CHOOSING A QUADRATIC COST CRITERION IN THE FREQUENCY DOMAIN

by

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ABSTRACT

Solution of the multi-input, multi-output least-squares optimal regulator problem requires the specification of two quadratic cost weighting matrices. The usual method of selecting these matrices involves repeated solution of the Ricatti equation and calculation of the eigenvalues of the resulting closed loop system matrix. A more direct method of obtaining the optimal system eigenvalues, which provides greater insight into the selection process, is presented here. This method bypasses the Ricatti equation and gives a scalar "characteristic-squared" equation for the eigenvalues of the optimal system as a function of the two cost weighting matrices. The results are then specialized for "block companion" systems and an example worked illustrating application of root locus techniques. Computational aspects are discussed and the zeros of the optimal system are briefly examined.

THESIS SUPERVISOR: S. K. Mitter
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Chapter I

INTRODUCTION

One of the "standard" problems of optimal control theory is the infinite-time, linear time invariant plant, quadratic cost optimal output† regulator problem. The exact statement of the problem is as follows:

Optimal Regulator Problem

Given the fixed, completely controllable, completely observable linear time invariant plant

$$(1)\qquad d\underline{x}/dt = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \qquad \underline{y}(t) = \underline{C}\,\underline{x}(t)$$

$\underline{A}$: n x n $\quad$ $\underline{x}$: n x 1
$\underline{B}$: n x r $\quad$ $\underline{u}$: r x 1
$\underline{C}$: m x n $\quad$ $\underline{y}$: m x 1

and the quadratic cost criterion

$$J = \int_0^\infty (\underline{u}'(t)\underline{R}\,\underline{u}(t) + \underline{y}'(t)\underline{Q}\,\underline{y}(t))\,dt$$

where $\quad \underline{R} = \underline{R}'$ is positive definite

and $\quad \underline{Q} = \underline{Q}'$ is positive definite‡,

find the optimal control $\underline{u}^*(t)$ that minimizes J.

The controllability and observability conditions in the above problem are needed to insure the existance of a unique solution. The system given by (1) is completely controllable if and only if the n x nr matrix

$$\left[\underline{B} \,\vdots\, \underline{A}\,\underline{B} \,\vdots\, \ldots \,\vdots\, \underline{A}^{n-1}\,\underline{B}\right]$$

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† The optimal state regulator problem is included as a special case.

‡ The condition on $\underline{Q}$ can be weakened slightly, see Chapter IV.

has rank n and is completely observable if and only if the n x nm matrix

$$\left[\underline{C}' \,\vdots\, \underline{A}'\underline{C}' \,\vdots\, \cdots \,\vdots\, (\underline{A}')^{n-1}\,\underline{C}'\right]$$

has rank n.

The solution of the optimal regulator problem, under the stated conditions, is well known (see, for example, Athans and Falb[1], Ch. 9) and is given by:

Solution to Optimal Regulator Problem

The optimal control for the output regulator problem is

$$\underline{u}^*(t) = -\underline{R}^{-1}\,\underline{B}'\,\underline{K}\,\underline{x}(t)$$

where $\underline{K}$ is the unique positive definite symmetric solution of the algebraic Ricatti equation

$$\underline{A}'\,\underline{K} + \underline{K}\,\underline{A} - \underline{K}\,\underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K} + \underline{C}'\,\underline{Q}\,\underline{C} = \underline{0} \quad .$$

The optimal trajectory is the solution of the homogeneous time invariant differential equation

$$d\underline{x}/dt = (\underline{A} - \underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K})\,\underline{x}(t)$$

with initial condition $\underline{x}(0) = \underline{x}_o$. Furthermore, the optimal system is strictly stable, i.e. all eigenvalues of the matrix $(\underline{A} - \underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K})$ have negative real parts.

For a given fixed plant $(\underline{A}\ ,\ \underline{B}\ ,\ \underline{C})$, the solution to the optimal regulator problem is completely and uniquely determined once a cost criterion $(\underline{R}\ ,\ \underline{Q})$ is specified. The problem of designing a regulator for the plant therefore reduces to the problem of choosing $\underline{R}$ and $\underline{Q}$. If $\underline{R}$ and $\underline{Q}$ are specified a priori or can be deduced on physical grounds, the problem is effectively solved. More often, however, there is no firm a priori basis for choosing $\underline{R}$ and $\underline{Q}$ and they are considered adjustable design parameters that are varied until "satisfactory performance" is achieved. The use of optimal control theory in this case can be questioned on the grounds that since <u>any</u> valid choice of $\underline{R}$ and $\underline{Q}$ yields a solution "optimal" in some sense, the <u>real</u> criterion is "satisfactory performance" and not mathematical optimality. This is a valid point, but the use of optimal control theory can be defended in at least three ways. First, it automatically yields a stable closed loop system, even if the plant is unstable by itself. Second, Kalman[2] has shown that all systems designed this way share certain desirable properties such as reduced sensitivity to plant variations. Finally, optimal control theory provides a method of computing the required feedback matrix.

