

# Tuning Lasers for Fun and Profit: Ion Traps as Quantum Computers

Kenan S. Diab, [kdiab@mit.edu](mailto:kdiab@mit.edu)

Peer editor: Haofei Wei, [hwei@mit.edu](mailto:hwei@mit.edu)

May 25, 2008

## Abstract

We begin by describing the basic capacities that a real quantum computer must have in order to achieve universal computation. Then, we describe a possible physical implementation of a quantum computer that uses hyperfine splittings in heavy ions for qubit creation and manipulation. We conclude with a discussion of the relative strengths and weaknesses of this method, putting emphasis on methods of dealing with decoherence.

## 1 Introduction

In the fast-paced world of computer science, the desire for faster hardware and more efficient algorithms has always been a major catalyst for progress and growth in the field. It should not come as much of a surprise, then, that computer scientists are beginning to draw upon physics to find new ideas and paradigms for improving their craft. Among these borrowed tools is quantum mechanics. Quantum superposition and entanglement provide exciting new ways to manipulate information by liberating computation from the constraint of representing data in a strictly binary way. As has been documented by countless authors [1], the additional degrees of freedom offered by quantum mechanics allows for exponential improvements in the complexity required to solve broad classes of problems. For example, the best known classical algorithm known for integer factorization, the generalized number field sieve, runs in

$O\left(\exp[c(\log n)^{\frac{1}{3}}(\log \log n)^{\frac{2}{3}}]\right)$  time [2], while the quantum counterpart, Shor's algorithm, requires just  $O((\log n)^3)$  time. [3]

Despite its promise, however, quantum computation, has mostly remained a theoretical curiosity, owing to the difficulty in finding stable quantum systems that can be easily manipulated. Many models have been proposed, and some have been implemented in limited ways, but there still does not exist a practical, robust realization of quantum computation's promise. Recently, however, one model has emerged as a particularly promising candidate for quantum computing discovered so far: ion traps. Ion traps leverage the hyperfine electronic states of ions confined in a strong harmonic potential to represent information. Computation is achieved by entangling these states with the ions' vibrational states, which can be altered by lasers relatively easily. In this paper, I will describe precisely how this model works. After outlining the basic features a quantum computer must have, I describe how an ion trap works, and then I show how these basic features can be implemented. Then, I discuss the advantages and disadvantages of ion traps, giving quantitative detail about the stability of the model against unwanted interactions with the traps' environment (i.e. ambient decoherence).

## 2 Quantum computing basics

We start by examining some basic ideas in quantum computing. In a quantum computer, the fundamental unit of information is a *qubit*, which is simply a two-state system; the state of the qubit can be any vector in its two-dimensional Hilbert space. The axioms of quantum mechanics demand that the value stored in the qubit be a linear superposition of the system's two states. Choosing these states  $\{|0\rangle, |1\rangle\}$  as a basis for this system, we can write the state  $|\psi\rangle$  of the system as  $z_1|0\rangle + z_2|1\rangle$  for complex  $z_1$  and  $z_2$  satisfying  $|z_1|^2 + |z_2|^2 = 1$ . Noting that overall phase factors do not change the physical state of the qubit, we can choose  $z_1$  real so that  $|\psi\rangle = \cos \theta |0\rangle + e^{i\phi} \sin \theta |1\rangle$  for real  $\theta$  and  $\phi$ . This furnishes a geometric description for the state space of a qubit: it can be represented as a sphere with radius 1. The internal degrees of freedom  $\theta$  and  $\phi$  specify a location on the sphere. Note that this two dimensional vector space is much larger than the state space of a classical bit, which can only occupy the two poles of the sphere,  $|0\rangle$  and  $|1\rangle$ .

When we combine qubits together into a composite system, commonly known as a

*register*, the axioms of quantum mechanics require that we combine the state spaces of the individual qubits by taking tensor products. This naturally gives rise to entangled states, or states which cannot be factored into a tensor product of states in the state spaces of the individual qubits. For example, the state  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \in H_1 \otimes H_2$ , where  $H_1$  and  $H_2$  are the state spaces for the individual qubits, but  $|\Psi\rangle$  cannot be factored into a product of the form  $|\psi\rangle_1 \otimes |\psi\rangle_2$ , where  $|\psi\rangle_1 \in H_1$ ,  $|\psi\rangle_2 \in H_2$ .

