Dynamic Leadtime Management in Supply Chains

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Abstract

One way to hedge against random fluctuations of demand in supply chains is to keep inventories at various points in the chain. Another is to actually modify the leadtimes in the system dynamically. Working with multiple suppliers, using multiple transportation options, having the option to expedite certain processes, or having different possible routes for a unit to go through the supply chain are all examples of having flexibility in the supply chain leadtimes. In this paper, we study a serial inventory system with complete leadtime flexibility. For expositional purposes, we use a terminology that corresponds to making shipment (routing) decisions in the supply chain, however, the model can represent many different types of leadtime flexibility, such as the ones described above. In particular, we allow the option of shipping units from any stage in the supply chain to any downstream stage. This option gives the manager an additional flexibility that can be utilized as the uncertain demand is revealed. The shipments incur origin-destination dependent costs. Under a certain supermodularity assumption on the shipment costs, we show that optimal policies can be characterized as extended echelon base stock policies, which is a natural generalization of the well known echelon base stock policies.

1 Introduction

One way to hedge against random fluctuations of demand in supply chains is to keep inventories at various points in the chain. Accordingly, there is a large body of academic work on the subject of multi-echelon inventory management. Most of this literature assumes that leadtimes in the supply chain are exogenously determined. However, another widely used method for responding to demand fluctuations is to actually change the leadtimes in the system dynamically. Working with multiple suppliers, using multiple transportation options, having the option to expedite certain processes, or having different possible routes for a unit to go through the supply chain are all examples of having flexibility in supply chain leadtimes. For example, consider a supply chain where products typically go through a national warehouse, a regional warehouse, and are then sold to end customers by a retailer. If at some point the inventory level at the retailer gets dangerously low due to an unexpected rise in demand, the manager at the retailer may want to place an order at the national warehouse directly, bypassing the regional warehouse. This may be a faster and worthwhile way of replenishing inventory, albeit possibly at a higher cost. Another common situation is one involving alternative transportation options. For example, Hewlett Packard's Network Server Division manufactures a major subassembly of network servers in Singapore and ships it to its distribution centers, where the assembly is completed due to customer specifications. Two modes of transportation with different leadtimes and costs are available (air and ocean) between the factory and the distribution centers (See Beyer & Ward 2001). Both of these examples involve dynamically modifying the *leadtimes* in the supply chain, in response to demand fluctuations. This paper investigates optimal policies for such multi-echelon systems, where inventories and leadtimes are managed simultaneously.

In addition to supply chains, this model may be applicable in a manufacturing setting as well. Suppose that as part of a manufacturing operation, a product needs to go through several processes sequentially, and that there is both work in process inventory as well as finished goods inventory in the system. At certain points of the manufacturing process, there may be the possibility to speed up certain processes, by renting additional machines, hiring temporary employees, reprioritizing, or outsourcing. Such situations can be modeled as one involving dynamic management of leadtimes, and can be handled using the model in this paper.

In order to make the exposition clear, while developing the results, we use a terminology that corresponds to a particular type of leadtime flexibility. In particular, we consider the question of managing a multi-echelon inventory system, where units do not have to go through the supply chain over a predetermined path, and shipping (or routing) decisions are made in a dynamic fashion. However, the results that we obtain at the end are applicable to settings with other types of leadtime flexibility as well, such as systems with multiple suppliers or alternative transportation options. Section 6.2 includes a discussion on such alternative interpretations.

We analyze a multi-echelon inventory system with complete flexibility in the way a unit goes through the stages in the system. Consider the classical serial inventory system of Clark & Scarf (1960). In this system, units go through a series of inventory stages, until they are ready to satisfy customer demand. Every unit goes through all the stages in the same sequence. Shipment from a stage to the next one has a certain cost rate associated with it. In this paper, we present a model where units do not have to go through all the stages of the system, and can be transferred from any stage to any downstream stage, by incurring a certain cost, that depends on both the origin and the destination of the unit. This option gives the controller an additional flexibility that can be utilized as the uncertain demand is revealed. The queueing analogy is that the classical serial inventory system of Clark & Scarf (1960) corresponds to a tandem queueing network, whereas the inventory system studied in this paper corresponds to a feed-forward network of queues.

The paper that is most closely related to our work is Lawson & Porteus (2000). Our model effectively generalizes the one studied by Lawson & Porteus (2000), by allowing a more general cost structure. The term *dynamic leadtime management* was used in that paper to reflect flexibility in the system leadtime, that is, the time it takes for a unit to go through all the stages in a serial system. Their model is closely related to ours, even though it is presented in a somewhat different manner. Similar to our model, they consider a serial system. But in their model, the units have to go through *all* the stages. They allow the transition from a stage to another to be *expedited*, making it essentially instantaneous. Since a unit can be expedited through several stages within the same period, such a transition can also be viewed as transferring the unit from a stage to another by skipping others in between, just like in our model. The main difference is that Lawson & Porteus (2000) use an *additive* cost structure, meaning that the cost of expediting through several stages is equal to the sum of the one-step expediting costs. Our model assumes a more general, "supermodular" cost structure, which includes the special case of additive costs. (See Section 4 for details). In fact, it is general enough to include every cost structure where the cost of a shipment is a convex function of the number of stages that the unit bypasses.

The main contribution of this paper is a characterization of optimal policies for this multi-echelon inventory system. These optimal policies have different interpretations for systems with different types of leadtime flexibility. The analysis is based on the idea of decomposing the overall problem into a series of single-unit single-customer subproblems, similar to Muharremoğlu & Tsitsiklis (2001). The resulting optimal policies are what we call "extended echelon base stock policies". These policies are a natural generalization of echelon base stock policies. Echelon base stock policies are based on one threshold value for each stage, which is compared to the echelon inventory position of the stage. They have been shown to be optimal for the usual serial system, by Clark & Scarf (1960) for finite horizon, and by Federgruen & Zipkin (1984) for infinite horizon models. Chen & Zheng (1994) gave a streamlined alternative proof, which is also valid in continuous time. Note that in these traditional serial systems, a unit can only be sent to the next stage, whereas we allow for a unit to be sent to any downstream stage. Extended echelon base stock policies are also threshold policies, but they involve a threshold for every pair of origin-destination stages (i.e., for every origin stage z and destination stage w with $z > w$). This means that there are $(M+1) \cdot M/2$ threshold values, where M is the number of stages in the system. We call these values *extended echelon base stock levels*. Given the extended echelon base stock levels and the echelon inventory positions in the system at a given time, the optimal ordering levels are easily determined. We derive this result by using the single unit approach, i.e., by showing that the problem can be decomposed into a series of subproblems each of which involves a single unit and a single customer. This approach also gives rise to an algorithm for computing the optimal extended echelon base stock levels, by simply solving a single unit single customer subproblem.

There is a sizeable body of work on problems of optimizing a single stage system with two supply options, one faster than the other. Many of these problems can be seen as special cases of our formulation, as demonstrated in Section 6.2. These different supply options are referred to as dual supply modes, expediting, negotiable leadtimes, or emergency orders in different papers. Barankin (1961) studies a single-period inventory model where two supply options are available, with time lag of one and zero periods, respectively. Daniel (1963) extends this to the multi-period setting. Fukuda (1964) further extends the result to the case when the leadtimes are k and $k+1$ respectively. If the leadtimes differ by more than one period, Whittemore & Saunders (1977) give conditions under which it is optimal to use only one supply mode. Chiang & Gutierrez (1998) study a system where emergency orders can be placed more frequently than the regular orders (e.g., regular orders can be placed once a week, whereas emergency orders can be placed on any day). Zhang (1996) and Gallego et al. (2002) study system with three supply modes. There are numerous other papers on single stage systems with expediting, some involving continuous review, some with yield randomness, some that evaluate the cost of a particular policy, and some that find the optimal parameters for a particular class of policies. These include Moinzadeh & Nahmias (1988), Moinzadeh & Schmidt (1991), Huggins & Olsen (2002), Anupindi & Akella (1993) and Swaminathan & Shanthikumar (1999).

The rest of this paper has six sections. We start by reviewing a background result on decomposable systems in Section 2. In Section 3, the problem formulation is given. Sections 4 and 5 analyze the finite and infinite horizon versions of the problem, respectively. This is followed by Section 6, which discusses some extensions, alternative interpretations of the formulation and explores the model of Lawson & Porteus (2000) through the perspective of this paper. Section 7 contains some brief concluding remarks.

2 Preliminaries

In this section, we state a result on the decoupled nature of optimal policies for *decomposable* systems. We refer the reader to Muharremoğlu & Tsitsiklis (2001) for a detailed description of *decomposable* systems and *decoupled* policies. Here, we simply give a brief description of such systems and policies and provide the necessary result.

Loosely speaking, a decomposable system is a system consisting of multiple (countably infinite) non-interacting subsystems, driven by a common source of uncertainty, that evolves independently of the subsystems and is modulated by an exogenous Markov chain s_t. The state of a decomposable system at time t can be represented as $x_t = (s_t, x_t^1, x_t^2, \ldots)$, where s_t is the state of the exogenous Markov chain, and x_t^i is the i^{th} component of the state. Each state component x_t^i evolves independently of the others, under the influence of a control variable u_t^i, the only coupling arising through the exogenous process s_t. Furthermore, the cost incurred by the overall system at a given time is the sum of the costs incurred by all of the subsystems, where the cost for a particular subsystem can be written as a function of the state of the subsystem (s_t, x_t^i) and the control u_t^i applied to it. Since the dynamics are non-interacting and the costs are additive, it should be clear that each subsystem can be controlled separately, without any loss of optimality.

