# Lecture 3: Hydrostatic Equilibrium of an Ideal Gas or Fluid

An *ideal gas or an ideal fluid* is defined with the following assumptions:
a. There are no interparticle forces.

b. It is a classical (non-relativistic) system

c. It is incompressible.

The key variables for an ideal fluid are:

$$\begin{array}{lll} \text{density, } \rho_0 & [\rho_0] & \equiv \ \mathrm{g\,cm^3} \\ \text{velocity, } \boldsymbol{v}(\boldsymbol{r}, t) & [v] & \equiv \ \mathrm{cm\,s^{-1}} \end{array}$$

## Equation of Continuity

The equation of continuity is equivalent to the *Conservation of Mass Principle*. In an ideal fluid, mass is neither created nor destroyed.

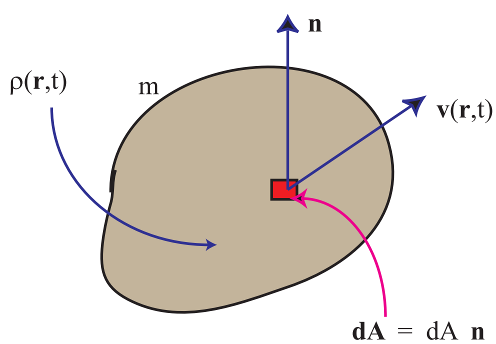


Figure 1: A volume element of a fluid. An infinitesimal area element with velocity $\boldsymbol{v}(\boldsymbol{r},\mathrm{t})$ is shown.

Conservation of mass requires that at all times

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0 \qquad \text{where} \qquad M = \int \mathrm{d}V\, \rho(\boldsymbol{r}, t)$$

Taking the *total derivative* of $\rho(\boldsymbol{r}, t)$ with respect to time,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}V\, \rho(\boldsymbol{r}, t) = \frac{\partial}{\partial t} \int \mathrm{d}V\, \rho \ + \oint \mathrm{d}\boldsymbol{A} \cdot \boldsymbol{j} = 0 \tag{1}$$

Here we have defined the mass flux,

$$\boldsymbol{j(r,t)} = \rho \boldsymbol{v} \qquad \text{with dimensions} \qquad [j] = [\rho][v] = \mathrm{g\,cm^{-3} \cdot cm\,s^{-1} = g\,cm^2\,s}$$

Applying the divergence theorem on Eqn. 1,

$$\int \mathrm{d}V \, \frac{\partial \rho}{\partial t} + \int \mathrm{d}V \, \boldsymbol{\nabla} \cdot \boldsymbol{j} = \int \mathrm{d}V \left\{ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} \right\} = 0 \tag{2}$$

This gives us *the continuity equation*

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0} \tag{3}$$

Substituting for the mass flux into the Eqn. 3,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = \frac{\partial \rho}{\partial t} + \rho \, \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \rho + \rho (\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0$$

This emphasizes the fact that the conservation of mass always holds.

Consider the total time derivative of $\rho(\boldsymbol{r}, t)$ with implicit time dependence in position:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \rho \cdot \boldsymbol{v}$$

$$\Rightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \rho \tag{4}$$

$\frac{\partial \rho}{\partial t}$ gives us the usual stationary time derivative (Eulerian derivative), i.e. the time derivative of density when the observer is at rest. On the other hand, Eqn. 4 gives us the moving time derivative (Lagrangian time derivative). This is the time derivative seen by the observer moving along with the fluid flow.

For an *incompressible fluid* (or Euler fluid) , the density $\rho$ does not vary in position or time.

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \rho = 0$$

Thus, The continuity equation (Eqn. 3) reduces to

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{5}$$

**Note**

Eqn. 5 is reminiscent of Gauss's Law for $\boldsymbol{E}$ and $\boldsymbol{B}$ in the absence of sources.

A more rigorous criterion for incompressibility is $\rho(\boldsymbol{r}, t) = \rho_0$. Though Eqn. 5 is sufficient, the latter criterion is easier to deal with.

## Hydrostatic Equilibrium

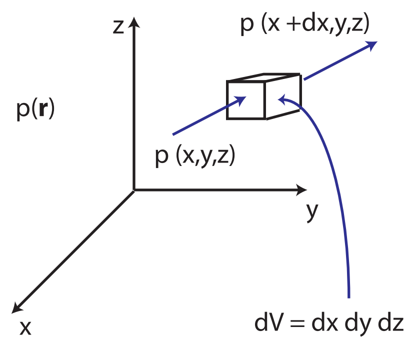


Figure 2: A region of an incompressible fluid is shown. An infinitesimal volume element is considered with pressures exerted at either face length d$x$ apart.

