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A nonlinear threshold model for community response to environmental hazards

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In Predescu et al., 2002, the authors present a model for the interaction between the number of mosquito breeding sites in a community and that community's "level of consciousness" of the environmental problem. We build upon this concept here by generalizing the model and incorporating a threshold relationship between the environmental hazard, x_n, and community response, z_n; the hazard must exceed a critical level ω before a community intervenes.

$$\begin{cases} x_{n+1} = ax_nh(z_n) + bg(z_n) \\ z_{n+1} = c(x_n)u(x_n - \omega) + dz_n. \end{cases}$$

The model exhibits three different regimes of behavior as a function of the parameters: a zero-consciousness equilibrium, a positive consciousness (and low hazard) regime, and an oscillatory intermediate regime. This paper establishes global and local stability results, characterizations of trajectories and invariant regions of phase space for each of the regimes, and presents a complete characterization of the behavior of the special case in which community awareness increases at a constant rate once the threshold is reached.

Keywords: difference equations, stability, oscillations, threshold, nonlinear, step function

AMS Subject Classification: 39A11, 40A05

1. Motivation and the threshold model

The literature on ecological population models is very rich, exploring the dynamics of single-species, predator-prey, seasonally-variant and harvested populations, among other structures (see, for example, [1]-[5]). A novel application of these population concepts is presented in Predescu et al. [6], which proposes a non-linear discrete-time model for the interaction between the number of mosquito breeding sites in a community and that community's "level of consciousness" of the public health danger of such sites. A conscious community can respond in two different ways: by eliminating existing breeding sites and by slowing the rate of creation of new sites. The community's consciousness is inspired by the presence of breeding sites, and has a natural rate of decay. Denote the number of breeding sites at time n by x_n, and the level of consciousness by z_n. In the model, the two variables are related by a pair of difference equations

$$\begin{cases} x_{n+1} = ax_nh(pz_n) + bh(qz_n) \\ z_{n+1} = cx_n + dz_n. \end{cases} \tag{1}$$

where a represents the fraction of breeding sites that survive between time points in the absence of intervention, b is the rate of creation of new sites (in the absence of intervention), c relates the sensitivity of the level of consciousness to the presence of breeding sites, and d measures the decay of consciousness between time points. The function $h(\cdot)$ is a continuous, monotonically-decreasing map from $[0, \infty)$ to $[0, 1]$ such that $h(0) = 1$ and $lim_{z\to\infty}h(z) = 0$. The fraction $1 - h(pz_n)$ measures the proportion of existing

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breeding sites that are actively destroyed by community intervention, while $h(qz_n)$ measures the reduction in the rate of creation of new sites (perhaps by residents taking extra care to avoid creating standing pools of water). The parameters p and q allow the response of a community through each of the two types of intervention to scale differently.

There are several assumptions in the model of system (1) that are worth considering. We often observe large groups of people exhibiting a *threshold* response to environmental problems; the status quo persists until a situation reaches a critical level, at which point the community acknowledges the problem and focuses its resources on preventing a disaster. This phenomenon also occurs in scientific communities where a large burden of proof is required before an intervention is recommended; Kriebel et al. present this idea as a bias toward high Type-II error in their discussion of the precautionary principle applied to environmental science [7]. This naturally suggests replacing the linear dependence cx_n in system (1) with a more general threshold dependence $c(x_n)u(x_n - \omega)$, where $c(\cdot)$ is a non-decreasing, nonnegative continuous function defined on $[0, \infty)$, and $u(\cdot)$ is a "step function," defined by

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Thus, ω represents the hazard threshold necessary to arouse consciousness. Without loss of generality, we will also assume that $c(x_n)$ is strictly positive for $x_n \in (0, \infty)$. The choice of an increasing $c(\cdot)$ reflects an increased willingness to engage in environmental protection behaviors in areas which are more substantially burdened by environmental problems, an assertion which has been documented by Inglehart in his comparative analysis of 43 countries [8].

Additionally, the assumption that the two types of community intervention have the same functional dependence on the level of consciousness is overly restrictive (and, in fact, unnecessary for proving most of the results in [6]); it is natural to replace the functions $h(pz_n)$ and $h(qz_n)$ in system (1) with the functions $h(z_n)$ and $g(z_n)$, respectively, where $g(\cdot)$ has the same requirements as $h(\cdot)$.

These conditions lead to the following discrete-time model:

$$\begin{cases} x_{n+1} = ax_n h(z_n) + bg(z_n) \\ z_{n+1} = c(x_n)u(x_n - \omega) + dz_n. \end{cases} \tag{2}$$

We assume that $b > 0$ and $0 < a, d < 1$. Observe that the nonnegative orthant is invariant under system (2); we will focus on initial conditions within this quadrant. If $c(x) = x$ and $\omega = 0$, we recover system (1).

In the following sections, we will explore the behavior of this threshold model in its full generality, and completely characterize a special case: $c(x) = c$, a positive constant. This special case represents a community's behavior when an environmental hazard prompts the activation of an intervention program that recruits new volunteers at a constant rate while the hazard is above the threshold, then is phased out once the hazard is no longer judged a threat.

2. Fixed points and local stability analysis

A pair of fixed points $(\bar{x}, \bar{z})$ of system (2) must satisfy

$$\begin{cases} \bar{x} = a\bar{x}h(\bar{z}) + bg(\bar{z}) \\ \bar{z} = c(\bar{x})u(\bar{x} - \omega) + d\bar{z}. \end{cases}$$

The step function requires us to analyze two separate cases.

- **Regime I:** $\omega > \bar{x}$. Then $\bar{z} = 0$ and $\bar{x} = b/(1 - a)$. This is a unique fixed point:

$$(x_I, z_I) \triangleq \left(\frac{b}{1 - a}, 0 \right). \tag{3}$$

This point represents the hazard level in its natural equilibrium, with no human intervention, and no community consciousness of the problem. This may be acceptable if a hazard level of x_I has a low public health impact.