Given a system with this mathematical structure, we can do computation by carrying the system through some physical process. Again, quantum mechanics constrains how this process may affect the system: the time evolution operator for the process must be unitary. Thus, we can consider a quantum computer to be *universal* in some sense if it has the capacity to carry out an arbitrary unitary transformation. In classical information theory, only three boolean gates (AND, OR, and NOT) are required to construct any arbitrary classical function. Miraculously, in quantum computing, only two gates are necessary: a gate that can perform an arbitrary unitary transformation on a single qubit<sup>1</sup>, and what is known as a controlled-NOT gate. A controlled-NOT takes a control qubit and a target qubit as input and flips the target qubit only if the control qubit is set to  $|1\rangle$ . Using the  $|\text{control}, \text{target}\rangle$  basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$  for the product space of these two qubits, we can write the controlled-NOT gate in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

To see why these two gates are sufficient, we appeal to a well-known fact: any unitary matrix  $U$  can be decomposed into a product of primitive unitary matrices  $V_i$ , each of which acts non-trivially on no more than two basis vectors. The proof is a tedious argument by explicit construction [4], and is omitted for the sake of brevity. Using this fact, we reduce the problem to showing that we can construct an arbitrary unitary transformation that acts non-trivially on only two basis vectors.

---

<sup>1</sup>It may strike some readers as odd that we treat all unitary single-qubit transformations as being a single kind of gate. On face, it appears as though this comprises a very broad class of transformations, but it turns out that operations on single qubits are easy to do. This will become clear in our discussion of ion traps in the following section.

Suppose we have an  $n$ -qubit register and a unitary transformation that acts non-trivially on the basis states  $|s\rangle = |s_1 s_2 \cdots s_n\rangle$  and  $|t\rangle = |t_1 t_2 \cdots t_n\rangle$ , where the strings  $\{s_i\}$  and  $\{t_i\}$  are the binary representation of the two basis states. Then, we can write a sequence of binary numbers  $a_1, a_2, \cdots, a_k$ , where  $a_1 = s$ ,  $a_k = t$ , and each adjacent pair of  $a_i$  differ in only one bit. For example, the numbers 000 and 111 can be associated with the sequence (000, 001, 011, 111). Hence, we can transform  $s$  into  $a_{k-1}$  using controlled-NOT gates, holding all bits except one constant at each step in the sequence. Then, we apply the required unitary transformation on  $a_{k-1}$  and  $t$ , controlling all the qubits except the one qubit which differs. Finally, we apply the controlled-NOT gates in reverse order to return  $a_{k-1}$  to  $a_1 = s$ .

To clarify how this process works, we give an example: Suppose we have a state  $a|000\rangle + b|001\rangle + c|011\rangle + d|111\rangle$  and a unitary single-qubit transformation  $A$ . We would like to apply this transformation on the basis states  $|000\rangle$  and  $|111\rangle$ , so that  $a$  is sent to  $a'$  and  $d$  is sent to  $d'$ . We can't do this outright because the space spanned by  $|000\rangle$  and  $|111\rangle$  does not correspond to the space of a single-qubit in the register. Hence, we apply the algorithm given above, which is diagrammed in figure 1:

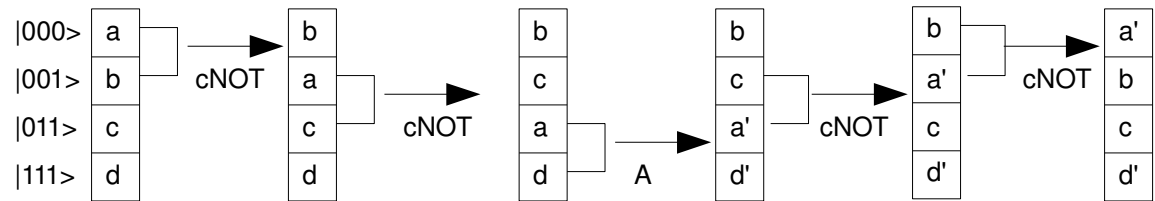


Figure 1: A primitive unitary operator built from universal gates.

