_k-bimodules, we have $T_k \oplus eT_k \cong T_k$, where the forward map is $(m, en) \mapsto xm + en$ and its inverse is $m \mapsto (ym, em)$. Thus

$$(-) \otimes_k eT_k \oplus (-) \otimes_k T_k \cong (-) \otimes_k eT_k$$

as functors $\mathrm{Perf}(k) \to \mathrm{Perf}(T_k)$. Thus for any additive invariant, $(-) \otimes_k T_k$ is nulhomotopic on $E(k) \to E(T_k)$, which proves that $E(k[x, x^{-1}]) \cong \Sigma E(k) \oplus E(T_k)$.

An actual computation 2: haven't we seen this before?

If we use the computation of $E(T_k) = E(k\langle x, y\rangle/(yx-1))$ given above, we obtain the Fundamental Theorem of Algebraic K-theory:

$$E(k[x, x^{-1}]) \cong \Sigma E(k) \oplus E(k) \oplus NE(k) \oplus NE(k)$$

for any localising invariant E.

Intermission

Time for a pause.

(Even if you're reading these slides after the talk – it would be a good time to make a cup of tea.)

Further directions

So we have a great result if $A' \otimes_A B \to B'$ is an equivalence, but what if that map only kind-of an equivalence?

Definition

A map $f \colon X \to Y$ of spectra is called *n-connective* if its fibre F is n-connective. Specifically, $\pi_k(F) \simeq 0$ for all $k \leq n$.

This may not be the same definition for everyone, including the use of connective for *connected*, but I'm keeping Land-Tamme's notation to avoid (my own) confusion. We can use this concept to denote different degrees of excision being satisfied.

n-cartesian squares

Suppose that we have a commutative square of spectra

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

Then this square is cartesian if and only if the canonical map $W \to X \times_Z Y$ is an equivalence. But if this map is only n-connective, we call the square *n-cartesian.* This means we have a long exact sequence that terminates on the left:

$$\pi_n(W) \to \pi_n(X) \oplus \pi_n(Y) \to \pi_n(Z) \to \pi_{n-1}(W) \to \cdots$$

Supposing that you like coefficients, for an abelian group Λ, we say that a map is Λ-n-connective if its fibre tensored with the Moore spectrum $M\Lambda$ is n-connected (and a similar definition for Λ-n-cartesian).

What's the point of n-cartesian squares?

We know that for a Milnor square, we obtain a sequence very much like the one on the previous slide up to $K_1(A)$ but no further; we should conclude that there is a 1-connective (but not generally higher) map floating around somewhere.

Lemma (2.4, Land-Tamme)

Let Λ be a localisation of $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$. Suppose that $A \to B$ is a map of connective $\mathbb{E}_1$-rings which induces an isomorphism $\pi_0(A) \xrightarrow{\sim} \pi_0(B)$ and is Λ-n-connective for some $n \geq 1$. Then the induced map on (non-connective) $K(A) \to K(B)$ is Λ-$(n+1)$-connective.

There is something to prove here: we reduce to proving this for the connective K-theory via the $+$-construction for discrete rings, as an isomorphism of $\mathbb{E}_1$-rings on π_0 gives an isomorphism on all negative K-groups by a result of Blumberg-Gepner-Tabuada.

Connective invariants

We turn the above lemma into a definition, because there are at least three examples.

Definition

A spectrum-valued localising invariant E is said to be Λ-k-*connective* if for every map $A \to B$ of connective $\mathbb{E}_1$-rings which is a π_0-isomorphism and Λ-n-connective for some $n \geq 1$, the induced map $E(A) \to E(B)$ is Λ-$(n+k)$-connective.

We omit Λ in the case $\Lambda = \mathbb{Z}$. So K-theory is Λ-1-connective, THH is Λ-0-connective, and TC is 1-connective. What if something is k-connective for all k? That would mean that it's an invariant that only depends on the underlying discrete ring, like negative K-theory. More on that soon.

The point

Theorem (2.7-8, Land-Tamme)

Assume that ($\square$) is a pullback square of $\mathbb{E}_1$-ring spectra all of which are connective such that $A' \otimes_A B \to B'$ is a π_0-isomorphism (which hereafter we will call ($\square_E$). If $A \otimes_A B \to B'$ is Λ-n-connective for some $n \geq 1$ and if E is a Λ-k-connective localising invariant, then

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(B') \end{array}$$

is Λ-$(n+k-1)$-cartesian.

