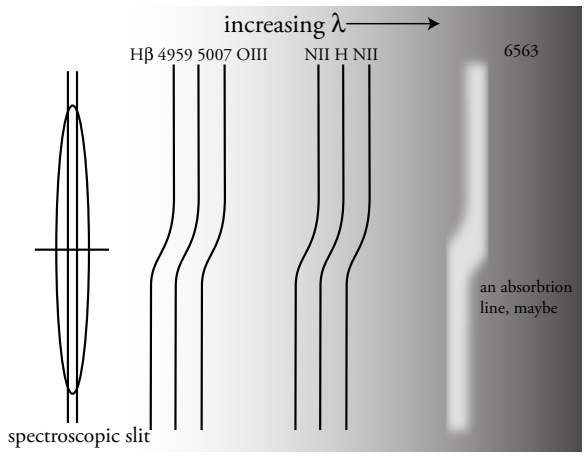
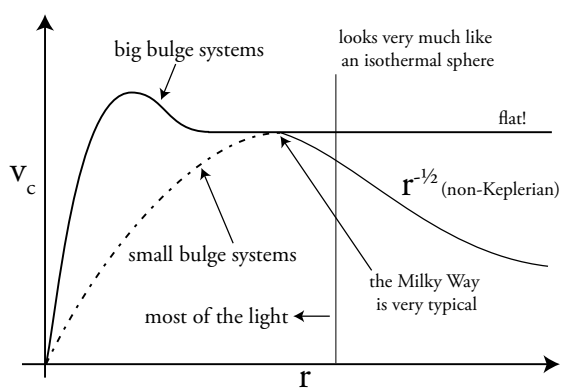
Galaxy Disks: rotation and epicyclic motion

1. Last time, we discussed how you measure the mass of an elliptical galaxy. You measure the width of the line and apply the radial Jeans equation, making some assumptions about anisotropy.

2. Galaxy disks are much easier: they are mostly ordered, with very little randomness. Once again, you use a spectrograph.



Ionized gas– HII regions and diffuse (ambient) starlight.



$v_c \sim 220 km/s$

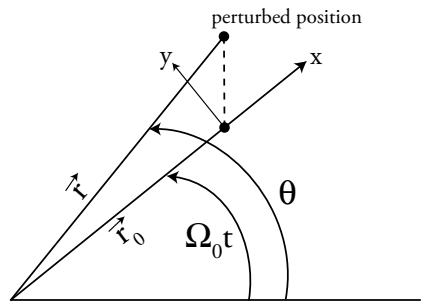
   (a) As you might expect, v_c increases as L increases. Empirical Tully-Fisher relation: $L \sim v_c^4$. Can be used as standard candles.

   (b) The corresponding Faber-Jackson relation for ellipticals is $L \sim \sigma^4$.

3. Of course, disks are not infinitely cool– there's a velocity dispersion tensor with compenents σ_{rr}^2, $\sigma_{\theta\theta}^2$, and σ_{zz}^2 in polar coordinates.

(a) Now you know how to integrate an orbit numerically, but it turns out that if you assume that the deviations from circularity are small, you can treat the problem analytically. A perturbation approach.

(b) Let's go to a coordinate system in which a particle in circular motion would be at rest.



Let

$$\Omega(r) = \frac{v_c(r)}{r}$$

$$\vec{r} = \vec{r}_o + x\hat{x} + y\hat{y}$$

$$x \approx r - r_o$$

$$y \approx r_o(\theta - \Omega_o t)$$

4. Now you know that in an inertial frame, the equation of motion is given by

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi$$

(a) In the absence of a potential, $\dot{\vec{r}} = constant$.

(b) This is not so in an accelerating frame. In a frame rotating with an angular velocity of $\vec{\Omega}_o$,

$$\ddot{\vec{r}} = -\vec{\nabla}\Phi - \underbrace{2\vec{\Omega}_o \times \vec{v}}_{Coriolis force} - \underbrace{\vec{\Omega}_o \times (\vec{\Omega}_o \times \vec{r})}_{centrifugal force}$$

The Coriolis force explains why storms rotate counterclockwise in the northern hemisphere. The centrifugal force explains why the Earth bulges at the equator.

(c) Both of these forces are fictitious, in the sense that we've chosen a non-intertial coordinate system.

(d) This is developed in section 7.2 of Symon's "Classical Mechanics" and in appendix 1.D.3 of Binney & Tremain.

5. We will substitute our expression for $\vec{r}$ into this equation, but first notice that

$$\vec{\nabla}\Phi = \frac{v_c^2}{r} = \Omega^2\vec{r} \approx \left(\underbrace{\Omega_o^2 r_o + \Omega_o^2 dx}_{\Omega_o^2\vec{r}} + 2r_o\Omega_o\frac{d\Omega}{dr}x \right)\hat{x}$$

where I've expanded the product in a 2D Taylor series around r_o.