For any time horizon T, any time $t < T$, and any state x of the overall system, let $J_{t,T}^{\pi}(x)$ be the cost-to-go under policy π (from time t until the end of the horizon T), and let $J_{t,T}^{*}(x)$ be the optimal cost-to-go. Similarly, let $\hat{J}_{t,T}^{*}(s, x^i)$ be the optimal cost-to-go for the i^{th} subsystem, when the state of the subsystem is (s, x^i).

Definition 2.1. A policy π for a decomposable system is said to be *decoupled* if it can be represented in terms of mappings $\hat{\mu}_t(\cdot)$, so that

$$u_t^i = \hat{\mu}_t(s_t, x_t^i), \qquad \forall\ i,\ t,$$

where u_t^i is the control applied to subsystem i at time t under policy π.

Lemma 2.1. *Consider a decomposable system.*

1. *For any $x = (s, x^1, x^2, \ldots)$ and any $t \le T$, we have*

$$J_{t,T}^{*}(x) = \sum_{i=1}^{\infty} \hat{J}_{t,T}^{*}(s, x^i).$$

2. *There exists a decoupled policy π^* which is optimal, that is,*

$$J_{t,T}^{\pi^*}(x) = J_{t,T}^{*}(x), \qquad \forall\ t,\ \forall\ x.$$

3. *For any s, x^i, and any remaining time k, let $\hat{U}^*_k(s, x^i)$ be the set of all decisions that are optimal for a subproblem, if the state of the subproblem at time $T-k$ is (s, x^i). A policy $\pi = \{\mu^i_t\}$ is optimal if and only if for every i, t, and any $x = (s, x^1, x^2, \ldots)$ for which $J^*_{t,T}(x) < \infty$, we have*

$$\mu^i_t(x) \in \hat{U}^*_{T-t}(s, x^i).$$

3 Problem Formulation

We consider a single-product serial inventory system consisting of M stages, indexed by $1, \ldots, M$. Customer demand can only be satisfied by units at stage 1. Any demand that is not immediately satisfied is backlogged. Inventories are replenished by sending units to downstream stages in the supply chain. Units can be sent from any stage z ($z = 2, \ldots, M$) to any downstream stage $w < z$, by incurring a certain cost. Stage M receives replenishments from an outside supplier with unlimited stock. For notational simplicity, we label the outside supplier as stage $M+1$. We assume that inventory is reviewed periodically, so that a discrete time model can be used. The order leadtime for every origin-destination pair is assumed to be equal to one period. (See Section 6.1 for a discussion of other leadtime models).

We assume that demand is independent and identically distributed over time, but note that our approach allows the analysis to be easily extended to the case where demand is Markov-modulated, similar to Muharremoğlu & Tsitsiklis (2001). We assume that demand takes on nonnegative integer values, and that the expected demand per period is finite. Let d_t be the demand at time t.

The system involves inventory holding, ordering, and backorder costs. In particular, we assume:

(a) For each stage z, there is an inventory holding cost rate h_z that gets charged at each time period to each unit at that stage. We assume that the holding cost rate h_{M+1} at the external supplier is zero.

(b) There is a backorder cost rate b which is charged at each time step for each unit of backlogged demand.

(c) For each order from stage z to stage w, there is a cost of $c_{z,w}$ per unit.

We assume that the holding cost and backorder cost parameters are positive, and that the shipping cost is non-negative.

For the rest of the paper, we assume a particular structure on the ordering (or shipping) costs, which we call supermodularity. We will show that when the shipping costs have this property,

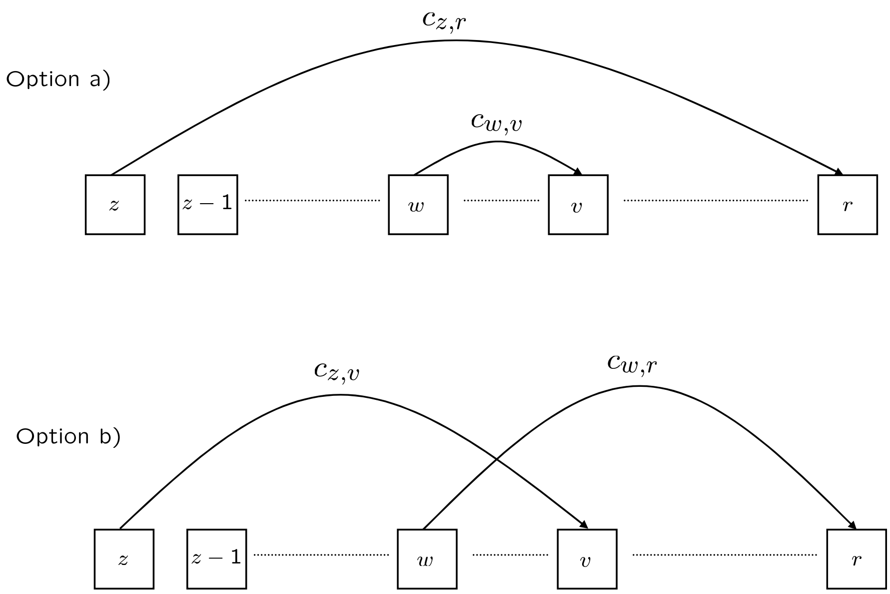

Figure 1: The supermodularity assumption means that if two units are to be shipped from two given stages z and w to two other stages v and r, then the option where the unit originating from the earlier stage ends up at the later destination is at least as costly as the other option. In this figure, this means that option a) is at least as costly as option b).

optimal policies can be characterized as *extended echelon base stock policies*. (When the shipping costs do not have this structure, optimal policies can be much more complicated.)

Assumption 3.1. The shipping costs are *supermodular*, that is, for any stages $z > w \geq v > r$, we have (see Figure 1):

$$c_{z,r} + c_{w,v} \geq c_{z,v} + c_{w,r}$$

Note that in the case of additive costs, as in Lawson & Porteus (2000), the inequality is satisfied as an equality, hence additive costs are supermodular. In cases where the cost of a shipment is a convex function of the number of stages that the unit skips, the supermodularity assumption is also satisfied. However, if a single shipment over a long distance is less expensive than the sum of two small shipments, then the supermodularity assumption is violated when $w = v$.

The detail-oriented reader may have noticed that the model has not been specified in full detail: we would still need to describe the relative timing of observing the demand, fulfilling the demand, placing orders, receiving orders, and charging the costs. Different choices with respect to these details result, in general, to slightly different optimal costs and policies. Whatever specific choices are made, the arguments used for our subsequent results remain unaffected. For specificity, however, we make one assumption about delivery of units to customers. We assume that if a customer arrives during period t, a decision to give a unit to the customer can only be made at time $t+1$, or later.

4 Finite Horizon Analysis

In this section, we study the finite horizon version of our model. In particular, we characterize the structure of optimal policies for the overall problem as *extended echelon base stock policies*, as defined later in this section.

The analysis is based on the idea of decomposing the overall problem into a series of single-unit single-customer subproblems, which was developed in Muharremoğlu & Tsitsiklis (2001). The approach can be outlined as follows: We define an expanded state and control space for the overall problem, where we treat every unit and every customer as distinct objects (including the units and the customers that are not yet in the system). Next, we argue that units and customers can be paired up and *committed* to each other, without any loss of optimality. Given this commitment of particular units to particular customers, the overall system becomes decomposable into subproblems consisting of single unit-customer pairs. Since there exists a decoupled policy for this decomposable problem, the different unit-customer pairs can be controlled independently of each other. Then, the structure of one of these subproblems (all of which are identical) that includes a single unit and a single customer is characterized. Finally, the relationship of the optimal subproblem policies to the overall problem is established, by characterizing the structure of the policies for the overall system that result from applying the subproblem optimal policy to every unit-customer pair.

The resulting policies are not as simple as echelon base stock policies, but are quite manageable, in the sense that they can be summarized by $(M+1)\cdot M/2$ threshold values for each time period). These values can be interpreted as extended base stock levels, that depend on both the stage from which units are to be shipped as well as the stage that is the intended destination. Given these threshold levels and the echelon inventory positions for each stage, the optimal ordering levels at a certain time can be computed via a simple expression, which is given at the end of this section.

4.1 The Expanded State Space

Since we would like to treat each unit and each customer as distinct objects, we start by associating a unique label with each unit and customer. At any given time, there will be a number of units at each stage. Because we assume that each shipment (regardless of the origin and destination) takes a single period, once the shipments from the previous period are delivered, there are no outstanding orders in transit. In addition, conceptually, we have a countably infinite number of units at the outside supplier, which we call stage $M+1$.

Definition 4.1. *The location of a unit*: If a unit is at the outside supplier, we define its location to be $M+1$. If the unit is in the system and has not yet been given to a customer, we define its location to be equal to the label of the stage it is at (e.g., if the unit is at stage z, then the the location of the unit is z). Finally, for any unit that has been given to a customer, we define its location to be 0. Thus, the set of possible locations is $\{0, 1, \ldots, M+1\}$.

We index the countably infinite pool of units by the nonnegative integers. We assume that *the indexing is chosen at time 0 in increasing order of their location*, breaking ties arbitrarily.