Exactly what constitutes "satisfactory performance" depends on the particular application of the regulator, but most relevant dynamic response characteristics can be related to the eigenvalues of the closed loop system. These closed loop eigenvalues are given by the roots of the characteristic equation

$$\det(s\,\underline{I} - \underline{A} + \underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K}) = 0$$

where s is a complex variable and $\underline{I}$ is the identity matrix.

Unfortunately, the Ricatti equation cannot be directly solved in closed form for $\underline{K}$ if $n>2$. Of course, for any given $\underline{R}$ and $\underline{Q}$ the Ricatti equation can be solved numerically and the result used to calculate the closed loop system eigenvalues. A "satisfactory" design may be reached after several tries, but for large values of plant order n the computation time required to solve the Ricatti equation may severly limit the practical number of tries. A method of determining the closed loop eigenvalues without having to solve the Ricatti equation is therefore sought.

Results of the desired type have previously appeared for certain simple cases. For single-input, single-output systems we can take, without loss of generality, $\underline{R} = 1$ and $\underline{Q} = q$, a positive scalar. Willis and Brockett[3] have shown that in this case the optimal system eigenvalues are the left half plane roots of the characteristic-squared equation

$$1 + q\ g(-s)\ g(s) = 0$$

where $$g(s) = \underline{c}\ (s\ \underline{I} - \underline{A})^{-1}\ \underline{b} \ .$$

This equation can be solved by Chang's root-square locus technique[4] to yield a plot of the optimal system eigenvalues as a function of q .

A similar result has been obtained by Chammas[5] for the multi-input, multi-output case under the condition that $\underline{R} = \underline{I}$ and $\underline{Q} = \underline{q}\ \underline{I}$. By manipulating the Ricatti equation, Chammas shows that the optimal system eigenvalues are the left half plane roots of the characteristic-squared equation

$$\det(\underline{I} + q\ \underline{G}'(-s)\ \underline{G}(s)) = 0$$

where $$\underline{G}(s) = \underline{C}\ (s\ \underline{I} - \underline{A})^{-1}\ \underline{B}\ .$$

The root-square locus technique cannot be applied in this case for m>2 because the expansion of the determinant contains, in general, all powers of q up to the m-1 . Numerical calculation the roots and plotting as a function of q is still possible and generally easier than repeatedly solving the corresponding Ricatti equation.

In the next chapter, Chammas' result is extended to the most general case of positive definite $\underline{R}$ and $\underline{Q}$ matrices. The method of approach differs substantially from that used by Chammas but the final result is basically similar, i.e. a scalar characteristic-squared equation whose left half plane roots are the optimal system eigenvalues.

In Chapter III the general result of Chapter II is specialized to the case of "block companion" systems and the characteristic-squared equation is put in a form more suitable for numerical calculation. The treatment of block companion systems is carried further in Chapter IV by means of a numerical example. The use of the root-square locus is illustrated and the zeros of the optimal system briefly examined.

Chapter V contains conclusions and Chapter VI recommendations for future work. Finally, two appendices are included. Appendix A collects some determinant identities that are used throughout the work and Appendix B contains the definition of block companion systems.

Chapter II

DERIVATION OF THE GENERAL CHARACTERISTIC-SQUARED EQUATION

A. SOLUTION OF THE OPTIMAL REGULATOR PROBLEM

In this section, the solution of the optimal regulator problem using Pontryagin's Minimum Principle is outlined. Only the steps that will be needed later are presented, details can be found in Chapter 9 of Athans and Falb[1]. For convenience of reference, the statement of the optimal output regulator problem is repeated below.

Optimal Regulator Problem

Given the fixed, completely controllable, completely observable linear time invariant plant

(2.1) $d\underline{x}/dt = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$ $\qquad \underline{A}: n \times n \quad \underline{x}: n \times 1$

$\qquad \underline{B}: n \times r \quad \underline{u}: r \times 1$

(2.2) $\underline{y} = \underline{C}\,\underline{x}$ $\qquad \underline{C}: m \times n \quad \underline{y}: m \times 1$

and the quadratic cost criterion

(2.3) $J = \int_0^\infty (\underline{u}'\,\underline{R}\,\underline{u} + \underline{y}'\,\underline{Q}\,\underline{y})\,dt$

where $\underline{R} = \underline{R}'$ is positive definite

and $\underline{Q} = \underline{Q}'$ is positive definite

find the optimal control $\underline{u}^*$ that minimizes J.

To apply the Minimum Principle, first the scalar Hamiltonian function H is formed

$$H = \underline{u}'\,\underline{R}\,\underline{u} + \underline{x}'\,\underline{C}'\,\underline{Q}\,\underline{C}\,\underline{x} + \langle \underline{A}\,\underline{x}, \underline{p} \rangle + \langle \underline{B}\,\underline{u}, \underline{p} \rangle$$

where $\langle\ ,\ \rangle$ stands for inner product and the costate vector $\underline{p}$

satisfies

(2.4) $d\underline{p}/dt = -\partial H/\partial \underline{x} = -\underline{C}'\ \underline{Q}\ \underline{C}\ \underline{x} - \underline{A}'\ \underline{p}$

with the boundary condition $\underline{p}(\infty)=\underline{0}$.