Note that the construction given in this argument is not even close to being optimal: indeed, it is completely untenable since the number of gates required in the construction for an  $n$ -qubit operation grows exponentially in  $n$ . [5] Regardless, it shows that any computation which is theoretically possible on a quantum computer requires only those operations. Although the question of how to design and efficiently implement quantum algorithms is interesting, it is beyond the scope of this paper. [6]

In order to actually implement the gates required for universality, it is necessary to find a suitable two-state system for qubit representation. There does not yet exist universal consensus as to what criteria make a system “suitable”, but one list of

criteria given by di Vincenzo captures many of them [7]:

1. Be a scalable physical system with well-defined qubits
2. Be initializable to a simple fiducial state such as  $|000\dots\rangle$
3. Have long decoherence times compared to gate times
4. Have a universal set of quantum gates
5. Permit high quantum efficiency, qubit-specific measurements

In the rest of the paper, I will argue that one possible system, the linear ion trap, not only accomplishes all these goals, but also accomplishes them more efficiently and reliably than most other potential quantum computing systems.

## 3 Linear Ion Traps

### 3.1 Qualitative overview

In a linear ion trap, also known as a Paul trap<sup>2</sup>, a quadrupole potential is constructed by positioning four rod-shaped electrodes in a square around the axis where the ions are to be trapped. The ends of the electrodes are held at a different voltage than the middle, creating a DC potential that radially confines the ions along the axis of the quadrupole. Since Earnshaw's theorem[8] implies that it is impossible to confine ions using only static fields, we must use some kind of alternating potential to confine the ions axially. To accomplish this, we ground a pair of electrodes diagonally opposite each other and apply an alternating potential to the other two. The combined action of these two potentials confines the ions in a string along the axis of the quadrupole. [9]

To construct qubits, we pick the electronic ground state and a hyperfine state of each ion as our two-state system. In order to allow the qubits for each ion to interact with each other, we can apply external electromagnetic fields via lasers to couple the electronic states to the center-of-mass vibrations of the whole trap. By using these vibrational modes as a kind of temporary quantum memory, we can build the fundamental quantum gates required for computation. Preparing the motional

---

<sup>2</sup>Wolfgang Paul was awarded the Nobel Prize in 1989 for developing the ion trap.

and electronic ground states of the ion trap can be achieved by Doppler cooling and sideband cooling. The output of a computation is obtained by measuring the population of the hyperfine states. [10]

To fill in the details of how these processes actually occur, we now need to examine the model quantitatively. [11]

### 3.2 Deriving the Hamiltonian of the system

We begin by modeling the system before the motional and electronic states are coupled. The simplest model for the effect of the quadrupole on the motional states of the ion is a three-dimensional anisotropic harmonic oscillator. The unperturbed Hamiltonian for such a system is given by simply summing the Hamiltonians for each individual ion:

$$V(x, y, z) = \sum_{i=1}^N m\omega_x^2 x_i^2/2 + m\omega_y^2 y_i^2/2 + m\omega_z^2 z_i^2/2$$

Usually in an ion trap, the radial confinement is very strong compared to the axial confinement. That is, we usually have that  $\omega_x \approx \omega_y \gg \omega_z$ . Hence, to first order, we assume that the ions are confined to move along the  $z$  axis, and so we ignore the  $x$  and  $y$  contributions to the Hamiltonian. Furthermore, for clarity of exposition, we will restrict ourselves to a single ion for now and give the generalization for  $N$  ions later. Hence, the motional state of an ion is simply described by a one-dimensional harmonic oscillator:  $H = m\omega_z^2 z^2/2 + p^2/2m$ . Using the usual ladder operators  $a^\dagger$  and  $a$  and shifting the energy spectrum by  $\hbar\omega_z/2$ , we write the motional piece of the Hamiltonian as  $\hbar\omega_z a^\dagger a$ .

The electronic piece of the Hamiltonian is difficult to precisely model because the perturbations to the energy states of the electrons are very complicated. For example, the Coulomb interactions between the electrons in singly-ionized strontium-38, a common ion used for quantum computing, are mathematically intractable. As a result, we are forced to make an approximation. Since we need two electronic states for representing qubits, we treat the atom as a two-state system where the separation between the energies is given by  $\hbar\omega_0$ , where  $\omega_0$  is experimentally determined. Adopting the Pauli matrices  $\sigma_i$  as the usual vehicle for describing a two-state system, we write the electronic piece of the Hamiltonian as  $H = \hbar\omega_0 \sigma_z/2$ . Note that we're essentially

assigning an “equivalent” spin to the entire ion so that the magnetic moment of the atom is  $\vec{\mu} = \gamma\vec{\sigma}/2$ , for some “equivalent” gyromagnetic ratio  $\gamma$ .