Let us now turn to the customer side of the model, which we describe using a countably infinite pool of past and potential future customers, with each such customer treated as a distinct object. At any given time, there is a finite number of customers that have arrived and whose demand is either satisfied or backlogged. In addition, conceptually, there is a countably infinite number of potential customers that may arrive to the system at a future period. Consider the system at time 0. Let k be the number of customers that have arrived, whose demand is already satisfied. We index them as customers $1, \ldots, k$, in an arbitrary order. Let l be the number of customer that have arrived, whose demand is backlogged. We index them as customers $k+1, k+2, \ldots, k+l$ in any order. The remaining (countably infinite) customers are assigned indices $k+l+1, k+l+2, \ldots$, in order of their arrival times to the system, breaking ties arbitrarily, starting with the earliest arrival time. Of course, we do not know the exact arrival times of future customers, but we can conceptually talk about a "next customer," a "second to next customer," etc. This way, we index the past and potential future customers. We now define a quantity that we call "the position of a customer."

Definition 4.2. *The position of a customer*: Suppose that at time t a customer i has arrived and its demand is satisfied. We define the position of such a customer to be -1. Suppose that the customer has arrived but its demand is backlogged. Then, we define the position of the customer to be 0. If on the other hand, customer i has not yet arrived but customers $1, 2, \ldots, m$ have arrived, then the position of customer i at time t is defined to be $i - m$. In particular, a customer whose position at time t is k, will have arrived by the end of the current period if and only if $d_t \geq k$.

Now that we have labeled every unit and every customer, we can treat them as distinguishable objects and, furthermore, we can think of unit i and customer i as forming a pair. This pairing is established at time 0, when indices are assigned, taking into account the initial locations and positions, and is to be maintained throughout the planning horizon.

To facilitate the single-unit analysis, we use the following state space: For each unit-customer pair i, $i \in \mathbb{N}$, we have a vector $\left(z_t^i, y_t^i\right)$, with $z_t^i \in Z = \{0, 1, \ldots, M+1\}$ and $y_t^i \in Y = \mathbb{N}_0$, where z_t^i is the location of unit i at time t, and y_t^i is the position of customer i at time t. The state of the system consists of a countably infinite number of such vectors, one for each unit-customer pair, i.e.,

$$x_t = \left(\left(z_t^1, y_t^1\right), \left(z_t^2, y_t^2\right) \ldots\right).$$

The control vector is an infinite sequence, $u_t = \big(u_t^1, u_t^2, \ldots\big)$, where the i^{th} component u_t^i corresponds to a holding or shipping decision for the i^{th} unit. If unit i is at location 0, it is already delivered to a customer and is unaffected by u_t^i. If unit i is in location $z > 1$, then u_t^i can take on a value in the set $\{1, 2, \ldots, z\}$. The decision $u_t^i = z$ corresponds to holding unit i at location z. The decision $u_t^i = w$ for some w with $1 \leq w < z$ corresponds to shipping unit i from location z to location w. Finally, if unit i is at stage 1, a decision $u_t^i = 0$ releases this unit so that it can be given to a customer. In case the number n of units released from stage 1 is larger than the number m of customers whose demand is backlogged, only m of these units are given to customers (i.e., move to location 0), and the remaining $n - m$ units stay at location 1. Otherwise, all n units are given to customers. The rules about which units are given to customers and which customers receive a unit are the following: If unit i is released to be given to a customer and customer i's demand is backlogged, then unit i is given to customer i. After all such matchings are done, if there are extra units and customers, the units and customers with the lowest indices are chosen until one side is empty.

4.2 Characterization of Optimal Policies

In this subsection, we characterize the structure of optimal policies for the overall system. To do this, we first show that the system decomposes into a series of single-unit single-customer subproblems. We then study the shape of optimal policies for these single-unit subproblems, and then determine the structure of the policies for the overall system that result from applying the optimal subproblem policy to every unit-customer pair.

To show that the system decouples, the first step is to show that we can restrict ourselves to the set of so-called *monotonic* policies and then to the set of *committed* policies. Given the restriction to committed policies, the system becomes equivalent to a decomposable system. This implies that there exists a *decoupled* policy that is optimal. In the interest of space, here we only give a verbal description of these three policy classes and refer the reader to Muharremoğlu & Tsitsiklis (2001) for precise definitions. The setting in that paper is different, but only slight and obvious modifications are necessary to write down the analogous definitions of these policy classes for this paper. Suffice it to say that monotonic policies are those that maintain the monotonic order among the units (i.e., units do not overtake each other under monotonic policies). Committed policies are those that release a unit only if the corresponding customer is there, hence the period when unit i is given to a customer and customer i receives a unit coincide. (One can think of this as "giving unit i to customer i", even though the only control that is exercised is to release the unit from stage 1, and the control does not specify which customer should "receive" this unit). Finally, decoupled policies are those that control each unit-customer pair independently of other units and customers (cf. Definition 2.1). This means that when deciding whether or not to ship unit i, a decoupled policy only considers the location of unit i, and the position of customer i and disregards all other units and customers.

The following results establish that we can indeed restrict policies for the overall system to the corresponding policy classes, without any loss of optimality. More precisely, we will say that a certain set of policies is optimal, if there exists a policy within that set which optimizes the expected cost over the time horizon of interest, for every monotonic initial state, that is, for every initial state in which the location of the i^{th} unit is nondecreasing in i.

Proposition 4.1. *The set of monotonic and committed policies is optimal.*

Proof. We first show that monotonic policies are optimal. Whenever some units are shipped from a particular stage, it does not matter which particular units are shipped, since all units are identical. Hence, one can always choose units so that the lower indexed ones will end up in lower locations. Hence, we can easily prevent overtaking of units that are in the same location at a certain time. There is still the issue that a unit may overtake another unit that is in a lower echelon. However, since the shipping costs are supermodular, any decision that results in such overtaking can be replaced with another that has at most the same cost and prevents overtaking, and the resulting states are identical for optimization purposes. For example, in Figure 1, option b) can be chosen instead of option a). Hence, we can restrict policies to the set of monotonic policies without sacrificing performance.

Now, we turn to the set of monotonic and committed policies. Consider a monotonic initial state. Under a monotonic policy, units are released from stage 1 to arrived customers in the order of the units' indices. Thus, for unit i to be given to a customer, the units $1, \ldots, i-1$ must have been given to other customers, now or in the past. (Otherwise unit i would move to location 0, while a lower-indexed unit would stay at location 1, which would contradict monotonicity.) Therefore, unit i is given to some customer only if customer i has already arrived. Hence the state evolution is the same as under an additional restriction that a unit i can be released from location 1 only when customer i has arrived, which corresponds to a monotonic and committed policy. $\square$

Proposition 4.2. *The set of committed and decoupled policies is optimal and*

$$J_T^*(x) = \sum_{i=1}^{\infty} \hat{J}_T^*\left(z^i, y^i\right)$$

for every monotonic state $x = \left\{\left(z^1, y^1\right), \left(z^2, y^2\right), \ldots\right\}$.

Proof. The original form of the problem, with all possible policies allowed is not decomposable, because of the coupling that arises when units are delivered (from location 1) to customers. Once we restrict to the set of committed policies, which we can do without sacrificing performance by

Proposition 4.1, this coupling is eliminated, and the system becomes equivalent to a decomposable system. Lemma 2.1 implies the optimality of decoupled policies. □

At this point, we have shown that the overall problem can be decomposed into a series of identical single-unit subproblems. Next, we analyze the structure of these subproblems in more detail, in order to characterize the structure of the resulting optimal policies for the overall problem.

Definition 4.3. For any k, z and y let $\hat{U}_k^*(z,y) \subset \{1, 2, \ldots, M+1\}$ be the set of all decisions that are optimal if a subproblem is found at state (z,y) at time $t = T-k$, that is k time steps before the end of the horizon. Also, let $\hat{U}_\infty^*(z,y)$ be defined similarly for an infinite horizon problem.

Lemma 4.1. *There exists an integer $Y_{\max}$ such that any optimal policy for a finite or infinite horizon subproblem will not ship the unit from stage $M+1$ if the position of the customer is larger than $Y_{\max}$. Formally, there exist some $Y_{\max}$ such that for every $y > Y_{\max}$ and every t, $\hat{U}_t^*(M+1,y) = \{M+1\}$ and $\hat{U}_\infty^*(M+1,y) = \{M+1\}$.*

Proof. The proof is parallel to the proof of Lemma 5.4 in Muharremoğlu & Tsitsiklis (2001) and is omitted. □

The above lemma tells us that under any optimal policy, a unit will not be ordered from the outside supplier, unless the position of the corresponding customer is $Y_{\max}$ or less. Assuming that the initial state of the system is such that no unit with a corresponding customer position larger than $Y_{\max}$ is in locations $1, \ldots, M$, this ensures that throughout the time horizon and under any optimal policy, a unit with a corresponding customer position greater than $Y_{\max}$ must be at the outside supplier. Hence, when trying to find an optimal decoupled policy, we only need to focus on states with a y component less than or equal to $Y_{\max}$.

In general, the sets $\hat{U}_k^*(z,y)$ may contain multiple elements, resulting in multiple optimal policies. In the what follows, we define a mapping $M_k(z,y)$ that selects a particular optimal policy with desirable structural properties.

Definition 4.4. For any k, $z > 0$ and $y \in \{0, 1, \ldots, Y_{\max}\}$, we define $M_k(z,y)$ as follows:

(i) Let $M_k(z,y) = \max\Big\{u \mid u \in \hat{U}_k^*(z, Y_{\max})\Big\}$, for $y = Y_{\max}$.