A necessary condition for optimality is that along the optimal trajectory

$$\partial H/\partial \underline{u} = \underline{0} = \underline{R}\ \underline{u} + \underline{B}'\ \underline{p}\ .$$

Since $\underline{R}$ is positive definite by assumption, $\underline{R}^{-1}$ exists and the above equation has the unique solution

(2.5) $\underline{u}^* = -\underline{R}^{-1}\ \underline{B}'\ \underline{p}\ .$

Substituting this expression for $\underline{u}$ in equation (2.1) gives

(2.6) $d\underline{x}/dt = \underline{A}\ \underline{x} - \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{p}\ .$

Now that $\underline{u}$ has been eliminated, equations (2.6) and (2.4) can be combined to obtain the reduced canonical equation

(2.7) $$\frac{d}{dt}\left[\begin{array}{c}\underline{x}\\ \hline \underline{p}\end{array}\right] = \left[\begin{array}{c|c}\underline{A} & -\underline{B}\ \underline{R}^{-1}\ \underline{B}' \\ \hline -\underline{C}'\ \underline{Q}\ \underline{C} & -\underline{A}'\end{array}\right]\left[\begin{array}{c}\underline{x}\\ \hline \underline{p}\end{array}\right] \triangleq \underline{F}\left[\begin{array}{c}\underline{x}\\ \hline \underline{p}\end{array}\right]\ .$$

The optimal trajectory is defined by the solution of equation (2.7) with the 2n boundary conditions $\underline{x}(0) = \underline{x}_0$ (arbitrary) and $\underline{p}(\infty) = \underline{0}$. It can be shown that the state $\underline{x}$ and the costate $\underline{p}$ are related by a constant matrix, that is $\underline{p} = \underline{K}\ \underline{x}$, where $\underline{K}$ is the unique positive definite symmetric solution of the algebraic Ricatti equation

(2.8) $\underline{A}'\ \underline{K} + \underline{K}\ \underline{A} - \underline{K}\ \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{K} + \underline{C}'\ \underline{Q}\ \underline{C} = \underline{0}\ .$

Using the relation $\underline{p} = \underline{K}\ \underline{x}$ in equation (2.6) yields the optimal closed loop system equation

(2.9) $d\underline{x}/dt = (\underline{A} - \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{K})\ \underline{x}$

which, along with the initial condition $\underline{x}(0) = \underline{x}_0$, provides an alternate description of the optimal trajectory.

Finally, equation (2.9) implies that the eigenvalues of the optimal closed loop system are the roots of

(2.10) $\det(s\,\underline{I} - \underline{A} + \underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K}) = 0$

and it can be shown that these roots have strictly negative real parts.

B. COMPUTATIONAL DIFFICULTIES

At first glance it may seem that the optimal closed loop system eigenvalues can be obtained, as a function of $\underline{R}$ and $\underline{Q}$, directly from equations (2.8) and (2.10). However, equation (2.8) is nonlinear in the unknown matrix $\underline{K}$ and in general has many solutions, only one of which is positive definite and symmetric. Even for fixed $\underline{R}$ and $\underline{Q}$, isolating the proper solution can be a major task and is usually not done directly in practice. Instead, a matrix differential equation related to equation (2.8) is numerically integrated until a "steady state" is reached. The matrix differential equation contains $n(n+1)/2$ distinct scalar differential equations, so the computational effort required increases rapidly with n.

If $\underline{R}$ and $\underline{Q}$ are treated as variables, isolation of the proper solution of equation (2.8) is a generally unsolvable algebraic problem. The numerical integration approach is little better, as the equations must generally be integrated again for each choice of $\underline{R}$ and $\underline{Q}$. The conclusion is therefore that equations (2.8) and (2.10) are not suitable for finding a general relation between the optimal system eigenvalues and the cost weighting matrices $\underline{R}$ and $\underline{Q}$.

A way to avoid the difficulty of having to solve the Ricatti equation is to ignore, for the moment, its existence and instead go back to equation (2.7). In the next two sections equation (2.7) will be used to derive a characteristic-squared equation, that is, an equation whose roots are the squares of the optimal system eigenvalues. This characteristic-squared equation is an explicit function of $\underline{R}$ and $\underline{Q}$ and is valid under the assumptions of the optimal regulator problem, but does not provide the feedback gains needed to implement the optimal solution. To get these gains the Ricatti equation must still be solved, once, after the final choice of $\underline{R}$ and $\underline{Q}$ is made.

C. PRELIMINARY RESULTS

Equations (2.7) and (2.9) are alternate descriptions of the optimal trajectory, so the n eigenvalues of the optimal system (i.e. the roots of equation (2.10)) must be included among the 2n eigenvalues of the matrix $\underline{F}$. Furthermore, it is known that the eigenvalues of the optimal system all have negative real parts. However, $\underline{F}$ could conceivably have more than n left half plane eigenvalues, which would make it impossible to separate the n optimal system eigenvalues from the n "extraneous" eigenvalues of $\underline{F}$. The following theorem provides enough information to show that this cannot happen.