Putting these pieces together, the total “free” Hamiltonian for the particle is  $H_0 = \hbar\omega_0\sigma_z/2 + \hbar\omega_z a^\dagger a$ . Before proceeding further, it will be useful to transform the spin raising and lowering operators,  $\sigma_+ \equiv (\sigma_x + i\sigma_y)/2$  and  $\sigma_- \equiv (\sigma_x - i\sigma_y)/2$ , and the vibration raising and lowering operators,  $a$  and  $a^\dagger$ , into the so-called *interaction picture*. In the interaction picture, the time evolution due to the free Hamiltonian, but not the perturbations, are shifted to operators. More precisely, let  $H = H_0 + H_1$  in the Schrodinger picture. Then every Schrodinger operator  $A$  gives an interaction operator  $\tilde{A} \equiv e^{iH_0t/\hbar} A e^{-iH_0t/\hbar}$ , and every Schrodinger state  $|\psi\rangle$  gives an interaction state  $|\psi_I\rangle \equiv e^{-iH_0t/\hbar} |\psi\rangle$ . Time evolution satisfies  $i\hbar \frac{d}{dt} |\psi_I\rangle = \tilde{H}_1 |\psi_I\rangle$ . The interaction picture, then, is a compromise between the Schrodinger and Heisenberg pictures, allowing us to analyze perturbations on a system more easily. We begin by finding  $\tilde{a}$ :

$$\begin{aligned} \tilde{a} &= e^{iH_0t/\hbar} a e^{-iH_0t/\hbar} \\ &= e^{i(\hbar\omega_0\sigma_z/2 + \hbar\omega_z a^\dagger a)t/\hbar} a e^{-i(\hbar\omega_0\sigma_z/2 + \hbar\omega_z a^\dagger a)t/\hbar} \\ &= e^{i\omega_z a^\dagger a t} a e^{-i\omega_z a^\dagger a t} \\ &= \left( \sum_{i=0}^{\infty} \frac{(i\omega_z t a^\dagger a)^i}{i!} \right) a \left( \sum_{i=0}^{\infty} \frac{(-i\omega_z t a^\dagger a)^i}{i!} \right) \end{aligned}$$

Recalling that  $[a^\dagger a, a] = a^\dagger a a - a a^\dagger a = (a a^\dagger - 1)a - a a^\dagger a = -a$ , we have that  $(a^\dagger a)^n a = a(a^\dagger a - 1)^n$ . Thus, we have

$$\begin{aligned} \tilde{a} &= \left( \sum_{i=0}^{\infty} \frac{(i\omega_z t a^\dagger a)^i}{i!} \right) a \left( \sum_{i=0}^{\infty} \frac{(-i\omega_z t a^\dagger a)^i}{i!} \right) \\ &= a \left( \sum_{i=0}^{\infty} \frac{(i\omega_z t (a^\dagger a - 1))^i}{i!} \right) \left( \sum_{i=0}^{\infty} \frac{(-i\omega_z t a^\dagger a)^i}{i!} \right) \\ &= a e^{i\omega_z t (a^\dagger a - 1 - a^\dagger a)} \\ &= a e^{-i\omega_z t} \end{aligned}$$

The same commutation relations can be used on the other three operators. Computing, we find:

$$\tilde{a}^\dagger = a^\dagger e^{i\omega_z t} \quad \tilde{\sigma}_+ = \sigma_+ e^{-i\omega_0 t} \quad \tilde{\sigma}_- = \sigma_- e^{i\omega_0 t}$$

Now, we are ready to turn on the external magnetic field. A laser with a variable frequency  $\omega$  and wavenumber  $k$  emits light described by  $\vec{B} = B_1 \hat{x} \cos(\omega t - kz - \phi)$ . Rewriting this in terms of complex exponentials and recalling that a magnetic dipole in an external field admits an interaction Hamiltonian of the form  $-\vec{\mu} \cdot \vec{B}$ , we obtain the interaction Hamiltonian:

$$H_{int} = -(\gamma B_1/2)(\sigma_+ + \sigma_-)(e^{i(\omega t - \phi)} e^{-ikz} + e^{-i(\omega t - \phi)} e^{ikz})$$