(ii) Starting with $z = M+1$ and $y = Y_{\max}$, compute the values of $M_k(z,y)$ recursively, first for decreasing values of y until $z = M+1$ and $y = 0$, and then by decreasing the value of z by 1 and going through values $y = Y_{\max}, Y_{\max}-1, \ldots, 1, 0$ for $z = M$, etc., using the following formula:

$$M_k(z,y) = \max\Big\{u \mid u \in \hat{U}_k^*(z,y) \text{ and } u \leq M_k(z', y+1) \text{ for any } z' \geq z\Big\} \tag{1}$$

The idea here is to choose a particular decoupled and committed policy, that is also monotonic. The recursion is defined in such a way that out of all possible choices from the $\hat{U}$ sets, we choose the one that moves a unit as little as possible, subject to not violating monotonicity. However, in order for this to work, and for $M_k(z, y)$ to be well-defined, we need the sets on the right-hand side of Eq. (1) to be nonempty. This is what we show next.

Lemma 4.2. *For every (k, z, y), $M_k(z, y)$ is well-defined.*

Proof. Suppose the statement is not true. Then, there exists some (k, z, y), such that the set of u's on the right-hand side of Eq. (1) is empty, and this is the first such (z, y) that is encountered in the recursion described in Definition 4.4. However, in this case, no monotonic policy can be optimal, as we will demonstrate next. To do this, we first identify a special monotonic state of the system. Recall that by the third part of Lemma 2.1, any optimal policy has to consist of actions that belong to the $\hat{U}$ sets. We then show that if the maximum is not attained, then any such policy (hence any optimal policy) will be forced to act in a non-monotonic fashion at the special monotonic state. Hence, no monotonic policy can be optimal, which is a contradiction.

To construct the monotonic state that demonstrates the contradiction, we describe a sequence of points in the (z, y) space. An example is given in Figure 2, and the figure may aid in following the construction. Let $(z_{\hat{y}}, \hat{y})$ be the first point in the recursion where $M_k(z, y)$ is not well-defined. Now, consider the following sequence of points $\{(z_{\hat{y}}, \hat{y}), (z_{\hat{y}+1}, \hat{y}+1), (z_{\hat{y}+2}, \hat{y}+2), \ldots, (z_{Y_{\max}}, Y_{\max})\}$ in the (z, y) space. Given (z_y, y), with $y \geq \hat{y}$, determine $(z_{y+1}, y+1)$ as follows: Look at the values of $M_k(z', y+1)$ for all $z' \geq z_y$ and choose a z' that results in the smallest value. Note that this value has to be greater than or equal to $M_k(z_y, y)$, by equation (1). Hence, in this sequence, the value of $M_k(z_y, y)$ is a non-decreasing function of y, with a number of jumps between which $M_k(z_y, y)$ stays constant. We want to concentrate on these jump points, because they provide us with some concrete information about the $\hat{U}$ sets. Let (z_{y_i}, y_i) be the i^{th} jump point, i.e., the i^{th} point in the sequence for which $M_k(z_{y+1}, y+1) > M_k(z_y, y)$. Let n be the number of jump points. For all the jump points, we have some information about the sets $\hat{U}_k^*(z_{y_i}, y_i)$. In particular, we know that any value in the set $\{M_k(z_{y_i}, y_i)+1, M_k(z_{y_i}, y_i)+2, \ldots, M_k(z_{y_i}+1, y_i+1)\}$ cannot be in the set $\hat{U}_k^*(z_{y_i}, y_i)$. This is clear by equation (1), since otherwise $M_k(z_{y_i}, y_i)$ would be higher. Now, consider a monotonic state, that has the following units with the corresponding customer positions:

(i) A unit at location $z_{\hat{y}}$ with a corresponding customer position of $\hat{y}$. Label this as unit c_0.

(ii) n units with unit location and customer position values $(z_{y_1}, y_1), (z_{y_2}, y_2), \ldots, (z_{y_n}, y_n)$. Label these units as units c_1 through c_n.

(iii) A unit at location $z_{Y_{\max}}$ with a corresponding customer position of $Y_{\max}$. Label this as unit c_{n+1}.

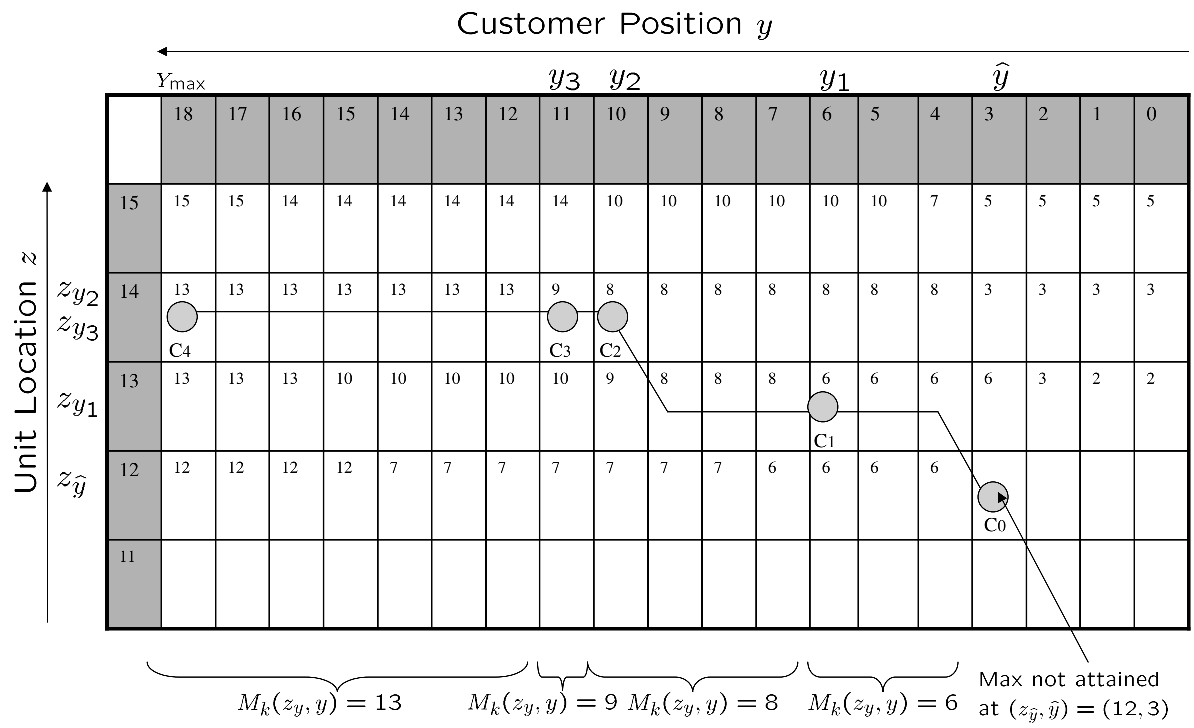

Figure 2: The figure shows an example of how to construct a special monotonic state that results in a contradiction, if the maximum in equation 1 is not attained (See the proof of Lemma 4.2). The table shows the $M_k(z, y)$ values. In the example, $(12, 3)$ is the first point where $M_k(z, y)$ is undefined. Then, a sequence of points is followed in the (z, y) space, as described in the proof. On this path, there are $n = 3$ jump points, at $(13, 6)$ (from 6 to 8), at $(14, 10)$ (from 8 to 9), and at $(14, 11)$ (from 9 to 13). These jump points, along with the beginning and end points of the path, determine the special monotonic state.

Note that these units have a location that increases with the position of the corresponding customers. For this reason, there exists a monotonic state that includes these units. Now, consider an optimal policy that is also monotonic. We examine the actions of such a policy on the units we have specified:

(a) Since $M_k(z_{Y_{\max}}, Y_{\max}) = \max\{u \mid u \in \hat{U}_k^*(z_{Y_{\max}}, Y_{\max})\}$, any optimal policy will ship unit c_{n+1} to location $M_k(z_{Y_{\max}}, Y_{\max})$ or lower.

(b) Because n is the last jump point, we have $M_k(z_{y_n}, y_n) < M_k(z_{y_n+1}, y_n{+}1) = M_k(z_{Y_{\max}}, Y_{\max})$, which implies that no action in the set $\{M_k(z_{y_n}, y_n) + 1, M_k(z_{y_n}, y_n) + 2, \ldots, M_k(z_{Y_{\max}}, Y_{\max})\}$ can be in $\hat{U}_k^*(z_{y_n}, y_n)$. This means that any monotonic and optimal policy will ship unit c_n to location $M_k(z_{y_n}, y_n)$ or lower.

(c) Now, no action in the set $\{M_k(z_{y_{n-1}}, y_{n-1}) + 1, M_k(z_{y_{n-1}}, y_{n-1}) + 2, \ldots, M_k(z_{y_n}, y_n)\}$ can be in $\hat{U}_k^*(z_{y_{n-1}}, y_{n-1})$. This means that any monotonic and optimal policy will ship unit c_{n-1} to location $M_k(z_{y_{n-1}}, y_{n-1})$ or lower.

(d) We can repeat the same argument for all the units c_i, and eventually obtain that any monotonic and optimal policy will ship unit c_1 to location $M_k(z_{y_1}, y_1)$ or lower. Note that $M_k(z_{y_1}, y_1) = M_k(z_{\hat{y}+1}, \hat{y} + 1)$, since this is the first jump point.

(e) Any monotonic policy will have to ship unit c_0 to location $M_k(z_{\hat{y}+1}, \hat{y} + 1)$ or lower, to preserve monotonicity. However, since $(z_{\hat{y}}, \hat{y})$ is a point for which the set in equation (1) is empty, this particular action u cannot be in $\hat{U}_k^*(z_{\hat{y}}, \hat{y})$. This means that no monotonic policy can be optimal, which contradicts Proposition 4.1, and concludes the proof of the lemma.