Theorem 1

If σ is an eigenvalue of $\underline{F}$, then so is $-\sigma$.

Proof:

See Kleinman[6], p.13 .

Since we already know that $\underline{F}$ has n left half plane eigenvalues, Theorem 1 implies that the n "extraneous" eigenvalues of $\underline{F}$ are the right half plane "mirror images" of the optimal system eigenvalues. This fact leads to two corollaries of Theorem 1.

Corollary 1

$m(s) = \det(s\underline{I} - \underline{F})$ can be factored uniquely into two "mirror image" polynomials $p(s)$ and $p(-s)$ having their roots confined, respectively, to the left and right half planes, i.e.

$$m(s) = (-1)^n \, p(s) \, p(-s)$$

where the roots of $p(s)=0$ have $Re(s)<0$.

Proof:

By definition,

$$m(s) = \det(s\underline{I} - \underline{F}) = \prod_{i=1}^{2n} (s - \sigma_i)$$

where σ_i $i=1,2,\ldots,2n$ are the eigenvalues of $\underline{F}$. By Theorem 1, the σ_i exist in pairs $(\lambda_i, -\lambda_i)$ $i=1,2,\ldots,n$. It is also known that at least n of the σ_i have negative real parts. Both conditions can be met only if exactly n of the σ_i have negative real parts, and these can be chosen to be the λ_i . Therefore

$$m(s) = \prod_{i=1}^{2n} (s - \gamma_i)$$

$$= \prod_{i=1}^{n} (s-\lambda_i)\ (s-(-\lambda_i))$$

$$= \prod_{i=1}^{n} (s-\lambda_i)\ (s+\lambda_i)$$

$$= (-1)^n \prod_{i=1}^{n} (s-\lambda_i)\ (-s-\lambda_i)$$

$$= (-1)^n \prod_{i=1}^{n} (s-\lambda_i) \prod_{j=1}^{n} (-s-\lambda_j)$$

$$= (-1)^n\ p(s)\ p(-s)$$

where $p(s) = \prod_{i=1}^{n} (s-\lambda_i)$ and $Re(\lambda_i) < 0$.

Corollary 2

$m(s) = \det(s\underline{I} - \underline{F})$ is an n^{th} degree polynomial in $z = s^2$.

Proof:

From the proof of Corollary 1

$$m(s) = \prod_{i=1}^{n} (s-\lambda_i)\ (s+\lambda_i)$$

$$= \prod_{i=1}^{n} (s^2 - \lambda_i^{\ 2})$$

$$= \prod_{i=1}^{n} (z - \lambda_i^{\ 2})$$

$$= m(z) \quad \text{where } z = s^2 .$$

If, as is usually the case, $\underline{F}$ is a real matrix then m(s) is a polynomial with real coefficients and has roots that are either real or in complex conjugate pairs. Complex eigenvalues of $\underline{F}$ will therefore exist in quadruples $(\lambda, \lambda^*, -\lambda, -\lambda^*)$ and the eigenvalues of $\underline{F}$ will have "mirror image" symmetry about both the real and imaginary axes. It also follows from this that the polynomials p(s) and p(-s) in Corollary 1 will have real coefficients in this case.

D. THE CHARACTERISTIC-SQUARED EQUATION

The results up to this point can be summarized and put in a more useful form by the following theorem.

Theorem 2

The eigenvalues of the optimal output regulator are the roots of p(s)=0 , or equivalently, the left half plane roots of the characteristic-squared equation

$$m(s) = \det(s\underline{I} - \underline{F}) = (-1)^n \, p(s) \, p(-s) = 0$$

in which m(s) is an n^{th} degree polynomial in $z=s^2$ and p(s) is the (unknown) characteristic polynomial of the optimal system.

Furthermore, for $r \le m \le n$, m(s) is conveniently expressed as

$$m(s) = (-1)^n \, d(s) \, d(-s) \, \det(\underline{I}_r + \underline{R}^{-1} \, \underline{G}'(-s) \, \underline{Q} \, \underline{G}(s)) \tag{2.11}$$

and for $m \le r \le n$ as

$$m(s) = (-1)^n \, d(s) \, d(-s) \, \det(\underline{I}_m + \underline{G}(s) \, \underline{R}^{-1} \, \underline{G}'(-s) \, \underline{Q}) \tag{2.12}$$

where

$$d(s) = \det(s\underline{I} - \underline{A})$$

is the open loop characteristic polynomial and

$$\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1} \, \underline{B}$$

is the open loop transfer matrix.

