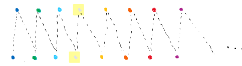
$L_{T(n)}K(R)$:

1) Purity

$R \in \mathrm{Alg}(Sp)$, $n \geq 1$,

Thm: (LMMT + CMNN) $L_{T(n)}K(R) \simeq L_{T(n)}K(L_{T(n) \oplus T(n-1)}R)$.

Today: $L_{T(n)}K(R) \simeq L_{T(n)}K(L_n^f R)$. $\quad L_{T(n)}K(R)$

$k(1)\downarrow$ 

Example: $R \in \mathrm{Alg}(Ab) \Rightarrow L_{T(k)}R = 0 \ \forall k \geq 1 \Rightarrow L_{T(n)}K(R) = 0 \quad n \geq 2$

$\rightsquigarrow$ Mitchel Thm.

A Key special case: $L_n^f(R) = 0 \Rightarrow L_{T(n)}K(R) = 0$.

We start by proving the Key special case when R is also connective.

Prop: If $R \in \mathrm{Alg}(Sp_{\geq 0})$, $L_n^f R = 0$ then $L_{T(n)}K(R) = 0$.

Proof: We first show $L_{T(n)}K(R) \simeq L_{T(n)}K(\pi_0 R)$.

We prove by descending induction: $L_{T(n)}K(R) \xrightarrow{\sim} L_{T(n)}K(\tau_{\leq m}R)$.

- Base case: $m = n$. want: $L_{T(n)}K(R) \xrightarrow{\sim} L_{T(n)}K(\tau_{\leq n}R)$

Enough: $L_{T(n)}\Sigma^\infty \Omega^\infty \tau_{\geq 1}K(R) \xrightarrow{\sim} L_{T(n)}\Sigma^\infty \Omega^\infty \tau_{\geq 1}K(\tau_{\leq n}R)$

($\Sigma^\infty \Omega^\infty$ detects $T(n)$-iso, $L_{T(n)}$ kill coconnectives)

$\Sigma^\infty \Omega^\infty \tau_{\geq 1}K(R) \simeq \Sigma^\infty BGL(R) \Rightarrow$ enough to show:

$L_{T(n)}\Sigma^\infty BGL(R) \xrightarrow{\sim} L_{T(n)}\Sigma^\infty BGL(\tau_{\leq n}R)$

Using what Ishan told us: enough to show that

1) $BGL(R) \to BGL(\tau_{\leq n}R)$ is a v_n-Periodic equiv.

2) $BGL(R) \to BGL(\tau_{\leq n}R)$ is $(n+1)$-connective.

Both follow from $\pi_0 GL(R) \simeq GL(\pi_0 R)$ and $\tau_{\geq 1}GL(R) \simeq \tau_{\geq 1}M(R)$

- Induction Step: use Land-Tamme.

$$\begin{tikzcd} \tau_{\leq m}R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi_0 R \arrow[d] \\ \tau_{\leq m-1}R \arrow[r] & \pi_0 R \oplus \Sigma^{m+1}\pi_m R \end{tikzcd} \overset{L.T.}{\rightsquigarrow} \begin{tikzcd} L_{T(n)}K(\tau_{\leq m}R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{T(n)}K(\pi_0 R) \arrow[d] \\ L_{T(n)}K(\tau_{\leq m-1}R) \arrow[r] & L_{T(n)}K(\tau_{\leq m-1}R \underset{\tau_{\leq m}R}{\odot} \pi_0 R) \end{tikzcd} \rightsquigarrow$$

$\rightsquigarrow$ by the hypothesis, enough to show

$\pi_0 R \xrightarrow{\sim} \tau_{\leq m}(\tau_{\leq m-1}R \underset{\tau_{\leq m}R}{\odot} \pi_0 R)$.

$$\begin{tikzcd} \pi_0 R \arrow[r] \arrow[dr] & \tau_{\leq m}(\tau_{\leq m-1}R \underset{\tau_{\leq m}R}{\odot} \pi_0 R) \arrow[d, "?"] \\ & \tau_{\leq m}(\pi_0 R \oplus \Sigma^{m+1}\pi_m R) \end{tikzcd}$$

Lemma:
$$\begin{tikzcd} A \arrow[r, "n\text{-Con.}"] \arrow[d, "m\text{-Con}"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd} \Longrightarrow A' \underset{A}{\odot} B \to B' \text{ is } (m+n+2)\text{-Connective.}$$

$A' \underset{A}{\otimes} B \nearrow$

$(\mathbb{Z}/p)$

$\mathbb{Z}/p \to R$

$K \quad \to K(R)$

Remains to show: $L_{T(n)}K(R) = 0$ for R p^∞-torsion discrete.

This reduces to the case $R = \mathbb{Z}/p$, and then use Quillen.

Proof of LMMT Theorem.

Need to get rid of 2-assumptions.