- **Regime II:** $\omega < \bar{x}$**.** Then $\bar{z} = c(\bar{x})/(1-d)$ and $\bar{x} = bg(\bar{z})/(1-ah(\bar{z}))$. Does this pair of simultaneous equations have a solution? Define the functions $r(\cdot)$ and $s(\cdot)$ as

$$r(z) \triangleq \frac{bg(z)}{1-ah(z)}$$
$$s(x) \triangleq \frac{c(x)}{1-d}.$$

Note that $r(z)$ is a continuous, decreasing function of its argument, which takes maximum value $r(0) = b/(1-a)$ and approaches zero as z approaches infinity. A fixed point $(\bar{x}, \bar{z})$ of system (2) that satisfies $\omega < \bar{x}$ is equivalent to finding a $\bar{z}$ such that $\bar{z} = s(r(\bar{z}))$. To determine whether such a $\bar{z}$ exists, consider the function $m(z) \triangleq s(r(z)) - z$. Observe that

$$m(0) = \frac{1}{1-d} c\left(\frac{b}{1-a}\right) > 0$$

and that as $z \to \infty$, $m(z)$ diverges to $-\infty$. Since $m(z)$ is strictly decreasing, it must have a unique zero at some positive value $\bar{z}$, and thus system (2) has a unique fixed point that satisfies $\omega < \bar{x}$. Denote this fixed point as the implicit solution to the following set of simultaneous equations:

$$(x_{II}, z_{II}) \triangleq (r(z_{II}), s(x_{II}))\,. \tag{4}$$

This fixed point represents persistent positive levels of consciousness and environmental hazard. Since $x_I = r(0) > r(z_{II}) = x_{II}$, the hazard level x_{II} is less than the hazard level of the "zero-consciousness" fixed point, x_I.

The non-empty interval between x_I and x_{II} implies the existence of a third relevant region of parameter space.

- **Regime III:** $x_{II} \leq \omega \leq x_I$**.** No fixed point exists.

Next, we consider the local stability of the fixed points of Regimes I and II.

2.1. *Local stability analysis of fixed points of Regimes I and II*

In each of Regimes I and II, the fixed point is bounded away from the threshold ω, and we'll assume that $r(\cdot)$ and $s(\cdot)$ are differentiable at the fixed points of each Regime. Thus, a local stability analysis via the Jacobian matrix can be performed. The Jacobian $J(x, z)$ is given by

$$J(x,z) = \begin{bmatrix} ah(z) & axh'(z) + bg'(z) \\ c'(x)u(x-\omega) + c(x)\delta(x-\omega) & d \end{bmatrix}$$

where $h'(\cdot)$ denotes the derivative of $h(\cdot)$ with respect to its argument and $\delta(\cdot)$ denotes the Dirac delta function.

- **Regime I:** $\omega > x_I$**.** Then

$$J(x_I, z_I) = \begin{bmatrix} a & axh'(z_I) + bg'(z_I) \\ 0 & d \end{bmatrix}$$

The eigenvalues of this matrix are a and d; by assumption, $0 < a, d < 1$. Thus, we have the following proposition.

Proposition 2.1: *The fixed point of Regime I is locally asymptotically stable for all choices of $c(\cdot)$.*

- **Regime II:** $\omega < x_{II}$. Then

$$J(x_{II}, z_{II}) = \begin{bmatrix} ah(z_{II}) & ax_{II}h'(z_{II}) + bg'(z_{II}) \\ c'(x_{II}) & d \end{bmatrix}$$

To characterize the local stability of this fixed point, we can apply the trace and determinant criteria set forth, for example, in [9]. Both the trace and determinant of $J(x_{II}, z_{II})$ are nonnegative. Since

$$\mathrm{tr}[J(x_{II}, z_{II})] = ah(z_{II}) + d < 1 + adh(z_{II}) < 1 + \det[J(x_{II}, z_{II})],$$

the fixed point cannot be a saddle point. If $\det[J(x_{II}, z_{II})] < 1$, the fixed point is locally asymptotically stable; if the determinant is greater than 1, the fixed point is a repeller. Observe that

$$\begin{aligned} \det[J(x_{II}, z_{II})] &= adh(z_{II}) - c'(x_{II})(ax_{II}h'(z_{II}) + bg'(z_{II})) \\ &= adh(z_{II}) - (1-d)s'(x_{II})r'(z_{II})(1 - ah(z_{II})) \end{aligned}$$

Some algebraic manipulation leads us to the following criterion for the asymptotic stability of (x_{II}, z_{II}).

Proposition 2.2: *If*

$$|s'(x_{II})r'(z_{II})| < \frac{1 - adh(z_{II})}{(1-d)(1 - ah(z_{II}))},$$

then the fixed point of Regime II is locally asymptotically stable. If the inequality is reversed, the fixed point is a repeller.

The criterion of Proposition (2.2) implies that if the level of consciousness increases too rapidly around the point x_{II} (represented by $s'(x_{II})$), or if the community intervention is too drastic around the point z_{II} (represented by $r'(z_{II})$), then the fixed point will not be stable. Not surprisingly, overcompensation at a critical point can lead to destabilizing behavior.

Corollary 2.3: *When $c(x) = c$, a positive constant, the fixed point of Regime II, $(x_{II}, z_{II}) = (r(z_{II}), c/(1-d))$ is locally asymptotically stable.*

Beyond local stability, it is indeed possible to determine the *global* asymptotic stability of the fixed point in Regime I, and for the special case of $c(x) = c$ in Regime II. First, we consider some properties of all trajectories of system (2).

3. Behavior of trajectories

The following two lemmas will be very useful throughout the analysis of system (2).

Lemma 3.1: *In any regime, $x_{n+1} > x_n$ if and only if $x_n < r(z_n)$.*

Proof: This observation is simple to demonstrate:

$$x_{n+1} = ax_nh(z_n) + bg(z_n) > x_n \Leftrightarrow x_n < \frac{bg(z_n)}{1 - ah(z_n)} = r(z_n).$$

□

Lemma 3.2: *In any regime, for any point (x_n, z_n) such that $x_n > \omega$, $z_{n+1} > z_n$ if and only if $z_n < s(x_n)$. If $x_n < \omega$, then $z_{n+1} < z_n$ for all values of z_n.*

Proof: When $x_n > \omega$:

$$z_{n+1} = c(x_n) + dz_n > z_n \Leftrightarrow z_n < \frac{c(x_n)}{1-d} = s(x_n).$$

When $x_n < \omega$:

$$z_{n+1} = dz_n < z_n.$$

□

Define z_I^s and x_I^r as follows

$$\begin{aligned} z_I^s &\triangleq s(x_I) \\ x_I^r &\triangleq r(z_I^s) = r(s(x_I)). \end{aligned}$$

Note that $z_I^s \geq z_{II}$ implies that $x_I^r \leq x_{II}$. Additionally, for $\omega < x_I$, define z_ω implicitly as

$$\omega \triangleq r(z_\omega).$$

Such a value exists because $r(\cdot)$ is continuous and ω is contained within its range. Since $r(\cdot)$ is a decreasing function, $\omega < x_{II}$ implies that $z_\omega > z_{II}$.

Lemmas 3.1 and 3.2 lead to the following proposition.