Proof:

The first part of the theorem is merely a restatement of Corollaries 1 and 2. The second part of the theorem is derived using the determinant identity (see Appendix A)

$$\begin{vmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{vmatrix} = \det(\underline{A}) \det(\underline{D} - \underline{C}\,\underline{A}^{-1}\,\underline{B})$$

to write

$$m(s) = \det(s\underline{I} - \underline{F}) = \begin{vmatrix} (s\underline{I} - \underline{A}) & \underline{B}\,\underline{R}^{-1}\,\underline{B}' \\ \underline{C}'\,\underline{Q}\,\underline{C} & (s\underline{I} + \underline{A})' \end{vmatrix}$$

(2.13) $$m(s) = \det(s\underline{I}-\underline{A}) \det((s\underline{I}+\underline{A})'-\underline{C}'\underline{Q}\,\underline{C}(s\underline{I}-\underline{A})^{-1}\underline{B}\,\underline{R}^{-1}\underline{B}').$$

Letting

$$d(s) = \det(s\underline{I}-\underline{A})$$

$$\underline{G}(s) = \underline{C}(s\underline{I}-\underline{A})^{-1}\underline{B}$$

equation (2.13) becomes

$$m(s) = d(s) \det((s\underline{I}+\underline{A})'-\underline{C}'\underline{Q}\,\underline{G}(s)\underline{R}^{-1}\underline{B}') .$$

Now factor $(s\underline{I}+\underline{A})'$ out of the determinant to get

$$m(s) = d(s) \det(-(-s\underline{I}-\underline{A})') \det(\underline{I}-\underline{C}'\underline{Q}\,\underline{G}(s)\underline{R}^{-1}\underline{B}'(s\underline{I}+\underline{A}')^{-1})$$

(2.14) $$m(s) = (-1)^n\, d(s)\, d(-s) \det(\underline{I}-\underline{C}'\underline{Q}\,\underline{G}(s)\underline{R}^{-1}\underline{B}'(s\underline{I}+\underline{A}')^{-1}).$$

Equation (2.14) requires expansion of the determinant of an n x n matrix. The order of the matrix can be reduced to r or m, whichever is less, by use of the determinant identity (see Appendix A)

(2.15) $$\det(\underline{I}_p - \underline{A}\,\underline{B}) = \det(\underline{I}_q - \underline{B}\,\underline{A})$$

if $\underline{A}$ is p x q and $\underline{B}$ is q x p .

Therefore, for $r \leq m \leq n$ we can let

$$"\underline{A}" = \underline{C}' \ \underline{Q} \ \underline{G}(s)$$

$$"\underline{B}" = \underline{R}^{-1} \ \underline{B}' \ (s \ \underline{I} + \underline{A}')^{-1}$$

$$"p" = n$$

$$"q" = r$$

in equation (2.15) to change equation (2.14) into

$$\begin{aligned} m(s) &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_r - \underline{R}^{-1}\underline{B}'(s\underline{I}+\underline{A}')^{-1}\underline{C}'\underline{Q} \ \underline{G}(s)) \\ &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_r + \underline{R}^{-1}\underline{B}'(-s\underline{I}-\underline{A}')^{-1}\underline{C}'\underline{Q} \ \underline{G}(s)) \\ &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_r + \underline{R}^{-1} \ \underline{G}'(-s) \ \underline{Q} \ \underline{G}(s)) \ . \end{aligned}$$

This proves (2.11). Likewise, for $m \leq r \leq n$ we can let

$$"\underline{A}" = \underline{C}' \ \underline{Q}$$

$$"\underline{B}" = \underline{G}(s) \ \underline{R}^{-1} \ \underline{B}' \ (s \ \underline{I} + \underline{A}')^{-1}$$

$$"p" = n$$

$$"q" = m$$

in equation (2.15) to change equation (2.14) into

$$\begin{aligned} m(s) &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_m - \underline{G}(s)\underline{R}^{-1}\underline{B}'(s\underline{I}+\underline{A}')^{-1}\underline{C}'\underline{Q}) \\ &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_m + \underline{G}(s)\underline{R}^{-1}\underline{B}'(-s\underline{I}-\underline{A}')^{-1}\underline{C}'\underline{Q}) \\ &= (-1)^n \ d(s) \ d(-s) \ \det(\underline{I}_m + \underline{G}(s) \ \underline{R}^{-1} \ \underline{G}'(-s) \ \underline{Q}) \ . \end{aligned}$$

This proves (2.12).

E. DISCUSSION

Theorem 2 provides expressions for the characteristic-squared equation of the optimal system involving only the cost weighting matrices, $\underline{R}$ and $\underline{Q}$, and the open loop frequency domain description of the plant, d(s) and $\underline{G}(s)$. This represents the desired extension of Chammas' result to the most general $\underline{R}$ and

$\underline{Q}$ matrices satisfying the assumptions of the optimal regulator problem.

In applying Theorem 2, it is generally neither feasible nor desirable to actually carry out the spectral factorization required to find the optimal system characteristic polynomial p(s). First, the coefficients in m(s) are in general complicated functions of the elements of $\underline{R}$ and $\underline{Q}$, making an analytic solution impossible. Second, interest centers on the roots of p(s)=0 and not on p(s) itself. Finally, m(s) is an n^{th} order polynomial in $z=s^2$, so no more effort is required to solve m(z)=0 than to solve p(s)=0 . Once the roots of m(z)=0 are found, it is only necessary to calculate their left half plane square roots to obtain the optimal system eigenvalues.