□

Let us label the set of subproblem states such that $z \le M$ and $y > Y_{\max}$ as *prohibited states*. Note that if the initial state of a subproblem is not prohibited, then the subproblem will never reach a prohibited state under an optimal subproblem policy, by Lemma 4.1. Let $\hat{\mu}_t(z, y) = M_{T-t}(z, y)$ for all t, z, and $y \le Y_{\max}$, and let $\hat{\mu}_t(M + 1, y) = M + 1$ for all t and all $y > Y_{\max}$. This results in functions $\hat{\mu}_t$ that satisfy $\hat{\mu}_t(z, y) \in \hat{U}_{T-t}^*(z, y)$ for all time periods and all subproblem states, except for the prohibited ones. Choosing the decision according to $\hat{\mu}_t$ for each unit at each time step constitutes a decoupled and committed policy. Furthermore, by our construction, $\hat{\mu}_t(z, y) \le \hat{\mu}_t(z', y')$ for any $z \le z'$ and $y \le y'$, hence this policy is also monotonic. This monotonic, committed and decoupled policy can be completely described by $(M + 1) \cdot M/2$ threshold values for each time period t, which we call the extended echelon base stock levels. We next define these values.

Definition 4.5. We define the extended echelon base stock levels $S_t^{z,w}$ as follows: Fix some z, w and t such that $z > w \ge 1$. There are two alternatives:

(i) If $\hat{\mu}_t(z, y) > w$ for all y, let $S_t^{z,w} = -\infty$.

(ii) Otherwise, let $S_t^{z,w} = \max\{y \mid \hat{\mu}_t(z, y) \leq w\}$.

The extended echelon base stock levels we have defined have some monotonicity properties, which reflect the monotonicity of $\hat{\mu}_t$:

Lemma 4.3. *(a) $S_t^{z,w} \geq S_t^{z,w-1}$ for all (z, w, t) such that $z > w > 1$.*

(b) $S_t^{z,w} \leq S_t^{z-1,w} + 1$ for all (z, w, t) such that $z - 1 > w \geq 1$.

Proof. Property (a) is an automatic consequence of the preceding definition. Suppose that property (b) fails to hold. Then, $S_t^{z,w} \geq S_t^{z-1,w} + 2$, which implies that $\hat{\mu}_t(z, S_t^{z-1,w} + 2) \leq w$. On the other hand, we have $\hat{\mu}_t(z - 1, S_t^{z-1,w} + 1) > w$. This contradicts the monotonicity of $\hat{\mu}_t$. (In particular, if the unit associated with a customer in position $S_t^{z-1,w} + 2$ is at stage z, and the unit associated with a customer in position $S_t^{z-1,w} + 1$ is at stage $z - 1$, the first will move to a stage w or lower,and will overtake the second which can only move to a stage higher than w.) □

Given the extended echelon base stock levels $S_t^{z,w}$, the whole function $\hat{\mu}_t$ is determined, since we know that $\hat{\mu}_t$ is non-decreasing step function in y and $S_t^{z,w}$ are exactly the points where the function increases. Between these step points, the function is constant. Hence, this subproblem policy partitions the customer position axis into segments that determine the stage that the unit will be shipped to if the corresponding customer's position falls within that segment. Figure 3 illustrates such a subproblem policy. We next study the structure of the policy that arises from applying this optimal subproblem policy $\hat{\mu}_t$ to every unit-customer pair. It turns out to be a generalization of echelon base stock policies, which we call *extended echelon base stock policies.*

Define I_t^z as the echelon inventory position at stage z at time t, which is the total number of units in stages $1, \ldots, z$, minus the number of backlogged customers. (Recall that we have unit leadtimes, so there are no outstanding orders to add to the inventory position). There is a one-to-one correspondence between the echelon inventory positions at the various stages and the corresponding customer positions for units at those stages. Consider a particular stage $z > 1$ and assume that $I_t^z > I_t^{z-1}$. (Otherwise, there are no units at stage z and there is no decision to make). There are $I_t^z - I_t^{z-1}$ units at stage z. The crucial observation is that since the state is monotonic, these units correspond to customers with positions $(I_t^{z-1} + 1)^+, (I_t^{z-1} + 2)^+, \ldots, (I_t^z)^+$. Given these customer positions and the extended echelon base stock levels, the destination of each of these units is easily determined. The overall policy resulting from applying the single unit policy to each unit-customer pair has a very simple and intuitive structure as illustrated in the top half of Figure 4. (The dependency on time t is suppressed in the figure). Here, the shipment decisions for units at stage 5 are illustrated. We are given the extended echelon base stock levels for units originating at stage 5, which are $S^{5,4} = 20$, $S^{5,3} = 12$, $S^{5,2} = 9$ and $S^{5,1} = 3$. One can think of

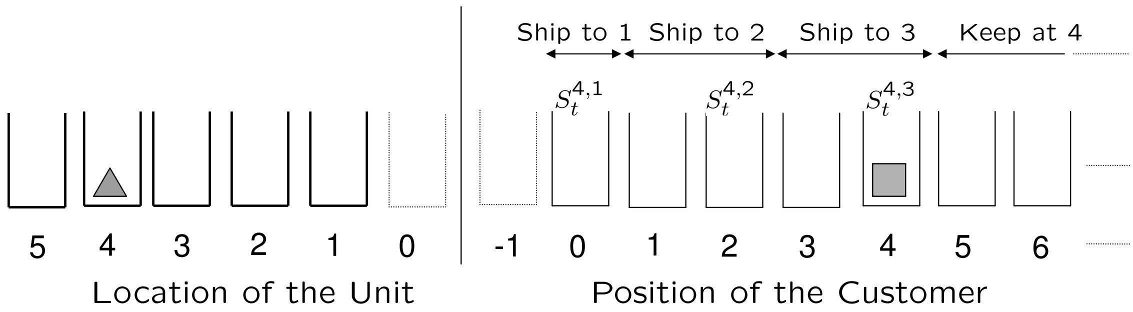

Figure 3: The figure shows the structure of the optimal policy for the single unit subproblem. Consider a certain stage z. The customer position axis is segmented into several intervals for this stage, where the end points of the intervals are the extended echelon base stock levels. For a unit at stage z, the decision of whether and where to ship the unit depends on in which interval the position of the corresponding customer falls. This figure depicts a situation, where the unit is at stage $z = 4$, the position y of the corresponding customer is 4, and $S_t^{4,1} = 0$, $S_t^{4,2} = 2$, $S_t^{4,3} = 4$. Since $S_t^{4,2} < y \le S_t^{4,3}$, the unit is shipped to stage 3.

these extended echelon base stock levels as forming a measuring stick. When the echelon inventory positions at stages 5 and 4 are compared against this measuring stick, the shipment decisions are easily determined. Two cases are depicted on the top half of Figure 4. The case depicted on the top left part of the figure corresponds to a situation where the echelon inventory position at stage 5 (I^5) is 24 and the echelon inventory position at stage 4 (I^4) is 5. This means that there are 19 units at stage 5. The policy dictates that out of these 19 units, 4 units will be kept at stage 5, 8 units will be shipped to stage 4, 3 units will be shipped to stage 3, and 4 units will be shipped to stage 2. The picture on the top right corresponds to another case where $I^5 = 17$ and $I^4 = -2$. We next give a formal definition of extended echelon base stock policies.

Definition 4.6. We say that a policy is of *extended echelon base stock* type if for every t and for all pairs of stages (z, w), with $z > w \ge 1$, there exist threshold values[1] $S_t^{z,w}$, which satisfy the monotonicity properties in Lemma 4.3, so that the quantity to be shipped from stage z to stage w at time t is[2]:

$$\Big((I_t^z \wedge S_t^{z,w}) - (I_t^{z-1} \vee S_t^{z,w-1})\Big)^+ .$$

The formula given in Definition 4.6 simply calculates the amount to be shipped from a stage z to a stage $w < z$, given the echelon inventory positions at stages z and $z - 1$, according to the procedure described in Figure 4. For example, let us consider the case shown on the top left part of the figure, and let us calculate the amount to be shipped from stage 5 to stage 3. We have,

$$\big((I^5 \wedge S^{5,3}) - (I^4 \vee S^{5,2})\big)^+ = ((24 \wedge 12) - (5 \vee 9))^+ = 3.$$

[1]The threshold values are allowed to be an integer or $-\infty$. Set $S_t^{z,0} = -\infty$ for all z and t.
[2]We define $A \vee B := \max\{A, B\}$ and $A \wedge B := \min\{A, B\}$