For convenience, define  $\Omega = \gamma B_1/2\hbar$ . This is called the Rabi frequency of the problem. To simplify this equation, we expand the exponential in  $z$  to first order so that  $e^{\pm ikz} \approx 1 \pm ikz$ . Qualitatively, this approximation is valid if the ions are well-localized in the traps so that the wavelength of the laser is much greater than the spread of the ground-state wavefunction. Quantitatively, we want  $\Delta z_0 \ll 1/k$ , where  $\Delta z_0 = \sqrt{\hbar/2M\omega_z}$  is the spread of the ground state wavefunction. Defining the parameter  $\eta = k\sqrt{\hbar/2M\omega_z}$ , the so-called Lamb-Dicke parameter, we restate this condition as  $\eta \ll 1$ . Assuming this condition is satisfied, the interaction Hamiltonian splits into two pieces: one piece picks up the constant term, and the other picks up the  $ikz$  term. We examine the first term, transforming into the interaction picture:

$$\tilde{H}_{int,1} = -(\hbar\Omega/2)(\sigma_+ e^{-i\omega_0 t} + \sigma_- e^{i\omega_0 t})(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)})$$

Now, we make another approximation. Typically,  $\omega$  is chosen to be around the same order of magnitude of  $\omega_0$  so that  $\omega + \omega_0 \gg \omega - \omega_0$ . Thus, the terms moving at frequency  $\omega + \omega_0$  oscillate very rapidly compared to the  $\omega - \omega_0$  terms and therefore average to zero over time. Throwing these terms out, and defining the difference frequency  $\delta = \omega - \omega_0$ , we have:

$$\tilde{H}_{int,1} = -(\hbar\Omega/2)(\sigma_+ e^{i(\delta t - \phi)} + \sigma_- e^{-i(\delta t - \phi)})$$

This is called the *rotating wave approximation*. Note that this piece contains no operators corresponding to the vibrational state of the ion. It describes only the internal electronic state of the ion. In the next section, we will see this piece allows to construct arbitrary unitary single-qubit transformations. Now, we examine the second piece of the interaction Hamiltonian:

$$\tilde{H}_{int,2} = (\hbar\Omega/2)(\sigma_+ e^{-i\omega_0 t} + \sigma_- e^{i\omega_0 t}) ik\tilde{z}(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)})$$

Substituting the harmonic oscillator position operator  $z = \sqrt{\hbar/2M\omega_z}(a + a^\dagger) = (\eta/k)(a + a^\dagger)$

$$\tilde{H}_{int,2} = (i\eta\hbar\Omega/2)(\sigma_+e^{-i\omega_0t} + \sigma_-e^{i\omega_0t})(ae^{-i\omega_zt} + a^\dagger e^{i\omega_zt})(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)})$$

Regrettably, there's not much we can do to simplify this equation, but we can identify some important special cases. If we tune the laser to  $\omega_0 + \omega_z$  and apply the rotating wave approximation, the spin and vibrational modes couple together:

$$\tilde{H}_{int,2} = (i\eta\hbar\Omega/2)(\sigma_+ae^{-i\phi} + \sigma_-a^\dagger e^{i\phi})$$

$\omega = \omega_0 + \omega_z$  is known as the blue side-band frequency. Similarly, when  $\omega = \omega_0 - \omega_z$ , the red side-band frequency, the modes couple as follows:

$$\tilde{H}_{int,2} = (i\eta\hbar\Omega/2)(\sigma_+a^\dagger e^{-i\phi} + \sigma_-ae^{i\phi})$$

To interpret what these Hamiltonians actually do, we associate the state where the ion is in the  $n^{\text{th}}$  electronic state and the  $m^{\text{th}}$  vibrational state by the ket  $|nm\rangle$ . Formally, this ket lies in the product space generated by the space of electronic states and the space of vibrational states. Recalling that  $n$  and  $m$  are only allowed to take on the values 0 and 1, we find the following relations:

$$\begin{aligned} \sigma_+a|11\rangle &= |00\rangle & \sigma_-a^\dagger|00\rangle &= |11\rangle \\ \sigma_+a^\dagger|10\rangle &= |01\rangle & \sigma_-a|01\rangle &= |10\rangle \end{aligned}$$

Matching these operators to the Hamiltonians corresponding to the blue and red side-band frequencies, we conclude that the blue side-band frequency induces transitions between  $|00\rangle$  and  $|11\rangle$ , and the red side-band frequency induces transitions between  $|01\rangle$  and  $|10\rangle$ . Thus, the second piece of the interaction Hamiltonian will play a crucial role in entangling the electronic and vibrational states when we construct a controlled-NOT gate.

The generalization of this entire discussion to  $N$  ions is trivial. The treatment of the electronic states is identical, since the electronic states of each ion form a separate qubit. The vibrational part needs only one minor modification: instead of using the vibrational modes of each ion, we use the shared vibrational mode of the center of mass. This takes the mass variable  $M$  to  $NM$ , so we should replace the Rabi frequency  $\Omega$  by  $\Omega/\sqrt{N}$ .