The conventional method of determining the optimal system eigenvalues also requires finding the roots of an n^{th} degree polynomial, $\det(s\underline{I} - \underline{A} + \underline{B}\,\underline{R}^{-1}\,\underline{B}'\,\underline{K})=0$. Using Theorem 2 can result in a savings of effort because finding n square roots is a much easier calculation than solving the Ricatti equation. Also, in some cases, the spectral factorization in Theorem 2 can be performed graphically using root locus techniques. This will be illustrated in Chapter IV.

The main obstacle in applying Theorem 2 is the need to expand the determinant in equation (2.11) or (2.12) in terms of the fixed elements of $\underline{G}(s)$ and the variable elements of $\underline{R}$ and $\underline{Q}$. The expansion can be done, tediously, by hand, but is not well suited to direct machine calculation. Some progress has been made overcoming this difficulty and a special case is completely solved in Chapter III. Also, for many practical cases the minimum of r and m is much less than n, so the order of the determinant will be much smaller than the dimension of the $\underline{K}$ matrix in the Ricatti equation. This may help offset the difficulty in expanding the determinant.

Chapter III

APPLICATION TO BLOCK COMPANION SYSTEMS

A. BLOCK COMPANION SYSTEMS

In order to get a formula for the characteristic-squared equation that is better suited for numerical calculation than those given in Theorem 2, attention is now limited to a special case. This special case comprises "block companion" plants with an equal number of inputs and outputs and diagonal $\underline{R}$ and $\underline{Q}$ matrices.

Block companion systems are defined in Appendix B where it is shown that for an m-input, m-output block companion system ($\underline{A}$,$\underline{B}$,$\underline{C}$) the open loop characteristic polynomial is

$$d(s) = \prod_{k=1}^{m} d_k(s) \tag{3.1}$$

and the open loop transfer matrix is

$$\underline{G}(s) = \begin{bmatrix} n_{11}(s)/d_1(s) & n_{12}(s)/d_2(s) & \cdots & n_{1m}(s)/d_m(s) \\ n_{21}(s)/d_1(s) & n_{22}(s)/d_2(s) & \cdots & n_{2m}(s)/d_m(s) \\ \vdots & \vdots & & \vdots \\ n_{m1}(s)/d_1(s) & n_{m2}(s)/d_2(s) & \cdots & n_{mm}(s)/d_m(s) \end{bmatrix} \tag{3.2}$$

where

$$d_k(s) = a_{0,k} + a_{1,k}\, s + \ldots + a_{n_k-1,k}\, s^{n_k-1} + s^{n_k} \tag{3.3}$$

$$n = \sum_{k=1}^{m} n_k \tag{3.4}$$

$$(3.5) \qquad n_{ij}(s) = c_{i,l_j} + c_{i,l_j+1}\, s + \ldots + c_{i,l_j+n_j-1}\, s^{n_j-1}$$

$$(3.6) \qquad l_j = 1 + \sum_{k=1}^{j-1} n_k \, .$$

These quantities can be obtained "by inspection" from the $\underline{A}$ and $\underline{C}$ matrices using the definitions

n = dimension of $\underline{A}$

n_k = dimension of k^{th} diagonal block of $\underline{A}$

$-a_{i-1,k}$ = i^{th} element of last row of k^{th} block of $\underline{A}$

c_{ij} = j^{th} element of i^{th} row of $\underline{C}$.

The $\underline{B}$ matrix does not appear explicitly in the above as it has a fixed form containing only ones and zeros.

Using equations (3.3) and (3.5) in equation (3.2), it is seen that the elements of $\underline{G}(s)$ are proper rational functions of s. The feature that distinguishes the $\underline{G}(s)$ of equation (3.2) from a general proper transfer matrix is the common denominator of each column. This fact simplifies the development of the next section and is one of the reasons for considering block companion systems.

B. EXPANSION OF THE DETERMINANT

The assumption that r=m implies that either form of Theorem 2 will require expansion of the determinant of an r x r=m x m matrix. The first form, equation (2.11), turns out to be more convenient, so let

$$m(s) = (-1)^n\, d(s)\, d(-s)\, \det(\underline{I} + \underline{R}^{-1}\underline{G}'(-s)\underline{Q}\,\underline{G}(s))$$

(3.7) $$m(s) = (-1)^n\, d(s)\, d(-s)\, \det(\underline{L}(s))$$

where

$$\underline{L}(s) = \underline{I} + \underline{P}\,\underline{G}'(-s)\,\underline{Q}\,\underline{G}(s)$$

$$\underline{P} = \underline{R}^{-1} = \mathrm{diag}(p_1, p_2, \ldots, p_m) = \mathrm{diag}(1/r_1, 1/r_2, \ldots, 1/r_m)$$

$$\underline{Q} = \mathrm{diag}(q_1, q_2, \ldots, q_m) .$$

Actually, equation (2.11) can be re-arranged to avoid inverting $\underline{R}$, i.e.

$$\det(\underline{R})\, m(s) = (-1)^n\, d(s)\, d(-s)\, \det(\underline{R} + \underline{G}'(-s)\underline{Q}\,\underline{G}(s))$$

but this has no advantage in the present case.