Since K-theory is 1-connective, we interpret the above as follows: if $A' \otimes_A B \to B'$ is a π_0-isomorphism and Λ-n-connective, then the long exact sequence on K-theory extends as far as $K_n(A;\Lambda)$.

Specific examples

A Milnor square of discrete rings gives us

$$\pi_0(A' \otimes_A B) \cong \mathrm{Tor}_0^A(A/I, B) \cong B/IB \cong B'$$

the requisite π_0-isomorphism. Because B' is discrete, the map $A' \otimes_A B \to B'$ is always 1-connective, we conclude the first result cited in this talk.

Another example: an *analytic isomorphism* along a multiplicatively closed set of elements $S \subset A$ is a map sending S to central elements of B that induces isomorphisms on the kernels and cokernels of multiplication by s for all $s \in S$. In this case, setting $A' = S^{-1}A$, $B' = S^{-1}B$ gives us a pullback of $\mathbb{E}_1$-ring spectra. But now, since $A \to A'$ is flat, $A' \otimes_A B \to B'$ is a equivalence, thus we get a full long exact sequence on K-theory due to Karoubi, Vorst, and Weibel.

Rephrasing slightly

If we're working with a Milnor square of discrete rings, then the map $A' \otimes_A B \to B'$ being n-connective is the same thing as the vanishing of $\mathrm{Tor}_i^A(A', B) = 0$ for $i = 1, \ldots, n-1$. If we want to use Λ coefficients for Λ a localisation of $\mathbb{Z}$, then use $\mathrm{Tor}_i^A(A', B \otimes_{\mathbb{Z}} \Lambda)$ instead.

If we want to use finite $\mathbb{Z}/m\mathbb{Z}$ coefficients, then we have a new condition: m-multiplication on $\mathrm{Tor}_i^A(A', B)$ must be an isomorphism for $i = 1, \ldots, n-2$ and surjective for $i = n-1$.

In either case, this gives us a very concrete description about how long the exact sequence extends to the left.

More flavours of invariants

We begin with a vocabulary-dump.

Definition

A localising invariant E is called *truncating* if it only depends on the underlying discrete ring: for every connective $\mathbb{E}_1$-ring spectrum A, $E(A) \to E(\pi_0(A))$ is an equivalence.

A localising invariant is called *nilinvariant* if for every nilpotent two-sided ideal $I \subset A$ in a discrete unital ring A, $E(A) \to E(A/I)$ is an equivalence.

Finally, to make the formal definition, an invariant is called *excisive* or to satisfy *excision* if it sends all pullback squares ($\square_E$) of connective $\mathbb{E}_1$-rings such that $\pi_0(A' \otimes_A B) \to \pi_0(B')$ is an isomorphism to pullback squares in spectra.

Truncating invariants are great

Theorem (3.3, Land-Tamme)

Any truncating invariant satisfies excision.

Any truncating invariant gives an equivalence $E(A' \odot_A^{B'} B) \to E(B')$ for a square of type $(\square_E)$. Since we know that $A' \odot_B^{B'} B$ is the correct replacement for B' to get a pullback square for any localising invariant (which truncating invariants are), we conclude excision.

Truncating invariants are great 2

Theorem (3.5, Land-Tamme)

Any truncating invariant is nilinvariant.

This has a different proof. By induction, assume $I^2 = 0$. We form two differential graded algebras $C(I, A) = [I \xrightarrow{i} A]$ and $C(I, A/I) = [I \xrightarrow{0} A/I]$. This gives us the requisite square

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ C(I,A) & \longrightarrow & C(I,A/I) \end{array}$$

This square is indeed of type $(\square_E)$, and the map $C(I, A) \to C(I, A/I)$ is a π_0-isomorphism. Thus the bottom horizontal map is an equivalence, which pulls back to an equivalence $E(A) \to E(A/I)$ as well.

So which invariants are truncating?

