Fractional Integration via Seven Easy Exercises

1. The gamma function

For $\mu > 0$ set

$$\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} \, dt \, .$$

Exercise 1:

Show that this integral makes sense and that

$$\Gamma(\mu + 1) = \mu \Gamma(\mu) \, .$$

Hint:

$$\Gamma(\mu+1) = -\int_0^\infty t^\mu \frac{d}{dt} e^{-t}\, dt = \int_0^\infty \left(\frac{d}{dt} t^\mu\right) e^{-t}\, dt\,.$$

Exercise 2:

Show that for n a positive integer $\Gamma(n+1) = n!$

Hint: Exercise 1 plus induction plus evaluation of the integral

$$\Gamma(1) = \int_0^\infty e^{-\frac{1}{2}}\, dt$$

to get the induction started.

2. The beta function

For $\mu, \nu > 0$ set

$$B(\mu,\nu) = \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\,dt\,.$$

Exercise 3: Show that

$$B(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\,.$$

Hint:

$$\Gamma(\mu)\Gamma(\nu) = \int_0^\infty \int_0^\infty t^{\mu-1}e^{-t}s^{\nu-1}e^{-s}\,dt\,ds$$

$$= \int_0^\infty \int_0^\infty e^{-(t+s)}t^{\mu-1}s^{\nu-1}\,dt\,ds\,.$$

By setting $x = s + t$ and $y = s$ show that this integral is equal to

$$\int_0^\infty e^{-x} \left(\int_0^x (x-y)^{\mu-1} y^{\nu-1}\, dy \right) dx\,.$$

Now make the change of variable, $y = xz$, (with x fixed) and write the inner integral as

$$x^{\mu+\nu-1} \int_0^1 (1-z)^{\mu-1} z^{\nu-1}\, dz\,.$$

Conclude that

$$\Gamma(\mu)\Gamma(\nu) = \left(\int_0^\infty x^{\mu+\nu-1} e^{-x}\, dx\right) B(\mu,\nu)$$

$$= \Gamma(\mu+\nu) B(\mu,\nu)\,.$$

Exercise 4: Compute the integral

$$\int_0^x (x-s)^{\mu-1} s^{\nu-1}\, ds\,.$$

More explicitly show that this integral is equal to

$$x^{\mu+\nu-1} B(\mu,\nu)\,.$$

Hint: With x fixed make the substitution $s = xt$.

3. Fractional integration

For $f : (0, \infty) \longrightarrow \mathbb{R}$ a bounded function and $\mu > 0$ set

$$J^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x - t)^{\mu - 1} f(t)\, dt\,.$$

Exercise 5: Show that $J^{\mu} f(0) = 0$ and

$$\frac{d}{dx} J^{\mu+1} f(x) = J^{\mu} f(x)\,.$$

Hint:

$$\frac{d}{dx} J^{\mu+1} f(x) = I + II$$

where

$$\mathrm{I} = \frac{(x-t)^{\mu} f(t)}{\Gamma(\mu+1)}\Big|_{t=x}$$

and

$$\mathrm{II} = \frac{1}{\Gamma(\mu+1)} \int_0^{\infty} \frac{d}{dx} (x-t)^{\mu} f(t)\, dt\,.$$

Exercise 6: From exercise 5 deduce that $J^{\mu+1}f = J(J^{\mu}f)$.

Exercise 7: Show that for $\mu, \nu > 0$

$$J^{\mu}(J^{\nu}f) = J^{\mu+\nu}f\,.$$

Hint: Rewrite the left hand side as

$$\frac{1}{\Gamma(\mu)\Gamma(\nu)}\left(\int_0^x (x-t)^{\mu-1}\int_0^t (t-s)^{\nu-1} f(s)\,ds\,dt\right)$$

and by reversing the integrations in s and t show that the expression in parentheses can be rewritten

$$\text{(I)}\qquad \int_0^x \left(\int_s^x (x-t)^{\mu-1}(t-s)^{\nu-1}\,dt\right) f(s)\,ds\,.$$

Now make the change of variables

$$t_{\text{new}} = t - s$$

in the t integral and rewrite it as

$$\text{(II)} \qquad \int_0^{x-s} ((x-s)-t)^{\mu-1} t^{\nu-1}\, dt\,.$$

Finally use exercise 4 to rewrite (II) in the form

$$(x-s)^{\mu+\nu-1} B(\mu,\nu)\,.$$

Conclude that the expression (I) is

$$B(\mu,\nu)\int_0^x (x-s)^{\mu+\nu-1} f(s)\,ds$$

and hence:

$$
\begin{aligned}
J^{\mu}(J^{\nu}f) &= \frac{B(\mu,\nu)}{\Gamma(\mu)\Gamma(\nu)}\int_0^x (x-s)^{\mu+\nu-1}f(s) \\
&= \frac{B(\mu,\nu)\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)}J^{\mu+\nu}f(x) \\
&= J^{\mu+\nu}f(x)
\end{aligned}
$$

by exercise 3.