Performing the indicated matrix operations yields

$$\underline{L}(s) = \begin{bmatrix} \dfrac{d_1(s)d_1(-s)+p_1 g_{11}(s)}{d_1(s)d_1(-s)} & \cdots & \dfrac{p_1 g_{1m}(s)}{d_m(s)d_1(-s)} \\ \dfrac{p_2 g_{21}(s)}{d_1(s)d_2(-s)} & \cdots & \dfrac{p_2 g_{2m}(s)}{d_m(s)d_2(-s)} \\ \vdots & & \vdots \\ \dfrac{p_m g_{m1}(s)}{d_1(s)d_m(-s)} & \cdots & \dfrac{d_m(s)d_m(-s)+p_m g_{mm}(s)}{d_m(s)d_m(-s)} \end{bmatrix}$$

where

(3.8) $$g_{ij}(s) = \sum_{k=1}^{m} q_k\, n_{ki}(-s)\, n_{kj}(s) .$$

Now note that $\underline{L}(s)$ can be factored into the product of three matrices, i.e.

$$\underline{L}(s) = \underline{D}(-s)\,\underline{F}(s)\,\underline{D}(s)$$

where

$$\underline{D}(s) = \mathrm{diag}(1/d_1(s),\ 1/d_2(s),\ \ldots,\ 1/d_m(s))$$

(3.9)

$$\underline{H}(s) = \begin{bmatrix} d_1(s)d_1(-s)+p_1g_{11}(s) & \cdots & p_1g_{1m}(s) \\ p_2g_{21}(s) & \cdots & p_2g_{2m}(s) \\ \vdots & & \vdots \\ p_mg_{m1}(s) & \cdots & d_m(s)d_m(-s)+p_mg_{mm}(s) \end{bmatrix}$$

Therefore

$$\det(\underline{L}(s)) = \det(\underline{D}(-s)\ \underline{H}(s)\ \underline{D}(s))$$
$$= \det(\underline{D}(-s))\ \det(\underline{H}(s))\ \det(\underline{D}(s))$$
$$= (1/d(-s)d(s))\ \det(\underline{H}(s))$$

so using equation (3.7)

$$m(s) = (-1)^n\ d(s)\ d(-s)\ \det(\underline{L}(s))$$
$$= (-1)^n\ d(s)\ d(-s)\ (1/d(-s)d(s))\ \det(\underline{H}(s))$$

(3.10) $$m(s) = (-1)^n \det(\underline{H}(s))\ .$$

The next step is to use equation (3.8) to write $\underline{H}(s)$ as the sum of m+1 matrices

(3.11) $$\underline{H}(s) = \sum_{k=0}^{m} \underline{H}_k(s)$$

where for k=0

(3.12) $$\underline{H}_0(s) = \mathrm{diag}(d_1(s)d_1(-s), d_2(s)d_2(-s), \ldots, d_m(s)d_m(-s))$$

and for k=1, 2, ..., m

(3.13) $$\underline{H}_k(s) = q_k \begin{bmatrix} p_1n_{k1}(-s)n_{k1}(s) & \cdots & p_1n_{k1}(-s)n_{km}(s) \\ p_2n_{k2}(-s)n_{k1}(s) & \cdots & p_2n_{k2}(-s)n_{km}(s) \\ \vdots & & \vdots \\ p_mn_{km}(-s)n_{k1}(s) & \cdots & p_mn_{km}(-s)n_{km}(s) \end{bmatrix} .$$

With $\underline{H}(s)$ written this way, the determinant of a sum of matrices formula of Appendix A can be applied to give

$$(3.14) \qquad \det(\underline{H}(s)) = \sum_{\substack{\alpha_0=0 \\ \alpha_0+\alpha_1+\dots+\alpha_m=m}}^{m} \sum_{\alpha_1=0}^{m} \dots \sum_{\alpha_m=0}^{m} \Delta h_0(\alpha_0)h_1(\alpha_1)\dots h_m(\alpha_m)$$

where $\Delta h_0(\alpha_0)h_1(\alpha_1)\dots h_m(\alpha_m)$ represents the sum of all possible determinants formed by choosing α_0 rows from $\underline{H}_0(s)$, α_1 rows from $\underline{H}_1(s)$, and so on, preserving the ordering of the rows.

In general, using equation (3.14) to calculate $\det(\underline{H}(s))$ requires the evaluation of m^{m+1} m x m determinants, in comparison to the single m x m determinant required to calculate $\det(\underline{H}(s))$ directly from equation (3.9). Even so, equation (3.14) has several advantages. First, each determinant in equation (3.14) is much simpler than the single determinant that results from using equation (3.9) directly. Second, it will be shown that most of the terms in equation (3.14) vanish in the present case. Finally, and perhaps most important, equation (3.14) yields a result that is easily programmed on a computer. Equation (3.13) shows that the i^{th} row of $\underline{H}_k(s)$ has the common factor $p_i q_k$ so the p's and q's will factor out of each determinant in equation (3.14), as literals, leaving determinants depending only on the fixed elements of the plant. These fixed polynomial determinants can be expanded numerically, multiplied by the appropriate literals, and then summed to give the complete expression. The fact that the determinants consist of polynomials is not a serious handicap as they can be handled numerically using, for example, the polynomial addition and multiplication subroutines included in the IBM System/360 Scientific Subroutine Package[7].