With these tools, we are now ready to show that ion traps allow for universal quantum computing. [12]

### 3.3 Proof of universality

#### 3.3.1 Single-qubit operations

We can easily construct arbitrary single qubit operations by setting  $\omega = \omega_0$ . Then,  $\delta = 0$ , and the first piece of the interaction Hamiltonian reduces to  $\tilde{H}_{int,1} = -\hbar\Omega/2(\sigma_+e^{-i\phi} + \sigma_-e^{i\phi})$ . This is time-independent, so we just exponentiate to get the time evolution operator. Define  $\theta = -\Omega t$ . Then, the time evolution operator is given by  $\exp(\frac{-i\theta}{2}(\sigma_+e^{i\phi} + \sigma_-e^{-i\phi}))$ . Collapsing the exponentials, we get  $\exp(\frac{-i\theta}{2}(\sigma_x \cos \phi + \sigma_y \sin \phi))$ . This is a rotation about the axis  $\hat{n} = (\cos \phi, \sin \phi, 0)$ . Clearly, by choosing  $\phi$  and  $t$ , we can generate any rotation about an axis in the xy plane. We claim that this generates all possible unitary transformations on a single qubit:

We appeal to the structure of the group of all single-qubit operations  $SU(2)$ . It is well-known that a surjective homomorphism can be constructed from  $SU(2)$  to  $SO(3)$  with kernel  $\pm I$ . [13] Hence,  $SU(2)$  is a double covering of  $SO(3)$ , and elements in  $SU(2)$  can be identified as rotations in 3-space, up to a choice of sign. Then, using Euler angles, we can choose any two different rotation axes to generate all of  $SO(3)$ . In particular, up to a universal phase, we can decompose any operator in  $SU(2)$  into the form  $R_x(\alpha)R_y(\beta)R_x(\gamma)$ , where  $R_x(\theta)$  and  $R_y(\theta)$  denote rotations around the  $x$  and  $y$  axes, respectively. Choosing the right spin on these rotations yields all  $SU(2)$ , as desired.

#### 3.3.2 Controlled-NOT gate

To build a controlled-NOT gate, we need three components. First, we need a *Hadamard gate*, which acts on the electronic states of a single ion, effecting the transformation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is unitary, so it can be constructed using the technique of the previous section. Second, we need a controlled phase flip gate. In the  $|nm\rangle$  basis, this can be written as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

To build this, we introduce an auxiliary electronic state of the ion,  $|2\rangle$ . While not strictly necessary, it is simpler to explain the construction of this gate by using this auxiliary state. We prepare a laser that generates a magnetic field  $B'$  (and therefore gives the problem a different Rabi frequency  $\Omega'$ ), and we introduce new raising and lowering spin operators  $\sigma'_+$  and  $\sigma'_-$ . Thus,  $\sigma'_+|1\rangle = |2\rangle$  and  $\sigma'_-|2\rangle = |1\rangle$ . Just as in the construction of the electronic part of  $H_0$ , we can associate the energetic spacing between  $|1\rangle$  and  $|2\rangle$  with some frequency  $\omega_{aux}$  so that transitions between the product states  $|11\rangle$  and  $|20\rangle$  are induced by tuning the laser to  $\omega_{aux} + \omega_z$ . In particular, the Hamiltonian for such a transition takes on exactly the same form as the side-band transitions described earlier:  $\tilde{H}_{aux} = (i\eta\hbar\Omega'/2)(\sigma'_+ae^{-i\phi} + \sigma'_-a^\dagger e^{i\phi})$ . Now, by the previous section, we can perform arbitrary rotations in the xy-plane on the space spanned by those two states, as they form a single qubit. We rotate the space through an angle of  $2\pi$  in the x-direction, which sends  $|11\rangle$  to  $-|11\rangle$  (remember, spin states are spinors!), leaving all other states unchanged, as desired.

Finally, we need a swap gate, which swaps qubits between the electronic states and the vibrational states:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is just a rotation by  $\pi$  in the y-direction on the subspace spanned by  $|01\rangle$  and  $|10\rangle$ , which we achieve by tuning the laser to the red side-band frequency and applying the technique of the previous section. Note that we can invert this operation by rotation by  $-\pi$  instead of  $\pi$ .