A list of truncating invariants and credit to previous authors:

Invariant	Proved truncating	Proved excisive & nilinvariant
$\mathrm{K}^{\mathrm{inv}}$	Dundas-Goodwillie-McCarthy	Dundas-Kittang after Geisser-Hesselholt
$\mathrm{HP}(-/\mathbb{Q})$	Goodwillie	Cuntz-Quillen
$\mathrm{K}^{\mathrm{inf}}_{\mathbb{Q}}$	Goodwillie	Cortiñas
KH	'well-known'	Weibel
$L_{K(1)}K$ of $\mathbb{Z}/N\mathbb{Z}$-alg	Land-Tamme, Meier	–

In the last case using Quillen's computation that $L_{K(1)}(\mathbb{Z}/p\mathbb{Z}) \simeq 0$, we can conclude $L_{K(1)}(\mathbb{Z}/p^n\mathbb{Z}) \simeq 0$ as well. The prime p is the one implicit in $K(1)$.

In case I can't remember the abbreviations

$\mathrm{K}^{\mathrm{inv}}$ is the fibre of the cyclotomic trace $\mathrm{K} \to \mathrm{TC}$, which has been studied in more detail lately by Clausen-Mathew-Morrow.

HP is periodic cyclic homology, which is truncating holds once we rationalise.

$\mathrm{K}^{\mathrm{inf}}$ is so-called *infinitesimal* K-theory, which is the fibre of the trace from K-theory to negative cyclic homology. This is truncating once we rationalise.

KH is homotopy K-theory, defined by Weibel for discrete rings and extended to $H\mathbb{Z}$-linear ∞-categories by Tabuada. It is truncating in both cases.

The last main result

The last class of results is more algebro-geometric in nature. The results in this direction are inspired by contemporary work of Kerz-Strunk-Tamme. Let S be a qcqs base scheme, and let $\mathbf{Sch}_S$ denote the category of quasi-separated S-schemes of finite type. An *abstract blowup square* is a pullback

$$\begin{array}{ccc} D & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow p \\ Y & \xrightarrow[i]{} & X \end{array}$$

in $\mathbf{Sch}_S$ where i is a finitely-presented closed immersion, p is finitely-presented and proper, and $p\colon p^{-1}(X \setminus Y) \to X \setminus Y$ is an isomorphism. It is precisely these squares, along with Nisnevich coverings, that generate the cdh topology on $\mathbf{Sch}_S$.

The last main result

Every localising invariant satisfies Nisnevich descent, which was proved by Thomason(-Trobaugh). Thus a localising invariant E satisfies cdh descent if and only if E sends abstract blowup squares to cartesian squares.

The proof is too long to include at this point in the talk, but using it is proved that *truncating* invariants satisfy cdh descent. This is done by passing to *derived (non-abstract) blowup squares*, which any localising invariant sends to a cartesian square of spectra, and using truncating to make the conclusion. In this case, however, we can only consider *finitary* localising invariants, since there is a filtered colimit involved.

Concrete examples therefore include $\mathrm{K}^{\mathrm{inv}}$, KH, $\mathrm{K}^{\mathrm{inf}}_{\mathbb{Q}}$, and $\mathrm{HP}(-/k)$ for k a commutative $\mathbb{Q}$-algebra. Previous results in this direction were proved by Haesemeyer, Cisinski, Clausen-Mathew-Morrow, and others.

One last word

There is a *pro*-story which, as an amateur, I decided not to include in this talk. In the case that we work over a Noetherian scheme S, there are two pro-results which I can cite for interest (assuming I got to this point):

Theorem (A.7-8, Land-Tamme)

TC and THH both satisfy pro-descent for abstract blowup squares of noetherian schemes.

The interested listener is invited to delve into the literature for (lots) more information about pro-cdh-descent.

References & Thanks

The precursor: Georg Tamme. *Excision in algebraic K-theory revisited.* https://arxiv.org/abs/1703.03331.

The paper: Markus Land, Georg Tamme. *On the K-theory of pullbacks.* https://arxiv.org/pdf/1808.05559.pdf.

Tamme gave a nice talk about the main results in 2019 at MSRI; I took the notes for this talk in fact, and they along with a video are available here: https://www.msri.org/workshops/873/schedules/26332.

Any omissions in this talk were inspired by Tamme's omissions. Thank you for your attention!