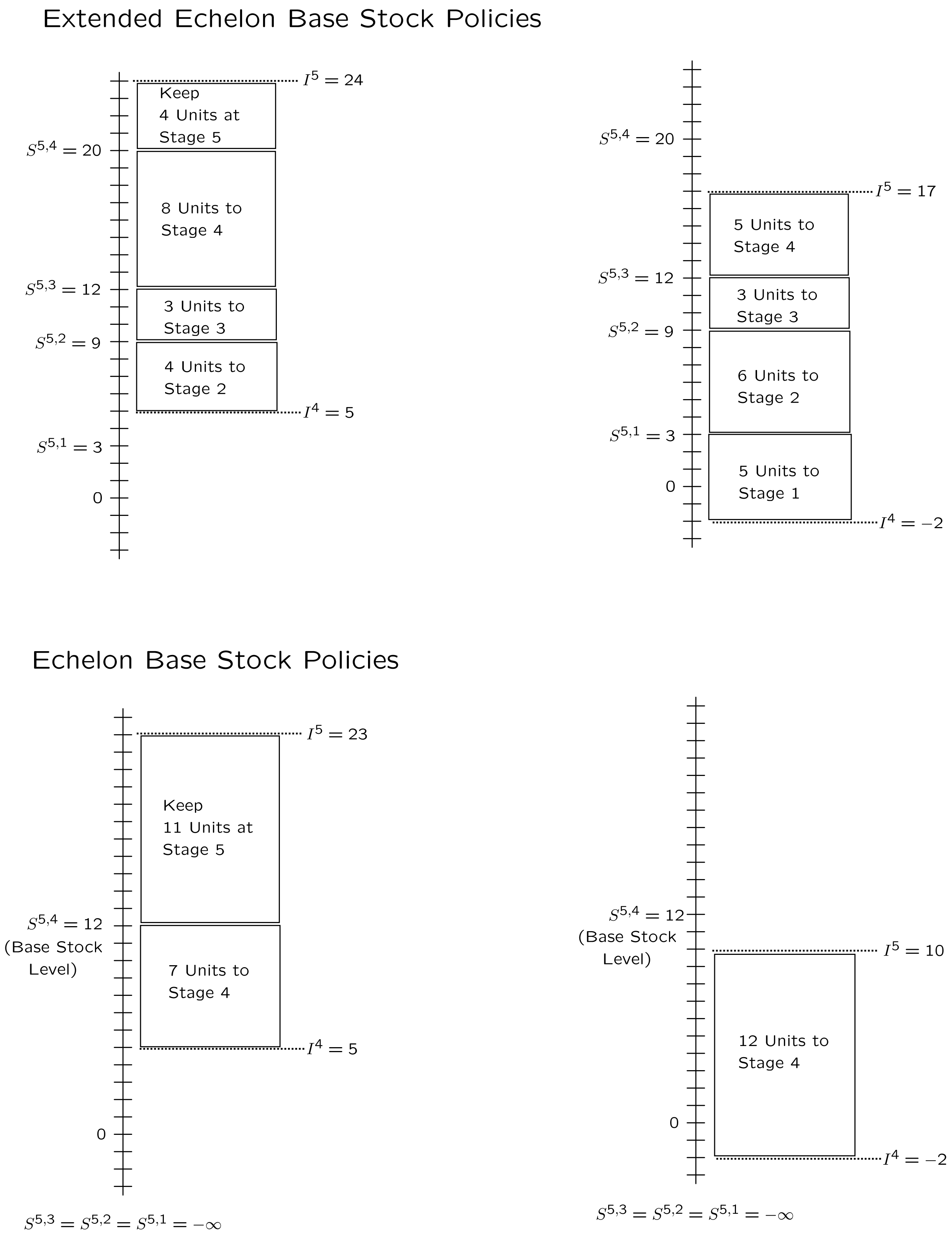

Figure 4: Illustration of Extended Echelon Base Stock Policies

For the amount shipped from stage 5 to stage 2, we have:

$$\big((I^5 \wedge S^{5,2}) - (I^4 \vee S^{5,1})\big)^+ = ((24 \wedge 9) - (5 \vee 3))^+ = 4.$$

Extended echelon base stock policies are very easy to implement. The informational requirements are minimal. Each stage z has a local manager responsible of shipments from stage z to downstream stages, who is given a set of extended echelon base stock levels for her stage at the beginning of the horizon. Then, as time progresses, the manager needs to monitor only the echelon inventory position and the local inventory level at her stage z. The echelon inventory position at stage $z-1$ is simply the difference between these two quantities. The amounts to be shipped from z to downstream stages depend only on the extended echelon base stock levels for stage z and the echelon inventory positions at stage z and $z-1$, which the manager has information about.

Note that an echelon base stock policy is a special case of an extended echelon base stock policy, with $S_t^{z,w} = -\infty$ for every $w < z-1$. In that case, the number of units shipped from stage z to stages other than the next stage $z-1$ are 0. The number of units shipped from stage z to stage $z-1$ are:

$$\Big((I_t^z \wedge S_t^{z,z-1}) - I_t^{z-1}\Big)^+$$

which is exactly the number of units shipped from stage z to stage $z-1$ under an echelon base stock policy with base stock level $S_t^{z,z-1}$ for stage $z-1$. The bottom half of Figure 4 illustrates how an echelon base stock policy is a special case of extended echelon base stock policies.

Theorem 4.1. *The set of extended echelon base stock policies is optimal for finite horizon problems.*

Proof. By an argument virtually identical to Lemma 2.1(3), choosing the decision according to $\hat{\mu}_t$ for every unit and every time step constitutes an optimal policy, for all initial states such that the corresponding subproblem states are not of prohibited type.[3] The resulting overall policy is an extended echelon base stock policy. □

5 Infinite Horizon Analysis

In the infinite horizon setting, we focus on stationary policies. A stationary policy is one of the form $(\mu, \mu, \ldots)$, where the decision at each time is a function of the current state but not of the current time.

Similarly, for the subproblems, we refer to a stationary policy of the form $(\hat{\mu}, \hat{\mu}, \ldots)$ as policy $\hat{\mu}$. Given a fixed discount factor $\alpha \in [0,1]$, let $\hat{J}_\infty^{\hat{\mu}}(z,y)$ and $\hat{J}_\infty^{*}(z,y)$ be the infinite horizon

[3]The difference here arises because Lemma 2.1(3) applies when $\hat{\mu}$ is defined at all states, whereas we defined $\hat{\mu}$ at all states except for a set of states that are impossible to reach under any optimal policy.

expected total discounted cost of policy $\hat{\mu}$, and the corresponding optimal cost, respectively. Let $\hat{J}_T^{\hat{\mu}}(z,y)$ be the expected total discounted cost of using the stationary policy $\hat{\mu}$ in a subproblem over a finite horizon of length T, given that the initial state of the subproblem is (z,y).

The next lemma from Muharremoğlu & Tsitsiklis (2001) relates the finite and infinite horizon versions of the subproblem.

Lemma 5.1. *For any fixed $\alpha \in [0,1]$, and any z, y, we have*

$$\lim_{T\to\infty} \hat{J}_T^*(z,y) = \hat{J}_\infty^*(z,y).$$

Furthermore, if $u \in \hat{U}_t^*(z,y)$ for infinitely many choices of t, then $u \in \hat{U}_\infty^*(z,y)$.

In Section 4.2, we showed that for finite horizon problems with horizon length T, an optimal subproblem policy with a particular structure can be chosen, by constructing a table $M_k(z,y)$ (like the one in Figure 2) for each $k \le T$. We then showed that this subproblem policy, when applied to all unit-customer pairs independently, results in an extended echelon base stock policy that is optimal. In particular, the $M_k(z,y)$ tables were constructed in a way such that a particular control u was chosen from within the $\hat{U}_k^*(z,y)$ sets for each z and $y \le Y_{\max}$ and remaining time $k \le T$, such that the resulting committed and decoupled policy for the overall system is an extended echelon base stock policy. The extended echelon base stock levels $S_t^{z,x}$ to be used at time $t = T-k$ only depend on the remaining time k, by construction. (Hence note that $S_{T-k}^{z,x} = S_{T'-k}^{z,x}$ for all $T' \ge T > k$).

Proposition 5.1. *There exists a $\bar{k}$, such that the stationary policy $\hat{\mu}$ that chooses its actions according to $\hat{\mu}(z,y) = M_{\bar{k}}(z,y)$ for all (z,y), is optimal for the infinite horizon subproblem.*

Proof. Consider the $M_k(z,y)$ tables for different values of k. First note that the table entries are values between 1 and $Y_{\max}$. Since there is a fixed number of entries, there is a finite number of possible tables, which means that at least one table is repeated for infinitely many choices of k. Let $\bar{k}$ be such that $M_{\bar{k}}(z,y) = M_k(z,y)$ for all (z,y), and for infinitely many choices of k. This means that, $M_{\bar{k}}(z,y) \in \hat{U}_k^*(z,y)$ for all (z,y) and for infinitely many k. By Lemma 5.1, $M_{\bar{k}}(z,y) \in \hat{U}_\infty^*(z,y)$ for all (z,y). Hence, $\hat{\mu}$ is optimal for the infinite horizon subproblem. □

Proposition 5.2. *Let μ^* be the stationary, decoupled, and committed policy for the overall problem that uses the optimal subproblem policy $\hat{\mu}$ of Prop. 5.1 for each unit-customer pair. Then, μ^* is a stationary extended echelon base stock policy.*

Proof. Note that $\hat{\mu}$ is constructed so that the prescribed actions are equivalent to the actions of a finite horizon policy, when $\bar{k}$ periods are left. In particular, the policy is to ship units according to extended echelon base stock levels $S_\infty^{z,x} = S_{T-\bar{k}}^{z,x}$, for some finite horizon problem with horizon length $T > \bar{k}$. The same levels are used in every period, so the policy is stationary. □

We have so far constructed a stationary extended echelon base stock policy μ^*. This policy is constructed as a limit of optimal policies for the corresponding finite horizon problems. It should then be no surprise that μ^* is optimal for the infinite horizon problem. Some careful limiting arguments are needed to make this rigorous. However, these arguments are virtually identical to the corresponding ones in Muharremoğlu & Tsitsiklis (2001) and for this reason, the proof of the following result is omitted.