To construct the controlled-NOT gate on ions  $i$  and  $j$ , we perform the following sequence of operations:

1. Hadamard gate on ion  $j$

2. Swap gate on ion  $j$
3. Controlled phase flip, with ion  $i$  as control and ion  $j$  as target
4. Inverse swap gate on ion  $j$
5. Hadamard gate on ion  $j$

The proof of this algorithm follows from simply multiplying all the matrices we've written down so far in this section. It is therefore omitted. Combining the result of this section and the previous section, we conclude that ion traps are capable of universal quantum computation.

## 4 Strengths and Weaknesses

We've demonstrated that ion traps are theoretically capable of doing quantum computing, so now we examine the practical issues involved with implementing them, discussing the advantages and disadvantages to the ion trap method.

### 4.1 Robustness against decoherence

The biggest challenge in building a working quantum computer is the problem of *decoherence*, or the wavefunction collapse that occurs when a particle in a quantum state measurably interacts with its surroundings. The properties of quantum mechanics that make quantum computing useful in the first place, entanglement and superposition, are unfortunately governed by rules that make useful quantum states fragile and difficult to produce. If an entangled state interacts at all with its environment, it will immediately collapse into a classical state. This is extremely problematic: after all, the reason we need controlled-NOT gates in the first place is to produce entangled states. The causes of decoherence are myriad, and solutions to problems of decoherence have been hard to come by. Quantum error correction codes exist, allowing for some tolerance of decoherence, but it is still of paramount importance that the physical model underlying any quantum computer be inherently resistant to decoherence. Ion traps are unquestionably one of the best models for doing quantum computation for this reason. In this section, I will describe one way in which ion traps exhibit this property: the electronic states ion traps use are incredibly resistant to spontaneous decay. [14]

In ion traps, the electronic states selected for computation are usually hyperfine states. The Hamiltonian describing the spontaneous decay of hyperfine electronic states is called the Jaynes-Cummings Hamiltonian. Its derivation is a problem of quantum electrodynamics and is therefore beyond the scope of this paper, but we state it without proof:

$$H_I = g(a^\dagger \sigma_- + a \sigma_+)$$

$g$  is a coupling factor which depends on the system being treated. Again, finding its specific form is purely a problem in quantum electrodynamics, so it will not be derived here. To focus on the inherent stability of the hyperfine state, independent of the coupling to the vibrational mode of the system, we will assume that only a single excitation of the system is possible, (i.e. we exclude the  $|11\rangle$  state). Then, recalling our symbol  $\delta$  for the laser detuning  $\omega - \omega_0$ , we can write the Jaynes-Cummings Hamiltonian in the  $\{|00\rangle, |01\rangle, |10\rangle\}$  basis:

$$- \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{pmatrix}$$

Calculating the time evolution operator  $U = e^{-iH_I t}$  is a routine exercise in diagonalizing the matrix and reverting to the initial basis states. Defining the Rabi frequency for this problem  $\Omega = \sqrt{g^2 + \delta^2}$ , we find:

$$\begin{aligned} U &= e^{-i\delta t} |00\rangle \langle 00| + (\cos \Omega t + \frac{i\delta}{\Omega} \sin \Omega t) |01\rangle \langle 01| \\ &\quad + (\cos \Omega t - \frac{i\delta}{\Omega} \sin \Omega t) |10\rangle \langle 10| - \frac{ig}{\Omega} \sin \Omega t (|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

Since the Jaynes-Cummings Hamiltonian given does not allow for decay from  $|10\rangle$  to  $|00\rangle$ , the probability for decay is  $p = |\langle 10|U|01\rangle|^2$ . To lowest order in  $g$ , we calculate:

$$p = g^2 \frac{4 \sin^2 \frac{t\delta^2}{2}}{\delta^2}$$

The specific form of  $g^2$  is dictated by quantum electrodynamics:

$$g^2 = \frac{\omega_0^2 |\langle 0|\vec{\mu}|1\rangle|^2}{2\hbar\omega\epsilon_0 c^2}$$