Proposition 3.3: *If $\omega < x_I$, the closed rectangle $(x, z) = [\min(x_I^r, \omega), x_I] \times [0, \max(z_I^s, z_\omega)]$ is invariant and attracting.*

Proof: If $\min(x_I^r, \omega) = x_I^r$, then $\max(z_I^s, z_\omega) = z_I^s$ (and vice versa). Observe that x_{n+1} is increasing in x_n and decreasing in z_n, and that z_{n+1} is increasing in both x_n and z_n; to demonstrate invariance, we need only check the corners of the rectangle.

If $\omega < x_I^r$, then the rectangle is $(x, z) = [\omega, x_I] \times [0, z_\omega]$.

$$\begin{aligned} x_{n+1} &\leq ax_I h(0) + bg(0) = ax_I + b = x_I \\ x_{n+1} &\geq a\omega h(z_\omega) + bg(z_\omega) = \omega \\ z_{n+1} &\leq c(x_I) + dz_\omega = (1-d)z_I^s + dz_\omega < z_\omega \\ z_{n+1} &\geq c(\omega) + d \cdot 0 > d \cdot 0 = 0. \end{aligned}$$

If $x_I^r < \omega < x_I$, then the rectangle is $(x, z) = [x_I^r, x_I] \times [0, z_I^s]$.

$$\begin{aligned} x_{n+1} &\leq ax_I h(0) + bg(0) = x_I \\ x_{n+1} &\geq ax_I^r h\left(z_I^s\right) + bg\left(z_I^s\right) = x_I^r \\ z_{n+1} &\leq c(x_I) + dz_I^s = (1-d)z_I^s + dz_I^s = z_I^s \\ z_{n+1} &\geq 0 + d \cdot 0 = 0. \end{aligned}$$

To show that this region is attracting, we first observe that there are no fixed points of system (2) on the boundaries of the rectangle; thus, it is not possible for trajectories to accumulate at the boundary. For any initial conditions (x_0, z_0), there must exist an m such that $x_m < x_I$. To see this, observe that for any n, $x_n > x_I$ implies that $x_n > r(z_n)$ for all possible associated z_n, and thus by Lemma 3.1, $x_{n+1} < x_n$. Since it is not possible for trajectories to accumulate at the boundary $x_n = x_I$, such an m must exist. By a parallel argument, once the x-component of a trajectory is below x_I, there exists a $p > m$ such that $z_p < \max(z_I^s, z_\omega)$, since $\max(z_I^s, z_\omega) > s(z_n)$ for any (x_n, z_n) such that $x_n < x_I$. It remains to show that the x-component of this trajectory must exceed $\min(x_I^r, \omega)$; again, this observation follows from the fact that for any n such that $z_n < \max(z_I^s, z_\omega)$ and $x_n < \min(x_I^r, \omega)$, the x-component of the trajectory is strictly increasing. □

Corollary 3.4: *In Regime III, the closed rectangle* $(x, z) = [x_I^r, x_I] \times [0, z_I^s]$ *is invariant and attracting.*

Proof: In Regime III, $x_{II} < \omega < x_I$. Since $x_I^r < x_{II}$ in all regimes, the corollary follows. □

Corollary 3.5: *For* $c(x) = c > 0$*, the closed rectangle* $(x, z) = [\omega, x_I] \times [0, z_\omega]$ *is invariant and attracting for Regime II, and the closed rectangle* $(x, z) = [x_{II}, x_I] \times [0, z_{II}]$ *is invariant and attracting for Regime III.*

Proof: When $c(x) = c$, $x_{II} = x_I^r$, $z_I^s = z_{II}$ and the result follows. □

4. Global stability in Regime I

This section will demonstrate the global asymptotic stability of the fixed point of Regime I.

Proposition 4.1: *In Regime I, for any* (x_0, z_0)*, there exists an* N *such that* $x_n < \omega$ *for all* $n \geq N$*.*

Proof: Since the functions $h(\cdot)$ and $g(\cdot)$ map the non-negative axis to the interval $[0, 1]$, we can compute a simple upper bound on x_{n+1} by observing that

$$x_{n+1} = a x_n h(z_n) + b g(z_n) \leq a x_n + b. \tag{5}$$

The upper bound in equation (5) has the following closed-form expression:

$$x_n \leq \left(x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} = (x_0 - x_I) a^n + x_I. \tag{6}$$

The upper bound on x_n converges to x_I. Since Regime I is defined by the inequality $\omega > x_I$, we can conclude that for any small ϵ, there must exist an N such that $x_n \leq x_I + \epsilon < \omega$ for all $n > N$. □

Next, we present a useful lemma.

Lemma 4.2: *Define*

$$A(n) = \sum_{j=0}^{n-1} c_{n,j} a^j$$

where $0 < a < 1$ *and the coefficients* $c_{n,j}$ *have the following properties:*

(i) $0 \leq c_{n,j} \leq 1$ *for every* n *and* j

(ii) there exist constants p *and* q *in the open interval* $(0, 1)$ *such that* $\lim_{n\to\infty} c_{n,j} = pq^j$ *for every* j*.*

Then

$$\lim_{n\to\infty} A(n) = \frac{p}{1 - aq}.$$

Proof: See Appendix A. □

Theorem 4.3: *Every trajectory in Regime I converges to the fixed point* (x_I, z_I)*.*

Proof: By Proposition 4.1, we can constrain our analysis to the set of initial conditions (x_0, z_0) such that $x_0 < \omega$; the trajectory of x will remain below ω. In this case, z_{n+1} is given by $z_{n+1} = d z_n$, which has the closed-form expression $z_n = z_0 d^n$. Substituting this observation into the equation for x_{n+1} in system (2) yields a linear, non-autonomous difference equation:

$$x_{n+1} = ax_n h(d^n z_0) + bg(d^n z_0). \tag{7}$$

Equation (7) has the following closed-form expression (given, for example, in [9]):

$$x_n = \prod_{i=0}^{n-1} ax_0 h(z_i) + \sum_{k=0}^{n-1} \left(\prod_{i=k+1}^{n-1} ah(z_i) \right) bg(z_k) = a^n x_0 \prod_{i=0}^{n-1} h(z_i) + b \sum_{j=0}^{n-1} a^j g(z_{n-j-1}) \prod_{i=n-j}^{n-1} h(z_i). \tag{8}$$

What is the limiting behavior of equation (8) as $n \to \infty$? Consider the first term, $a^n x_0 \prod_{i=0}^{n-1} h(z_i)$. Since $|a| < 1$ and $|h(z_i)| \leq 1$ for every i, this product converges to zero as $n \to \infty$.