Theorem 5.1. *The set of extended echelon base stock policies is optimal for both the discounted cost and the average cost criteria.*

6 Discussion

We start this section by discussing some extensions to the model. Afterwards, we provide some alternative interpretations of the results, corresponding to systems with different types of flexible leadtimes. Finally, we investigate the special case where the shipping costs are additive, as in Lawson & Porteus (2000). The assumption of additive costs simplifies the problem, and results in simpler policies. In particular, we show that our decomposition approach leads to results similar to the ones in Lawson & Porteus (2000).

6.1 Extensions

In this subsection, we discuss some possible extensions to our model. In particular, we focus on the leadtime and demand models.

In our model, the leadtimes for shipping from one stage to any other stage were assumed to be equal to one period. However, in many real world systems, longer leadtimes may be required. Such systems can be modeled within our framework, by converting them to equivalent unit leadtime systems. (This is done by introducing artificial locations between stages, and by setting appropriate shipping and holding costs to capture the dynamics of the system correctly.) However, our approach and results will only apply if the resulting shipping cost structures are supermodular. For instance, if we have two stages z and $z-1$ with a multi period leadtime between them, any model with a finite cost and faster shipping option (where the difference is more than one period) from a stage $w \geq z$ to a stage $v \leq z-1$ would result in a structure that is not supermodular and cannot be handled by our model. In this case, it may be optimal for units to overtake each other and optimal policies may potentially be more complicated than simple threshold policies. The only way to avoid this would be to allow *convertible* leadtimes, where a unit that is already in transit between z and $z-1$ via the slow transportation mode can be converted to the expedited mode while still in transit.

So, our analysis does not apply to situations that correspond to systems with shipping costs that are not supermodular. This raises the question whether another approach can yield the same result for such systems in general. The answer is no, since in such a case, optimal policies may be potentially much more complicated and dependent on the state of outstanding orders. This fact is illustrated in the next subsection, where systems with dual suppliers are considered using our model.

Another possible extension is to allow for stochastic leadtimes. Our method can handle stochastic leadtimes of the type used in Muharremoğlu & Tsitsiklis (2001). However, again, we need to make sure that in any such model, the supermodularity assumption is satisfied.

We can also handle the more general case of Markov-modulated demand, as in Muharremoğlu & Tsitsiklis (2001). In this case, the extended echelon base stock levels will of course depend on the state of the modulating Markov chain.

Finally, note that we have not given an explicit algorithm that determines the extended echelon base stock levels. However, since we have shown that the problem can be decomposed into single-unit subproblems, the extended base stock levels can easily be determined by solving a single unit subproblem. Such a problem is quite simple to solve using dynamic programming. Once the subproblem is solved and the optimal action sets $\hat{U}$ are determined, the extended echelon base stock levels are readily available.

6.2 Alternative Interpretations

While developing the results in previous sections, we used a terminology that corresponds to making shipping (or routing) decisions in a multi-echelon inventory systems. However, the system that we analyzed can also be used to model settings where different types of leadtime flexibility are present. In this subsection, we describe one such interpretation, that corresponds to using multiple suppliers in a single stage system with different leadtimes and costs.

Consider the manager of a single retail store that buys a durable product from suppliers and sells it to customers. In order to manage the uncertainty in customer demand, the manager keeps inventories of the product in the store. The store has a relationship with two different suppliers that provide the same product. One supplier is located closer to the store and can respond faster to orders. The other supplier is located further away and takes longer to fulfill orders; however, it charges a lower per unit price for the product. The manager can choose to place orders at either one of the suppliers at any time. The goal is to use inventories and the multiple suppliers to respond to fluctuations in customer demand, by managing the leadtime-cost tradeoff in the best way possible.

This problem is an example for a single stage inventory problem with dual supply options, which has been analyzed in numerous papers, some of which were listed in the introduction. The only available optimality results for this problem with simple policies correspond to cases where the suppliers' leadtimes differ by only one period. If the leadtime from the slow supplier and the fast supplier differ by more than one period, optimal policies are more complicated (than extended echelon base stock policies), as shown by Whittemore & Saunders (1977). This is illustrated in Figure 5, where two cases of a dual supplier problem are depicted. In case 1, the slow and fast suppliers have leadtimes of k and $k+1$, respectively, whereas in case 2, the leadtimes are $k-1$ and $k+1$. Let c^F be the cost that the fast supplier charges, and c^S the cost that the slow supplier charges, for a single unit of the product. To model these two situations as special cases of our formulation, we first visualize the two suppliers as a single aggregated supplier and insert $k-1$ artificial locations between the aggregated supplier and the retailer. We model the choice of different suppliers as using different paths while shipping units from the aggregate supplier to the retailer. Consider case 1. Units that are ordered from the slow supplier go through locations $k+2, k+1, k, k-1, \ldots, 1$, which takes $k+1$ periods, and incur a cost of c^S. Units that are ordered from the fast supplier skip location $k+1$, so that they are delivered in k periods, and incur a cost of c^F. To model the fact that a leadtime of less than k is impossible, we set some shipping costs equal to infinity, such as $c_{k+2,k-1}$, $c_{k+2,k-2}$, or $c_{3,1}$, etc. Also, we need to set the holding costs at the artificial stages to a large enough value to ensure that units are not delayed while in transit. With this formulation, the dual supplier problem is a special case of our formulation.

First, notice that supermodularity is satisfied in case 1. This means that our analysis is applicable, and extended echelon base stock policies are optimal. The interpretation of extended echelon base stock policies in this setting is as follows: There are two different reorder levels corresponding to the two suppliers, say R^F for the fast supplier and R^S for the slow supplier, where $R^S > R^F$. The optimal policy compares the inventory position I of the retailer (on hand inventory + outstanding orders - backlog) to these levels when making ordering decisions. In particular, the optimal ordering policy is:

If $I \geq R^S$, order nothing.
If $R^F \geq I > R^S$, order $I - R^S$ units from the slow supplier.
If $I > R^F$, order $R^F - R^S$ units from the slow and $I - R^F$ units from the fast supplier.

Now, notice that while supermodularity is satisfied in case 1, it is *not* satisfied in case 2, because we have:

$$c_{k+2,k-1} + c_{k,k} = c^F + 0 < c_{k+2,k} + c_{k,k-1} = \infty + 0$$

This implies that monotonic policies are not necessarily optimal for case 2, and the problem cannot be decomposed into single unit subproblems. Consequently, optimal policies are not guaranteed to be simple threshold type policies that depend only on the inventory position of the store. This is consistent with the existing results in the literature.

6.3 The Case of Additive Costs

The paper by Lawson & Porteus (2000) deals with a problem that is very similar to ours, i.e., a serial system with leadtime flexibility. They present their model a little differently, though. In

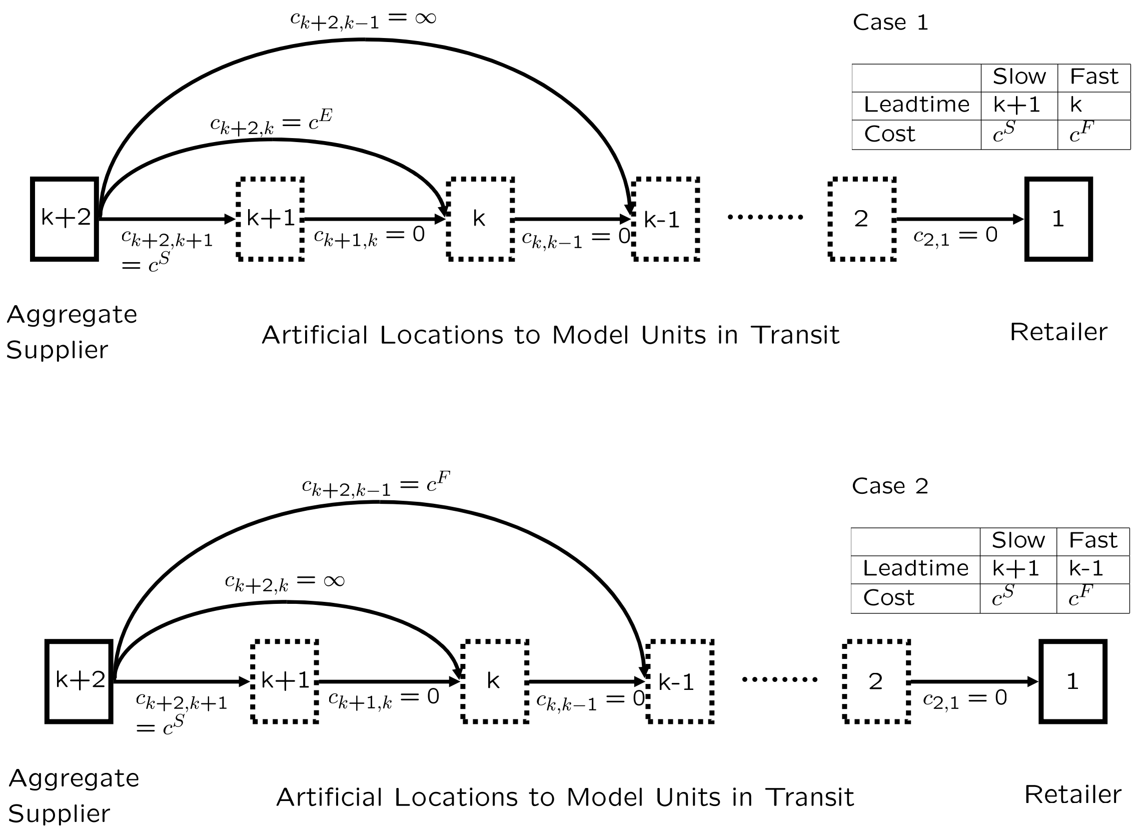

Figure 5: This figure depicts two cases of the single stage dual supplier problem. There is a retailer that can place orders from two different suppliers with different leadtimes and per unit costs In case 1, the difference between the slow and fast suppliers' leadtimes is one period, in case 2, it is two periods. When modelled as special cases of our formulation, by aggregating the suppliers into a single customer, inserting artificial locations to model units in transit and modelling the choice of the two suppliers as different paths from the aggregate supplier to the retailer, case 1 satisfies the supermodularity assumption, and case 2 doesn't. This means that monotonic policies are not necessarily optimal for case 2, and the system does not decompose into single unit subproblems, which explains why in dual supplier problems the structure of optimal policies is much more complicated when leadtimes differ by more than 1 period.