where  $\vec{\mu}$  is the atomic dipole operator defined earlier. The total probability of decay is given by integrating  $p$  over all optical modes. The specific form of the integral is complicated because of the need to account for polarizations, mode densities, and other optical parameters, and so it is omitted. After carrying out the integration, we differentiate the result with respect to time to find the probability of decay per second  $\gamma_{rad}$ . This turns out to be  $\frac{\omega_0^3 |\langle 0 | \vec{\mu} | 1 \rangle|^2}{3\pi\hbar\epsilon_0 c^5}$ . Taking the matrix element of the moment to be around  $\mu_B$ , the Bohr magneton, and inserting  $\omega_0/2\pi = 10\text{GHz}$  as a typical frequency associated with the energy spacing, we calculate  $\gamma_{rad} \approx 3 \cdot 10^6$  years. Measured decay times for hyperfine states are actually on the order of seconds or hours since our simplified model of the electronic structure of the ion neglects higher order perturbations that become relevant at this scale, but this is much better than other quantum computing models. For example, electronic states in quantum dots decohere in around  $10^{-6}$  seconds. Electronic states in gold electrons have decay times in the range of  $10^{-8}$  seconds. [15] It is because of this superior resistance to decoherence that ion traps have been recognized as useful for quantum computing.

## 4.2 Other considerations

Although decoherence is the arch-nemesis of all quantum computing, there are other criteria that can be used to evaluate quantum computing models. For example, it is important that initial states be easy to prepare. Unfortunately, vibrational ground states are very difficult to prepare in ion traps. Complicated optical techniques, such as Doppler cooling and sideband cooling, are required to cool the ions down to their vibrational ground state. Since the performance of these techniques are strongly dependent on the line-width of the laser, this requires not only a sophisticated setup, but also very high-performance optics. In contrast, schemes that rely on optical photons or optical cavity quantum electrodynamics have the capacity to produce initial states relatively easily. [16]

Another concern in quantum computing is scalability. Although simple quantum computers can be built out of many physical systems, few exhibit the potential for being effectively scaled up to useful sizes. Here, ion traps show great promise. Although *linear* ion traps may not be the best platform for quantum computing, as they require ever stronger potentials to achieve effective confinement as the number of ions increases, *planar* ion traps do not. By simply adding extra dimensions to the array and additional structure to the trap to confine the ions in discrete wells, it may

be possible to build huge arrays of ions for quantum computing. This is an advantage that none of the other models mentioned so far have. [17]

These are just a couple of possible criteria that may be used to evaluate the feasibility of a model for quantum computing, and it is still unclear how much each of them really matter. As the field advances, however, it will become clearer which criteria are most relevant in judging quantum computing models.

## 5 Conclusion

As we have seen, linear ion traps provide not only a theoretically complete method of doing quantum computing, but also a promising route for building practical quantum computers. Future improvements over the linear ion model are equally promising and are currently being pursued by many labs across the world<sup>3</sup>. Despite the seemingly slow rate of progress in this field, the potential benefits of quantum information theory are enormous and are undoubtedly worth pursuing.

## References

- [1] See, for example, G. Jaeger, *Quantum Information* (Springer, New York, NY, 2007)
- [2] Pomerance, Carl, A tale of two sieves, *Notices of the AMS* **43**, 12 (1996)
- [3] P.W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, appeared in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science* (Santa Fe, NM, 1994)
- [4] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (University Press, Cambridge, UK, 2005), page 189-191
- [5] Ref. [4], page 193
- [6] A good reference for reading about this further is A.O. Pittinger, *An Introduction to Quantum Computing Algorithms* (Birkhauser Press, Boston, MA, 2000)

---

<sup>3</sup>MIT is among them! (Prof. Isaac Chuang, RLE)

- [7] D. di Vincenzo, The physical implementation of quantum computation, *Fortsschritte der Physik* **48** (2000)
- [8] D.G. Griffiths, *Introduction to Electrodynamics*, (Prentice Hall, Upper Saddle River, NJ, 1999), page 115
- [9] D. Bouwmeester, et al, eds, *The Physics of Quantum Information* (Springer, New York, NY, 2000), page 164-167
- [10] J.I. Cirac and P. Zoller, Quantum computations with cold trapped ions, *Physical Review Letters* **74**, 20 (1995)
- [11] The construction given here in the following section closely follows M. le Bellac, *A Short Introduction to Quantum Information and Quantum Computation*, (Cambridge University, Cambridge, UK, 2006), page 114-119
- [12] The proof given here is similar to the one in Ref. [4], page 319-321
- [13] The results from quantum electrodynamics in this section are quoted from Ref [4], page 316
- [14] M. Artin, *Algebra*, (Prentice Hall, Upper Saddle River, NJ, 1999), page 276-277
- [15] Ref. [4], Figure 7.1, page 278
- [16] Ref. [9] page 133-162
- [17] Ref. [9], page 175-176