The second term, $b \sum_{j=0}^{n-1} a^j g(z_{n-j-1}) \prod_{i=n-j}^{n-1} h(z_i)$ is more complicated. Denote the summation by $A(n)$ and define

$$c_{n,j} = g(z_{n-j-1}) \prod_{i=n-j}^{n-1} h(z_i). \tag{9}$$

We can rewrite $A(n)$ as

$$A(n) = \sum_{j=0}^{n-1} c_{n,j} a^j. \tag{10}$$

Observe the following properties of the coefficients $c_{n,j}$:

(i) $0 \leq c_{n,j} \leq 1$ for every n and j. Since each $c_{n,j}$ is a product of nonnegative elements that have magnitude less than or equal to 1, this observation follows.

(ii) $\lim_{n\to\infty} c_{n,j} = g(0)\, h(0)^j = 1$ for every j. To prove this, we note that for any finite j, $c_{n,j}$ is a product of $j+1$ terms, the first of which has the form $g(z_{n-r})$ while the next j have the form $h(z_{n-s})$ for r and s finite. Since z_{n-s} converges to 0 as $n \to \infty$, we can invoke the continuity of the function $h(\cdot)$ and conclude that each $h(z_{n-s})$ converges to $h(0)$. An analogous statement is true for $g(z_{n-r})$. Thus, the finite product $c_{n,j}$ converges to $g(0)\, h(0)^j = 1$ as $n \to \infty$.

These properties satisfy the criteria of Lemma 4.2, and allow us to conclude that the closed-form expression of equation (8) must converge to $bg(0)/(1 - ah(0)) = b/(1-a) = x_I$. □

Lemmas 3.1 and 3.2 allow us to obtain a broad qualitative picture of the behavior of trajectories in all regimes. For Regime I, a plot of x_n v. z_n is given in Figure 1 for one particular choice of $f(\cdot)$ and $g(\cdot)$, and for $c(x) = c$. The directions of change in the $x-$ and $z-$ directions are indicated by arrows, per the observations of the Lemmas 3.1-3.2. Several sample trajectories are also illustrated.

5. Behavior in Regime II

As discussed in Section 2.1, the stability of the fixed point of Regime II depends on the parameters of the system and the values that $h(\cdot)$, $g(\cdot)$ and $c(\cdot)$ take at the fixed point. Recall that in Proposition 3.3, we were able to identify an invariant and attracting region of phase space that contains this fixed point, which provides a useful bound on the behavior of trajectories.

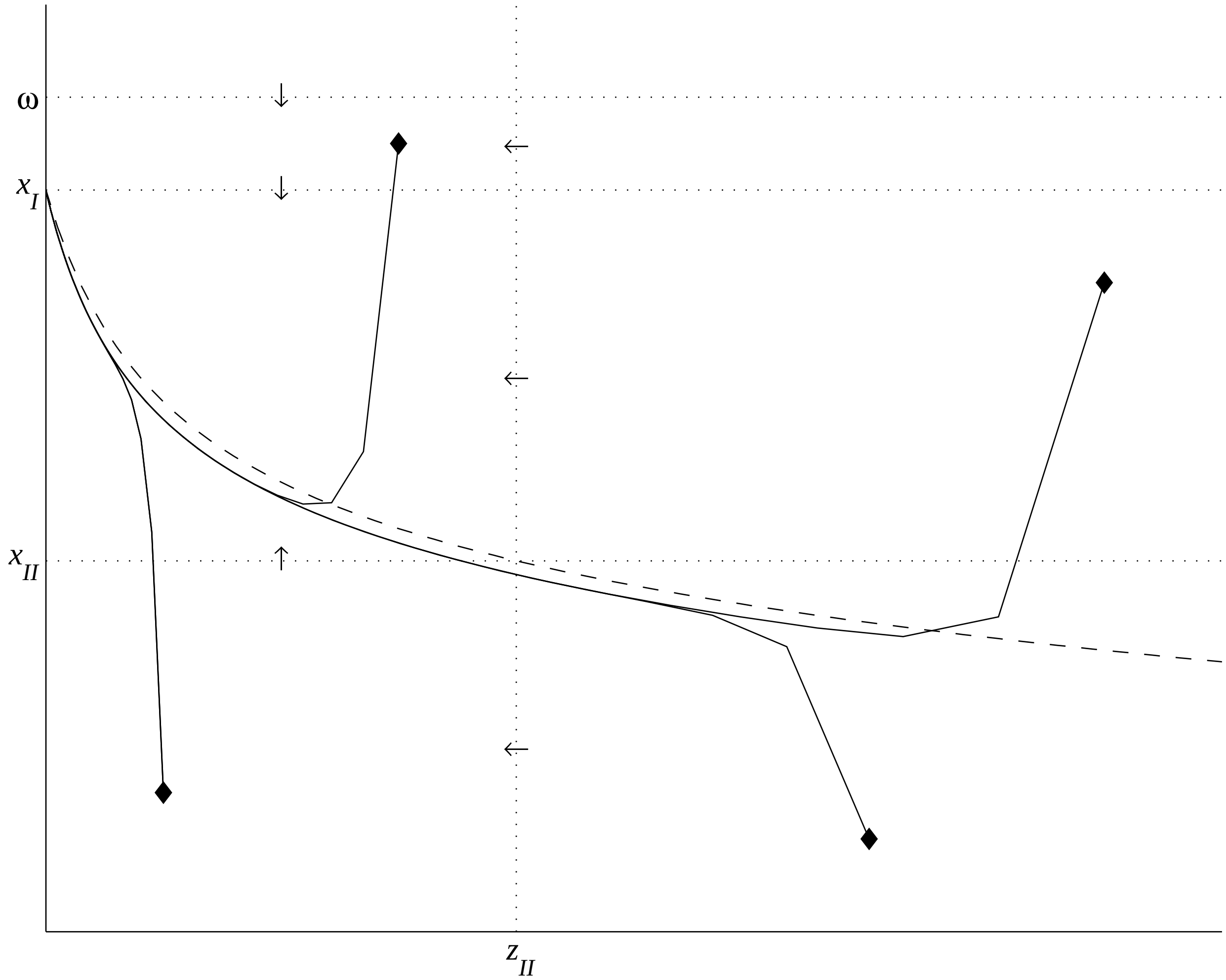

Figure 1. Regime I behavior, for $h(z) = 1/(1+z)$, $g(z) = 1/(1+\alpha z)$, and $s(x) = c(x)/(1-d) = c/(1-d) = z_{II}$. The function $r(z)$ is given by the dashed curve. The directions of trajectories across boundaries are indicated by arrows. Several example trajectories through the phase space are given by the solid lines, with the initial condition indicated by a diamond.
[$\omega = 9$, $\alpha = 0.1$, $a = 0.5$, $b = 4$, $c = 0.2$, $d = 0.9$.]