- One vector equation:

$$\ddot{x}\hat{x} + \ddot{y}\hat{y} \approx -\Omega_o^2 r\hat{x} - 2r_o\Omega_o\frac{d\Omega}{dr}x\hat{x} - \underbrace{2\Omega_o\dot{x}\hat{y} + 2\Omega_o\dot{y}\hat{x}}_{Coriolis} + \Omega_o^2\vec{r}$$

- 2 scalar equations: $\hat{x}$:

$$\ddot{x} - 2\Omega_o\dot{y} = 4\Omega_o A_o x$$

Here, "Oort's A" is defined as

$$A_o \equiv \left(-\frac{r}{2}\frac{d\Omega}{dr}\right)_{r_o}$$

$\hat{y}$:

$$\ddot{y} + 2\Omega_o\dot{x} = 0$$

- Anyone care to suggest a solution? Small motion about an equilibrium position. Try:

$$x = x_o\sin(\kappa t + \phi)$$

$$y = y_o\cos(\kappa t + \phi)$$

These give:

$$-\kappa^2 x_o(\sin\kappa t + \phi) + 2y_o\Omega_o\sin(\kappa t + \phi) = 4\Omega_o A_o x_o\sin(\kappa t + \phi)$$

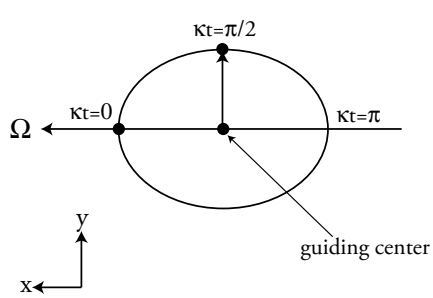
and

$$-\kappa^2 y_o\cos(\kappa t + \phi) + 2\kappa x_o\cos(\kappa t + \phi) = 0$$

$$\frac{y_o}{x_o} = \frac{2\Omega_o}{\kappa}$$

$$\kappa^2 = 4\Omega_o(\Omega_o - A_o)$$

6. What does motion look like? There are parametric equations for an ellipse in cartesian coordinates. Motion is said to be "retrograde."



7. This may be somewhat familiar. Kepler, when he introduced ellipses as the orbits of the orbits of the planets, was trying to avoid the use of epicycles. Here we are taking what looks like a giant step backwards and reintroducing them!

   (a) This is called the "epicyclic approximation" and κ is called the "epicyclic" frequency.

8. Case I: Galaxy v_c is constant;

$$\Omega(r) = \frac{v_c}{r}$$

$$A = -\frac{r}{2}\frac{d}{dr}\frac{v_c}{r} = \frac{1}{2}\frac{v_c}{r} = \frac{1}{2}\Omega$$

$$\kappa^2 = 4\Omega(\Omega - \frac{1}{2}\Omega) = 2\Omega^2$$

$$\kappa = \sqrt{2}\Omega$$

   (a) Leads to an orbit that isn't closed– a rosette.

   (b) The epicycle is complete *before* the guiding center completes an orbit. This is what you found.

   (c) $\frac{y_o}{x_o} = \sqrt{2}$– a bit squashed.

9. Case II: Kepler

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

$$A(r) = -\frac{1}{2}r\frac{3}{2}\frac{(GM)^{1/2}}{r^{3/2}} = \frac{3}{4}\Omega$$

$$\kappa^2 = 4\Omega(\Omega - A) = \Omega^2$$

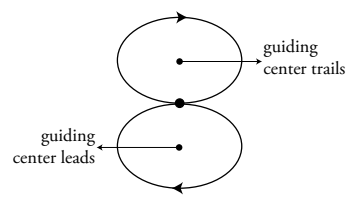
$$\kappa = \Omega$$

   (a) Here, one epicycle *per* guiding orbit; orbits close on themselves.

   (b) $\frac{y_o}{x_o} = \frac{2\Omega}{\kappa} = 2$– fairly flat: that's where Hipparchus went wrong. You can't completely avoid ellipses!

10. If you average over an orbit, you have

$$\frac{< \dot{y}^2 >_{orbit}}{< \dot{x}^2 >_{orbit}} = \frac{y_o^2}{x_o^2} = \frac{4\Omega^2}{\kappa^2}$$



But that's not the whole story. When we measure velocity dispersion at a point, we're averaging over stars.

$$\left(1 - \frac{A}{\Omega}\right) = \frac{< \sigma_{yy}^2 >_{stars}}{< \sigma_{xx}^2 >_{stars}} \frac{x_o^2}{y_o^2} = \frac{< \sigma_{\theta\theta}^2 >_{stars}}{< \sigma_{rr}^2 >_{stars}} = \frac{\kappa^2}{4\Omega^2}$$

(This comes from second moments of collisionless Boltzmann)

Case I:

$$\frac{< \sigma_{\theta\theta}^2 >_{stars}}{< \sigma_{rr}^2 >_{stars}} = \frac{1}{2}$$

We measure a larger σ in the radial direction, then in the tangential direction.