The pressure $p$ on a surface is defined as the force exerted on the surface per unit area. Considering the volume element of the incompressible fluid in Fig. 2, the infinitesimal forces exerted in the x-direction,

$$\begin{aligned} \mathrm{d}F_x &= -p\,(x+\mathrm{d}x,y,z)\,\mathrm{d}y\,\mathrm{d}z + p\,(x,y,z)\,\mathrm{d}y\,\mathrm{d}z \\ &= -\left(\frac{p\,(x+\mathrm{d}x,y,z) - p\,(x,y,z)}{\mathrm{d}x}\right)\mathrm{d}V \\ \Rightarrow \lim_{x\to 0}\frac{\mathrm{d}F_x}{\mathrm{d}V} &= -\frac{\partial p}{\partial x}\bigg|_{y,z} \end{aligned}$$

Thus, the derivative w.r.t. volume of the $x$-component of the force,

$$f_x = \frac{\mathrm{d}F_x}{\mathrm{d}V} = -\frac{\partial p}{\partial x}$$

And by a similar reasoning, for the $y$ and $z$-components,

$$f_y = -\frac{\partial p}{\partial y} \quad \text{and} \quad f_z = -\frac{\partial p}{\partial z}$$

This result can be expressed in vector form as

$$\boxed{\boldsymbol{f}_p = -\boldsymbol{\nabla} p} \tag{6}$$

The dimensions of $\boldsymbol{f}$ are

$$[f_p] = [\boldsymbol{\nabla} p] = [\nabla][p] = (\mathrm{cm}^{-1})\ (\mathrm{dyne\,cm}^{-2}) = \mathrm{dyne\,cm}^{-3}$$

In the case of gravity, the force is locally independent of volume, $\boldsymbol{F} = m\boldsymbol{g}$. In this case, $\boldsymbol{f}$ is simply the force per unit volume

$$\boldsymbol{f_g} = \rho \boldsymbol{g} \tag{7}$$

In hydrostatic equilibrium, $\boldsymbol{f_g} + \boldsymbol{f}_p = 0$. Thus, from Eqns. 6 & 7 we obtain the condition for equilibrium

$$\boxed{\boldsymbol{\nabla} p = \rho \boldsymbol{g}} \tag{8}$$

Note that Eqn. 8 is valid for both compressible and incompressible fluids.

***Example:*** **Pressure under the surface of a lake**

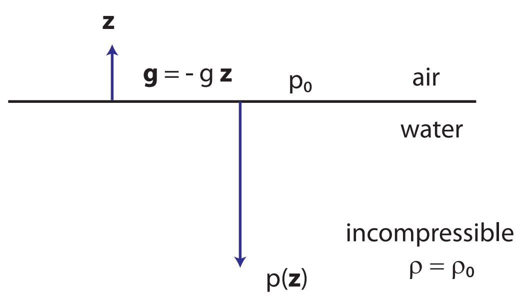


Figure 3: Hydrostatic equilibrium in a lake. The force due to gravity causes a pressure gradient with depth. The atmospheric pressure on the lake surface ($z = 0$) is $p_0$.

Assume that the water is incompressible ($\rho \equiv \rho_0$). The pressure should be isotropic in the $x$ and $y$ directions, i.e. the pressure is a function of only $z$. Hence, Eqn. 8 reduces to

$$\frac{\mathrm{d}p}{\mathrm{d}z}\,\hat{\boldsymbol{z}} \;=\; -\rho_0\, g\, \hat{\boldsymbol{z}} \qquad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}z} \;=\; -\rho_0\, g$$

Integrating up to z,

$$\begin{aligned} \int_0^z \mathrm{d}z\, \frac{\mathrm{d}p}{\mathrm{d}z} &= -\rho_0\, g \int_0^z \mathrm{d}z \\ \Rightarrow\; p(z) - p(0) &= -\rho_0\, g\, z \end{aligned}$$

Hence we obtain the useful result

$$\boxed{p(z) = p(0) - \rho_0\, g\, z} \tag{9}$$

### Non-equilibrium (Hydrodynamics)

If the forces do not balance, the net force per unit volume is given by *Newton's Second Law*

$$\begin{aligned}\rho \boldsymbol{a} &= \sum_i \boldsymbol{f}_i \\ &= -\boldsymbol{\nabla} p + \rho \boldsymbol{g}\end{aligned}$$

This gives us the *Euler equation*

$$\boxed{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g}} \tag{10}$$

As with the density functional in Eqn. 4, the total time derivative of the velocity can be expressed as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{11}$$

Thus the Euler equation can be reexpressed as

$$\boxed{\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho\,(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g}} \tag{12}$$