5.1. *Global stability of Regime II fixed point for* $c(x) = c$

Next, we focus on the behavior of system (2) for $c(x) = c$, a positive constant. With this choice of $c(\cdot)$, all trajectories (x_n, z_n) converge to the fixed point (x_{II}, z_{II}).

Theorem 5.1: *For $c(x) = c$, every trajectory in Regime II converges to the fixed point (x_{II}, z_{II}).*

Proof: Corollary 3.5 allows us to conclude that every trajectory eventually enters the invariant rectangle $(x, z) = [\omega, \infty) \times [0, z_\omega]$. For notational convenience, choose the time origin as a point at which the trajectory is contained in this interval and denote that point by (x_0, z_0). Then for $n \geq 0$, $z_{n+1} = c + dz_n$. This system has the following closed-form solution:

$$z_n \leq \left(z_0 - \frac{c}{1-d}\right) d^n + \frac{c}{1-d} = (z_0 - z_{II})d^n + z_{II}. \tag{11}$$

As $n \to \infty$, equation (11) converges to z_{II}. This expression for z_n allows us to treat the expression for x_{n+1} in system (2) as a linear function of x_n with time-varying coefficients; such an equation has the closed-form solution given in equation (8). Since z_n converges to z_{II}, we can apply Lemma 4.2 again to conclude that x_n converges to $bg(z_{II})/(1 - ah(z_{II})) = x_{II}$. □

5.2. *Behavior of trajectories in Regime II*

Figure 2 illustrates Regime II behavior for $c(x) = c$. The diagram indicates two attracting and invariant rectangles contained within the rectangle discussed in Corollary 3.5: the rectangle $(x, z) = [x_{II}, x_I] \times [0, z_{II}]$ and the rectangle $(x, z) = [\omega, x_{II}] \times [z_{II}, z_\omega]$. All trajectories in the phase plane must converge to one of these two rectangles. Depending upon the initial conditions, the trajectories of x_n may approach x_{II} from above or below (while concurrently approaching z_{II} from below or above, respectively).

Figure 3 presents an example of Regime II behavior for a choice of $c(\cdot)$ under which the fixed point is a repeller; this system demonstrates interesting attractive behavior.

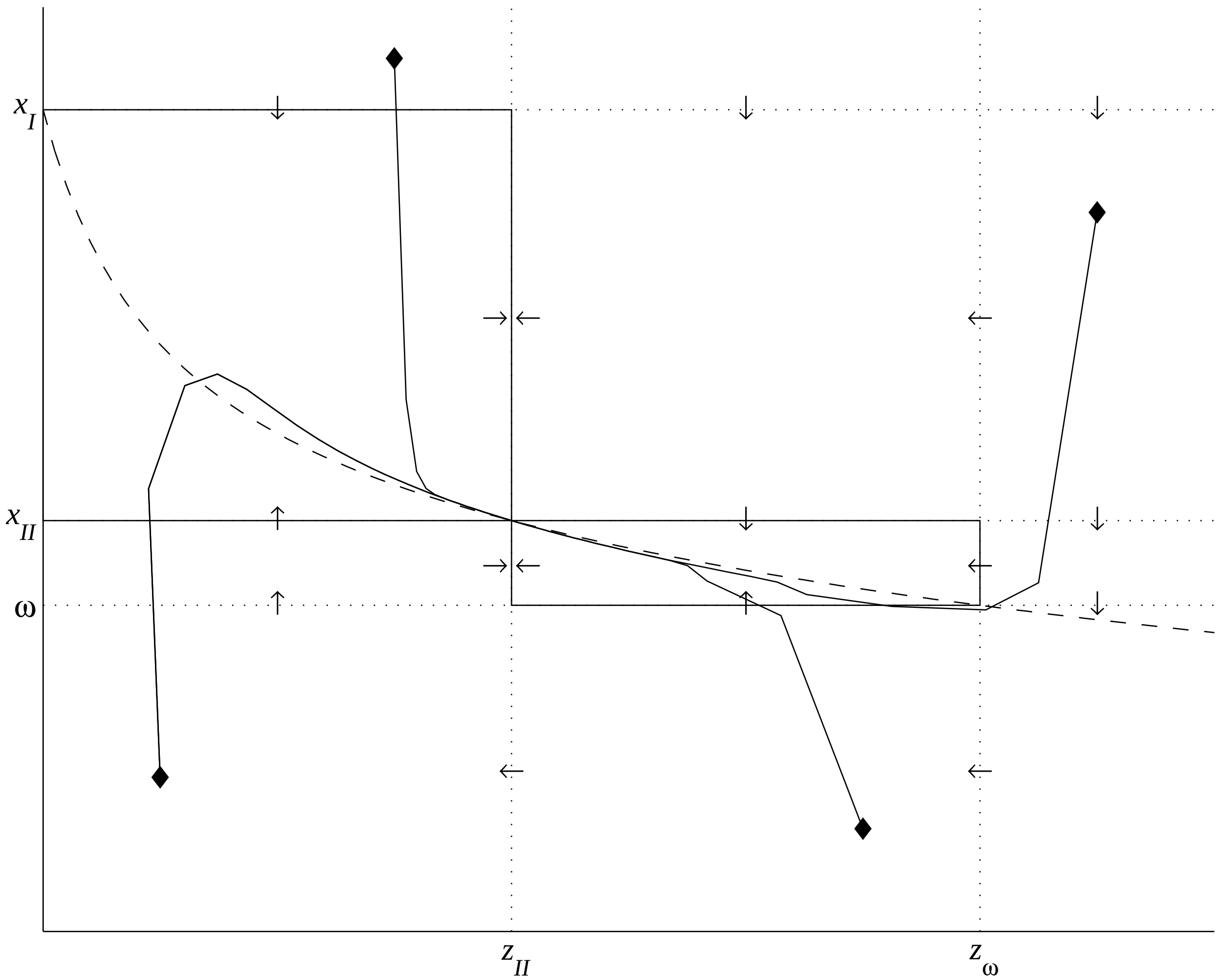

Figure 2. Regime II behavior, for $h(z) = 1/(1 + z)$, $g(z) = 1/(1 + \alpha z)$, and $s(x) = c(x)/(1 - d) = c/(1 - d) = z_{II}$. The function $r(z) = bg(z)/(1 + ah(z))$ is given by the dashed curve. Invariant rectangles are indicated in bold, and the direction of trajectories across boundaries are indicated by arrows. Several example trajectories through the phase space are given by the solid lines, with the initial condition indicated by a diamond. [$\omega = 3.17$, $\alpha = 0.1$, $a = 0.5$, $b = 4$, $c = 0.2$, $d = 0.9$.]