particular, they have a system where the units have to go through all the stages in this serial system. The transition from one stage to the next can take one period (via the regular shipment method) or can be instantaneous, which is called expedited delivery. The expedited delivery and the regular shipment from a stage to the next each have a particular cost associated with them. In addition, a unit can be expedited through several stages in the system within the same period. Such a transition can also be viewed as transferring the unit from a stage to another by skipping others in between, just like in our model. The main difference is that Lawson & Porteus (2000) use an *additive* cost structure, meaning that the cost of expediting through several stages is equal to the sum of the one step expediting costs. Our model assumes a more general cost structure, namely supermodularity, which includes additive costs as a special case.

Still, the policies reported in Lawson & Porteus (2000) are slightly different than our extended echelon base stock policies. The reason is one more difference between the two papers about the sequence of events within a period. In particular, in Lawson & Porteus (2000), in every period two sets of decisions are made. (See Figure 6). First, expediting decisions are made, which take place instantaneously. Here, any unit can be transferred from any stage to any downstream stage through a series of one stage expediting decisions, which is just like our expediting model. Then, regular order decisions are made, and these orders are not delivered until the beginning of the next period. Such a setting can easily be analyzed by our approach as well. We didn't use this setting in the previous sections just to simplify the exposition. We could model such a two step decision process by having alternating "expediting periods" and"regular order periods", where an actual period corresponds to one expediting and one regular order period in the model. In expediting periods, we would be able to send units from any stage to any downstream stage. There would be no customer arrivals and no holding or backorder costs in expediting periods, just origin-destination dependent shipment costs. Then, in regular order periods, we could only ship units to the next stage. The customer arrivals and the charging of holding and backorder costs would occur in regular order periods. The single unit analysis can easily be applied to this setting as well. The optimal policy in this case is to use extended echelon base stock policies in expediting periods and echelon base stock policies in regular order periods. In addition, the extended echelon base stock levels in this case have a very special structure due to the additivity of the expediting costs, as shown by the following proposition. Let $S_t^{z,w}$ be the extended echelon base stock levels for shipments originating at stage z destined to stage w at time t. (For $z > w \geq 1$).

Proposition 6.1. $S_t^{z,w} = S_t^{z-1,w}$ *for all* z, $w \leq z-2$, *and all* t.

Proof. Suppose that the proposition is not true. There are two cases: (In the rest of the proof, we suppress the dependency on t to simplify the expressions).

(a) Suppose that $S^{z,w} < S^{z-1,w}$ for some z and $w \leq z-2$. By the definition of the extended echelon base stock levels, we know that $M(z, S^{z,w}+1) = v$ for some $v > w$. By the definition

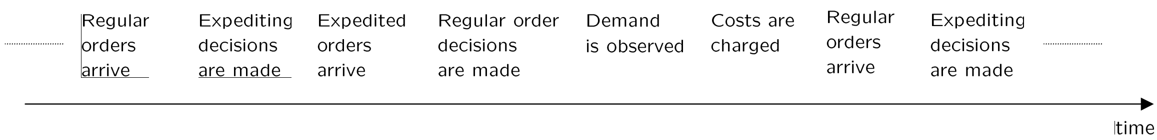

Figure 6: The sequence of events assumed by Lawson & Porteus.

of $M(\cdot)$, this means that $v \in \hat{U}^*(z, S^{z,w}+1)$, i.e., if the subproblem state is $(z, S^{z,w}+1)$, then it is optimal to ship the unit from z to v. This implies that

$$c_{z,v} + \hat{J}(v, S^{z,w}+1) \leq c_{z,r} + \hat{J}(r, S^{z,w}+1),$$

for any r. (Here, $\hat{J}(\cdot,\cdot)$ is the optimal cost-to-go function for the subproblem at the end of the expediting period.) The expediting costs are additive, meaning that $c_{z,r} = c_{z,v} + c_{v,r}$. Thus,

$$\hat{J}(v, S^{z,w}+1) \leq c_{v,r} + \hat{J}(r, S^{z,w}+1), \tag{2}$$

for any r.

By the definition of the extended echelon base stock levels and by the assumption that $S^{z,w} < S^{z-1,w}$, we have $M(z-1, S^{z,w}+1) = r$ for some $r \leq w$. This, together with the fact that $M(z, S^{z,w}+1) = v$ implies that $v \notin \hat{U}^*(z-1, S^{z,w}+1)$. Hence,

$$c_{z-1,v} + \hat{J}(v, S^{z,w}+1) > c_{z-1,r} + \hat{J}(r, S^{z,w}+1).$$

Since $c_{z-1,r} = c_{z-1,v} + c_{v,r}$, we obtain

$$\hat{J}(v, S^{z,w}+1) > c_{v,r} + \hat{J}(r, S^{z,w}+1). \tag{3}$$

Inequalities (2) and (3) are contradictory.

(b) Suppose that $S^{z,w} > S^{z-1,w}$ for some z and $w \leq z-2$. By the definition of the extended echelon base stock levels, we know that $M(z-1, S^{z,w}) = v$ for some $v > w$. By the definition of $M(\cdot)$, this means that $v \in \hat{U}^*(z-1, S^{z,w})$, i.e., if the subproblem state is $(z-1, S^{z,w})$, then the decision to ship the unit from $z-1$ to v is an optimal one. This implies that

$$c_{z-1,v} + \hat{J}(v, S^{z,w}) \leq c_{z-1,r} + \hat{J}(r, S^{z,w}),$$

for any r. The expediting costs are additive, meaning that $c_{z-1,r} = c_{z-1,v} + c_{v,r}$. Thus,

$$\hat{J}(v, S^{z,w}) \leq c_{v,r} + \hat{J}(r, S^{z,w}), \tag{4}$$

for any r.

By the definition of the extended echelon base stock levels and by the assumption that $S^{z,w} > S^{z-1,w}$, we have $M(z, S^{z,w}) = r$ for some $r \leq w$. This, together with the fact that $M(z-1, S^{z,w}) = v$ implies that $v \notin \hat{U}^*(z, S^{z,w})$. Hence,

$$c_{z,v} + \hat{J}(v, S^{z,w}) > c_{z,r} + \hat{J}(r, S^{z,w}).$$

Since $c_{z,r} = c_{z,v} + c_{v,r}$,

$$\hat{J}(v, S^{z,w}) > c_{v,r} + \hat{J}(r, S^{z,w}). \tag{5}$$

Inequalities (4) and (5) are contradictory.

□

Hence, we have shown that in the model where expediting costs are additive and expediting takes place instantaneously, we only need M threshold values for expediting decisions, instead of $M(M+1)/2$. In particular, we have values $\bar{S}^w$ such that $S^{M+1,w} = S^{M,w} = \cdots = S^{w+1,w} = \bar{S}^w$, for all w. These values are exactly the expediting base stock levels in Lawson & Porteus (2000). In addition, we have regular base stock levels for the regular order periods. These are the regular order base stock levels in Lawson & Porteus (2000). The policy of making expediting decisions according to the expediting base stock levels and then regular order decisions according to the regular order base stock levels is exactly what was coined as "top-down echelon base stock policies" in Lawson & Porteus (2000).

7 Conclusions

Having some sort of flexibility in leadtimes is a very commonly utilized method of managing uncertainty in supply chains. Working with multiple suppliers, using multiple transportation options, having the option to expedite certain processes, or having different possible routes for a unit to go through the supply chain are all examples of having flexibility in the supply chain leadtimes. In this paper, we introduced a formulation that can be used to model such problems where the objective is to control inventories and leadtimes simultaneously in order to minimize costs in the supply chain. Under a certain supermodularity assumption on the costs, we characterized the structure of optimal policies as extended echelon base stock policies, which is a natural generalization of the well known echelon base stock policies. Extended echelon base stock policies are quite intuitive and require minimal information sharing among different stages of the supply chain for implementation.

Our main analysis technique was to decompose the problem into a series of single-unit single-customer problems, and then to analyze the structure of the subproblems. This approach, besides being a proof technique, also gives rise to efficient algorithms for calculating the extended echelon base stock levels. In particular, to determine these threshold values, we only need to solve one subproblem with a single unit and a single customer, which can be achieved by solving a straightforward and simple dynamic programming problem.

While the set of supermodular costs is quite a large set, one can certainly think of many real world systems where this assumption would not be a reasonable one. We know that a complete relaxation of this assumption may lead to complicated policies that depend on the state of the outstanding orders. However, one interesting research question is whether there are other cost structures that still result in simple policies. Another equally intriguing question is, how do extended echelon base stock policies perform for systems where the supermodularity assumption is not satisfied? Finally, designing a system with leadtime flexibility may in itself involve a cost. What is the best way to manage the tradeoff between this cost and the benefits of having the flexibility?

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