1) $L_n^f R = 0$ $\qquad (\ ) \to R \to L_n^f R$

2) R is connective.

$\mathrm{Alg}(Sp) \to \mathrm{Cat}^{\mathrm{Perf}} \xrightarrow{L_{T(n)}K} Sp_{T(n)}$

$\mathcal{C}_{>n} \to Sp^{fin} \to \mathrm{Perf}(L_n^f \mathbb{S}) \quad \in \mathrm{Cat}^{\mathrm{Perf}}$

$\rightsquigarrow \otimes \mathrm{Perf}(R)$

$\mathcal{C}_{>n} \otimes \mathrm{Perf}(R) \to \mathrm{Perf}(R) \to \mathrm{Perf}(L_n^f R)$

$\rightsquigarrow$

$L_{T(n)}K(\mathcal{C}_{>n} \otimes \mathrm{Perf}(R)) \to L_{T(n)}K(R) \to L_{T(n)}K(L_n^f R)$ $\qquad L_n^f(\hom(x,Y)) = 0$

Want to prove: Vanishes. We use additive K-theory.

- $$\begin{tikzcd} \mathrm{Alg}(Sp) \arrow[d, "(*)\ \mathrm{Perf}"'] & \mathrm{Alg}(Sp^{cn}) \arrow[l, hook] \arrow[d, "(*)\ \mathrm{Proj}"] \\ \mathrm{Cat}^{\mathrm{Perf}} \arrow[d, "(*)\ K"'] & \mathrm{Cat}^{\mathrm{add}} \arrow[l, "(\mathcal{P}_\Sigma(-) \otimes Sp)^{\omega}"'] \arrow[d, "(*)\ K^{\mathrm{add}}"] \\ Sp \arrow[r, "\tau_{\geq 0}"] & Sp^{cn} \end{tikzcd}$$

(*) generates under filtered colimits

(*) Preserves filtered colimits.

$S^{\bullet}\mathcal{C}$

- for $\mathcal{C} \in \mathrm{Cat}^{\mathrm{Perf}}$, $\boxed{\Sigma \tau_{\geq 0}K(\mathcal{C}) \simeq |K^{\mathrm{add}}(S^{\bullet}\mathcal{C})|}$

Claim: $\mathcal{C} \in \mathrm{Cat}^{\mathrm{add}}$, $L_n^f \hom(x,Y) = 0 \ \forall x, Y \in \mathcal{C} \Rightarrow L_{T(n)}K^{\mathrm{add}}(\mathcal{C}) = 0$.

Proof: $\mathcal{C} \simeq \underset{i \in I}{\mathrm{colim}}\, \mathrm{Proj}(R_i)$, $R_i \in \mathrm{Alg}(Sp^{cn})$, $L_n^f R_i = 0$.

$L_{T(n)}K^{\mathrm{add}}(\mathrm{Proj}(R_i)) \simeq L_{T(n)}\tau_{\geq 0}K(\mathrm{Perf}(R_i)) \simeq L_{T(n)}K(\mathrm{Perf}(R_i)) = 0$.

Claim: $\mathcal{C} \in \mathrm{Cat}^{\mathrm{Perf}}$, $L_n^f \hom(x,Y) = 0 \ \forall x, Y \in \mathcal{C} \Rightarrow L_{T(n)}K(\mathcal{C}) = 0$.

Proof: $\Sigma L_{T(n)}K(\mathcal{C}) \simeq L_{T(n)}\Sigma \tau_{\geq 0}K(\mathcal{C}) \simeq L_{T(n)}|K^{\mathrm{add}}(S^{\bullet}\mathcal{C})| \simeq |L_{T(n)}K^{\mathrm{add}}(S^{\bullet}\mathcal{C})| = 0$. $\qquad L_{T(n)}K^{\mathrm{add}}(\mathcal{C}^{\Delta^k})$

☺

Main theorem from today's theorem:

Thm (Next week): $L_{T(n-1) \oplus T(n)}R = 0 \Rightarrow L_{T(n)}K(R) = 0$.

Fracture Square

$$\begin{tikzcd} L_n^f R \arrow[r] \arrow[d] & L_{T(n-1) \oplus T(n)}R \arrow[d] \arrow[ddr, bend left] & \\ L_{n-2}^f R \arrow[r] \arrow[drr, bend right] & L_{n-2}^f R \underset{L_n^f R}{\odot} L_{T(n-1) \oplus T(n)}R \arrow[dr] & \\ & & L_{n-2}^f L_{T(n-1) \oplus T(n)}R \end{tikzcd} \rightsquigarrow \begin{tikzcd} L_{T(n)}K(L_n^f R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{T(n)}K(L_{T(n) \oplus T(n-1)}R) \arrow[d] \\ L_{T(n)}K(L_{n-2}^f R) \arrow[r] & L_{T(n)}K(L_{n-2}^f L_{T(n-1) \oplus T(n)}R) \end{tikzcd}$$