6. Behavior in Regime III

Recall that Regime III is characterized by $x_{II} < \omega < x_I$, and that no fixed point exists. We can make several observations about the behavior of trajectories in this parameter regime.

Proposition 6.1: *If there exists an j such that $x_j < \omega$, then there exists an $k > j$ such that $x_k > \omega$. Similarly, for any $x_k > \omega$, there exists a $m > k$ such that $x_m < \omega$.*

Proof: Assume that $x_j < \omega$ and that the forward trajectory $x_n, n \geq j$ never exceeds ω. Then $z_{n+i} = d^i z_n$

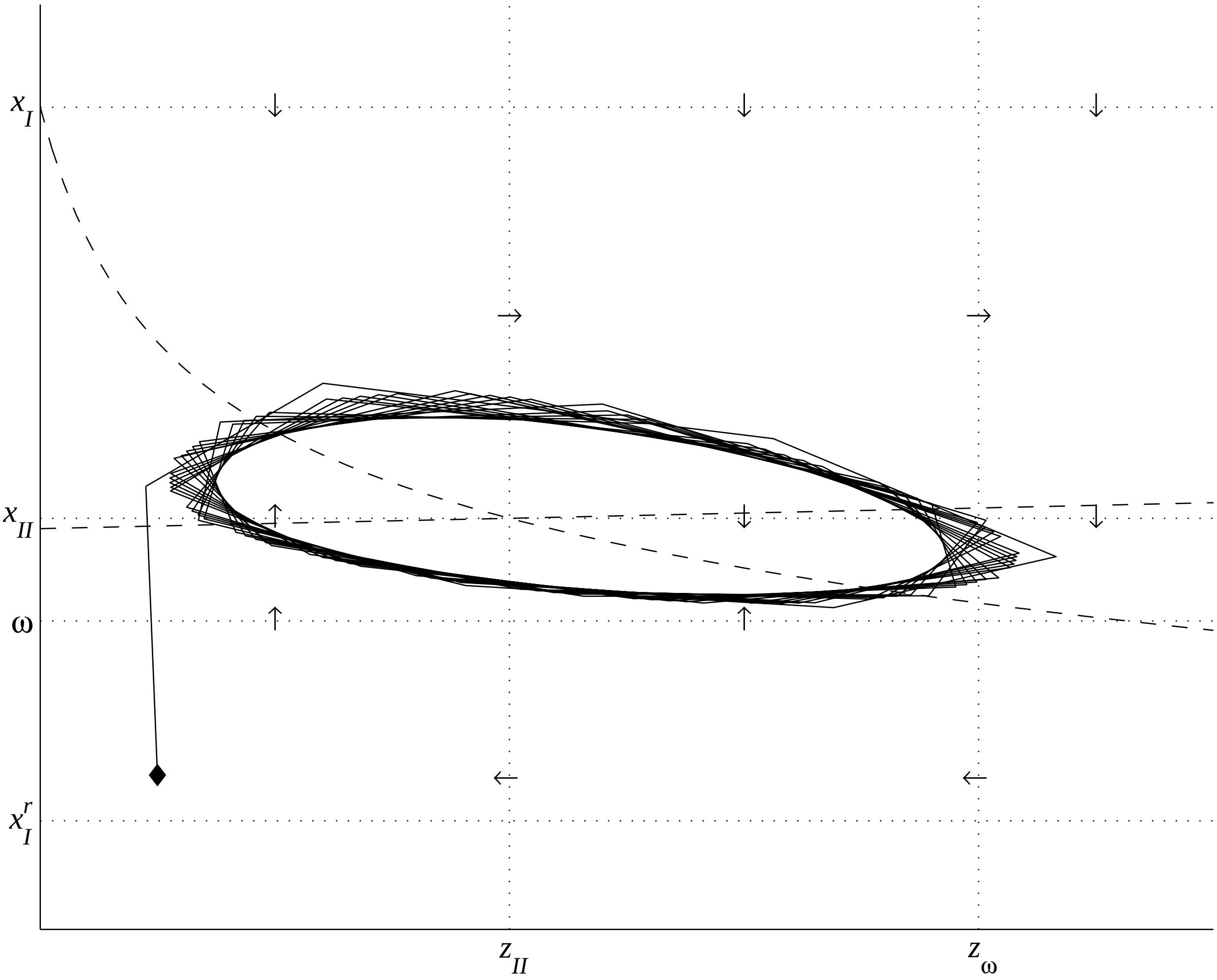

Figure 3. Regime II behavior, for $h(z) = 1/(1+z)$, $g(z) = 1/(1+\alpha z)$, and $c(x) = c_1 \arctan(x - x_{II}) + c_2$. The functions $r(z)$ and $s(x)$ are given by the dashed curves. The direction of trajectories across boundaries are indicated by arrows. One example trajectory through the phase space is given by the solid line, with the initial condition indicated by a diamond. [$\omega = 3$, $\alpha = 0.1$, $c_1 = 2$, $c_2 = 1.8$, $a = 0.5$, $b = 4$, $c = 0.2$, $d = 0.9$.]

for all $i \geq 0$, which converges to zero. Invoking Lemma 4.2, we can conclude that x_n converges to $x_I > \omega$, so there must exist a k such that $x_k > \omega$. Similarly, assume that $x_k < \omega$ and that the forward trajectory of x never drops below ω. By the same argument, z_n must converge to z_{II}, which requires that x_n converge to $x_{II} < \omega$, so there must exist an m such that $x_m < \omega$. □

Proposition 6.2: *If there exists an j such that $z_j < z_\omega$, then there exists an $k > j$ such that $z_k > z_\omega$. Similarly, for any $z_k > z_\omega$, there exists a $m > k$ such that $z_m < z_\omega$.*

Proof: Proposition 6.1 implies that the trajectory of x makes an infinite number of transitions from below ω to above ω. For each of these transitions, there must exist a time index j such that if $x_j < \omega$, $x_{j+1} > \omega$. Then

$$\omega < x_{j+1} = a x_j h(z_j) + b g(z_j) < a\omega h(z_j) + b g(z_j) \Rightarrow \omega < r(z_j) \Rightarrow z_j < z_\omega.$$

Every transition of x across ω from below must be followed by a transition across ω from above. If $x_k > \omega$ and $x_{k+1} < \omega$, then

$$\omega > x_{j+1} = a x_j h(z_j) + b g(z_j) > a\omega h(z_j) + b g(z_j) \Rightarrow \omega > r(z_j) \Rightarrow z_j < z_\omega.$$

Thus, every transition of x across ω is associated with a transition of z across z_ω, and the result follows. □

In Regime III, we'll define semicycles of x with respect to the threshold ω.

Definition 6.3: In Regime III, a positive semicycle of x with respect to ω is a sequence of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to ω, such that

$$\text{either } l = 0 \text{ or } l > 0 \text{ and } x_{l-1} < \omega$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} < \omega.$$

Proposition 6.4: *Once the trajectory has entered the invariant rectangle described in Corollary 3.4, all positive semicycles of x with respect to ω have length at least two.*

Proof: Let $x_{l-1} < \omega$ and $x_l > \omega$. Then $z_l = dz_{l-1} < z_{l-1}$, and

$$x_{l+1} = ax_l h(z_l) + bg(z_l) = ax_l h(dz_{l-1}) + bg(dz_{l-1}) > ax_{l-1}h(z_{l-1}) + bg(z_{l-1}) = x_l > \omega.$$

□

Propositions 6.1-6.2 allow us to conclude that trajectories of x in Regime III will oscillate about the threshold ω, while remaining bounded. Similarly, trajectories of z will oscillate about the value z_ω, while remaining bounded. An example of this behavior is depicted in Figure 4. As ω is varied, the period and character of the oscillations change, but simulations indicate that when $c(x) = c$, for any fixed ω, the trajectory settles into a unique limit cycle, regardless of initial conditions.

7. Conclusions and future work

This paper has explored the behavior of system (2), and identified three different regimes of behavior characterized by the threshold level ω. When $\omega > x_I$, the fixed point given by equation (3) is globally asymptotically stable. When $\omega < x_I I$, we have characterized a local asymptotic stability condition for the fixed point of equation (4), and shown that for the special case of $c(x) = c$, this fixed point is globally asymptotically stable. When $x_I I < \omega < x_I$, the system exhibits sustained oscillations.

The original model presented in [6] was formulated for the control of mosquito-borne diseases such as dengue, but the generalization represented by system (2) is readily applicable to many other environmental problems. One can imagine a similar model used to describe the quantity of toxic effluent that feeds into a community's water supply, or the growth of a landfill. Additionally, in order to apply a model that depends on a community's "level of consciousness," one must identify an appropriate proxy for this variable. Depending upon the available sociological data, it may be appropriate to use a measure like the amount of money allocated for hazard mitigation, survey results of intervention activities, or the number of citizens enrolled in environmental protection groups.

There are many directions in which to continue this investigation.

- Improving the characterization of the oscillatory behavior of Regimes II and III. Can we obtain attractivity criteria for locally stable fixed points in Regime II?
- Using the model to choose public health protection strategies. As an example, consider the mosquito breeding site control scenario. There are several ways in which a policy-maker can change the dynamics of this system to achieve different ends. The first is to keep all existing active control strategies in place (i.e., leave the functions $h(\cdot)$ and $g(\cdot)$ fixed) while making the community more aware of the public health threat (i.e., lowering ω). This will have an environmental impact only if the change in ω induces a transition from Regime I to Regime III, or from Regime III to Regime II. Otherwise, the cost and effort associated with lowering ω will be wasted. An alternate approach is to put resources into changing the shape of the functions $h(\cdot)$ and $g(\cdot)$. If a community is already operating in Regime II, then implementing stronger intervention measures will lower the equilibrium breeding site population. Before any of these

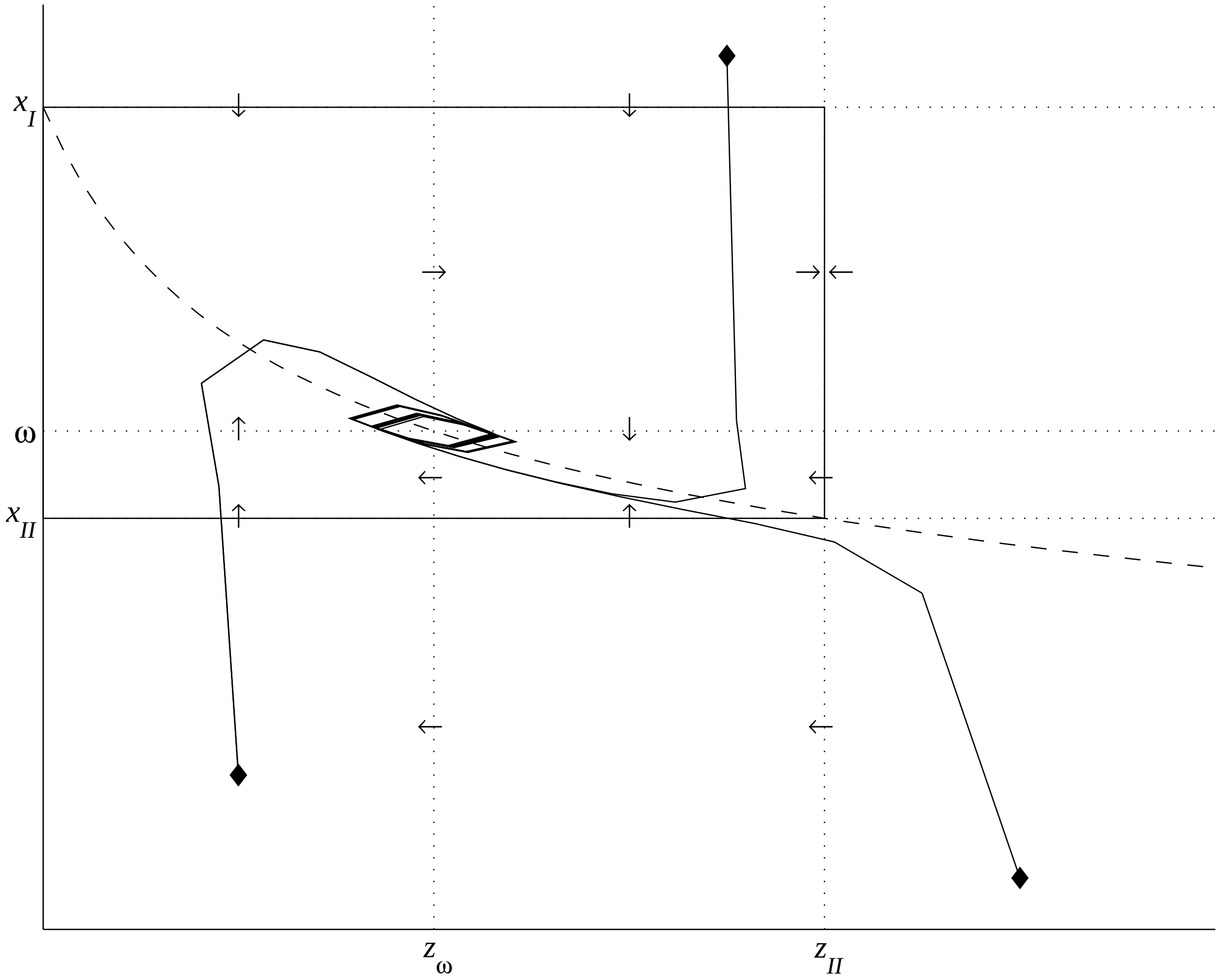

Figure 4. Regime III behavior, for $h(z) = 1/(1+z)$, $g(z) = 1/(1+\alpha z)$, $\alpha = 0.1$, and $s(x) = c(x)/(1-d) = c/(1-d) = z_{II}$. The function $r(z) = bg(z)/(1+ah(z))$is given by the dashed curve. Invariant rectangles are indicated in bold, and the direction of trajectories across boundaries are indicated by arrows. An example trajectories through the phase space is given by the solid line, with the initial condition indicated by a diamond. This trajectory converges to a period-6 oscillation with positive and negative semi-cycles of length 3 in both x and z. [$\omega = 4.85$, $\alpha = 0.1$, $a = 0.5$, $b = 4$, $c = 0.2$, $d = 0.9$.]

policy decisions can be made, however, it is essential to assess an acceptable range for the breeding site population; the choice of control measures to achieve this goal can then be optimized.

- Incorporating additional feedback mechanisms. In his 1994 study of the effectiveness of residential recycling programs, Everett notes that

> Willingness to participate in collective-action endeavors...depends on the potential participant's perception of the collective-action's likelihood of success, the sense of urgency associated with the collective activitiy, and the personal and collective benefits and costs of participation [10].

In this model, only the "sense of urgency," as represented by the hazard level, is a stimulus for increased participation. Considering the additional factors that bear on a community's perception of intervention efficacy, and their mathematical treatment, will lead to a more realistic representation.

- Adding a network structure. It is more realistic to consider the dynamics of these kinds of processes when they operate within a network of communities. It is likely that a community's struggle with an environmental problem will prompt neighboring communities to implement pre-emptive intervention measures, which may have the consequence of reducing the overall prevalence of the hazard. The interaction between communities could be achieved via a link between the levels of consciousness of two communities, or a link between the level of consciousness and a neighbor's level of intervention.

8. Acknowledgments

The model presented in this paper builds on the conceptual foundation established in Predescu et al. [6], and subsequent discussions with R. Levins and T. Awerbuch-Friedlander have been both insightful and encouraging. The commentary of G. Verghese significantly improved the manuscript. This work was supported by the Martin Family Fellowship for Environmental Sustainability at MIT.

Appendix A: Proof of Lemma 4.2

Proof: Choose an $\epsilon > 0$, and choose an integer L such that $\frac{2a^L}{1-a} < \min\{\epsilon, \frac{1}{1-a}\}$. Such an L must exist since the geometric series $G(n)$ is known to converge from below:

$$G(n) = \sum_{j=0}^{n-1} a^j = \frac{1-a^n}{1-a} \text{ and } \lim_{n\to\infty} G(n) = \frac{1}{1-a}. \tag{A1}$$

Find a $\delta > 0$ such that $\delta\frac{1}{1-a} + (2-\delta)\frac{a^L}{1-a} < \epsilon$. Since $\frac{2a^L}{1-a} < \frac{1}{1-a}$ and L has been chosen such that $\frac{a^L}{1-a} < \epsilon$, such a δ can be found. Consider the partial sum $A(L)$

$$A(L) = \sum_{j=0}^{L-1} c_{L,j} a^j.$$

Since each $c_{L,j}$ converges to pq^j as $L \to \infty$, there exists a N_j for each $c_{L,j}$ such that $|c_{n,j} - pq^j| < \delta$ for every $n > N_j$. Choose $N > \max_j\{N_j, L\}$ and consider $A(N)$:

$$A(N) = \sum_{j=0}^{N-1} c_{N,j} a^j \tag{A2}$$

The coefficients of the first L terms of equation (A2) are, by construction, within δ of pq^j. Thus,

$$\begin{aligned} A(N) = \sum_{j=0}^{N-1} c_{N,j}a^j &= \sum_{j=0}^{L-1} c_{N,j}a^j + \sum_{j=L}^{N-1} c_{N,j}a^j \\ &\leq \sum_{j=0}^{L-1}(pq^j+\delta)a^j + \sum_{j=L}^{N-1} a^j = \sum_{j=0}^{L-1} p(aq)^j + \delta\frac{1-a^L}{1-a} + \frac{a^L-a^N}{1-a} \\ &< \sum_{j=0}^{L-1} p(aq)^j + \delta\frac{1-a^L}{1-a} + \frac{a^L}{1-a}. \end{aligned}$$

$A(N)$ can also be bounded from below:

$$\begin{aligned} A(N) &= \sum_{j=0}^{L-1} c_{N,j}a^j + \sum_{j=L}^{N-1} c_{N,j}a^j \\ &\geq \sum_{j=0}^{L-1}(pq^j-\delta)a^j = \sum_{j=0}^{L-1} p(aq)^j - \delta\frac{1-a^L}{1-a}. \end{aligned}$$

Combining the bounds of equations (A3) and (A3) yields

$$\left| A(N) - \sum_{j=0}^{L-1} p(aq)^j \right| \leq \delta \frac{1 - a^L}{1 - a} + \frac{a^L}{1 - a}.$$

By the properties of the geometric series given in equation (A1),

$$\left| \frac{p}{1 - aq} - \sum_{j=0}^{L-1} p(aq)^j \right| = p \frac{(aq)^L}{1 - aq}.$$

Applying the triangle inequality to achieve the result:

$$\begin{aligned} \left| \frac{p}{1 - aq} - A(N) \right| &\leq \left| A(N) - \sum_{j=0}^{L-1} p(aq)^j \right| + \left| \frac{p}{1 - aq} - \sum_{j=0}^{L-1} p(aq)^j \right| \\ &\leq \delta \frac{1 - a^L}{1 - a} + \frac{a^L}{1 - a} + p \frac{(aq)^L}{1 - aq} \\ &\leq \delta \frac{1 - a^L}{1 - a} + \frac{a^L}{1 - a} + \frac{a^L}{1 - a} \\ &= \delta \frac{1}{1 - a} + (2 - \delta) \frac{a^L}{1 - a} \\ &< \epsilon. \end{aligned}$$

□

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