Now consider the terms in equation (3.14) with at least one $\alpha_k > 1$ $k=1,2,\ldots,m$, i.e. a determinant containing more than one row from the same $\underline{H}_k(s)$ $1 \le k \le m$. Suppose, for example, that the i^{th} and j^{th} rows of the determinant are from $\underline{H}_k(s)$, then

$$\det \begin{bmatrix} \vdots & & \vdots \\ q_k p_i n_{ki}(-s) n_{k1}(s) & \cdots & q_k p_i n_{ki}(-s) n_{km}(s) \\ \vdots & & \vdots \\ q_k p_j n_{kj}(-s) n_{k1}(s) & \cdots & q_k p_j n_{kj}(-s) n_{km}(s) \\ \vdots & & \vdots \end{bmatrix}$$

$$= q_k p_i n_{ki}(-s) q_k p_j n_{kj}(-s) \det \begin{bmatrix} \vdots & & \vdots \\ n_{k1}(s) & \cdots & n_{km}(s) \\ \vdots & & \vdots \\ n_{k1}(s) & \cdots & n_{km}(s) \\ \vdots & & \vdots \end{bmatrix}$$

$$= 0$$

because the i^{th} and j^{th} rows of the second determinant are identical.

Therefore all terms in equation (3.14) with $\alpha_k > 1$ $1 \le k \le m$ vanish, so equation (3.14) reduces to

$$\det(\underline{H}(s)) = \sum_{\alpha_0=0}^{m} \sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_m=0}^{1} \Delta h_0(\alpha_0) h_1(\alpha_1) \ldots h_m(\alpha_m)$$

$$= \sum_{\alpha_0=0}^{m-1} \sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_m=0}^{1} \Delta h_0(\alpha_0) h_1(\alpha_1) \ldots h_m(\alpha_m) + \det(\underline{H}_0(s))$$

(3.15) $$\det(\underline{H}(s)) = \sum_{\alpha_0=0}^{m-1} \sum_{\alpha_1=0}^{1} \cdots \sum_{\alpha_m=0}^{1} \Delta h_0(\alpha_0) h_1(\alpha_1) \ldots h_m(\alpha_m) + d(s)d(-s)$$

where the summations are restricted by the condition

$$\alpha_0 + \alpha_1 + \ldots + \alpha_m = m \quad .$$

For a fixed value of α_0 , $\Delta h_0(\alpha_0) h_1(\alpha_1) \ldots h_m(\alpha_m)$ now consists of the sum of all possible determinants formed by choosing α_0 rows from $\underline{H}_0(s)$ and replacing the remaining $k=m-\alpha_0$ rows with the corresponding rows from k <u>different</u> $\underline{H}_i(s)$'s . That is, letting H_{ij} denote the j^{th} row of $\underline{H}_i(s)$, a typical term of $\Delta h_0(\alpha_0) h_1(\alpha_1) \ldots h_m(\alpha_m)$ is

(3.16) $\det(\underline{V}(H_{i_1 j_1}, H_{i_2 j_2}, \ldots, H_{i_k j_k}))$ = determinant formed by replacing row j_1 of $\underline{H}_0(s)$ with row j_1 of $\underline{H}_{i_1}(s)$, row j_2 of $\underline{H}_0(s)$ with row j_2 of $\underline{H}_{i_2}(s)$, and so on, with $1 \leq j_1 < j_2 < \ldots < j_k \leq m$, $(i_1, i_2, \ldots, i_k)$ distinct with $1 \leq i \leq m$ and $1 \leq k \leq m$.

Using definition (3.16), equation (3.15) can be written as

(3.17) $$\det(\underline{H}(s)) = \sum_{k=1}^{m} \sum_{i,j} \det(\underline{V}(H_{i_1 j_1}, H_{i_2 j_2}, \ldots, H_{i_k j_k})) + d(s)d(-s)$$

where the inner summation extends over all combinations of i's and j's satisfying

$1 \le j_1 < j_2 < \ldots < j_k \le m$

$(i_1, i_2, \ldots, i_k)$ distinct

$1 \le i \le m$.

The matrix $\underline{V}(H_{i_1 j_1}, H_{i_2 j_2}, \ldots, H_{i_k j_k})$ contains two types of rows: k rows of the form (from equation (3.13))

$$[q_i p_j n_{ij}(-s) n_{i1}(s) \quad q_i p_j n_{ij}(-s) n_{i2}(s) \quad \ldots \quad q_i p_j n_{ij}(-s) n_{im}(s)]$$

$$= q_i p_j [n_{ij}(-s) n_{i1}(s) \quad n_{ij}(-s) n_{i2}(s) \quad \ldots \quad n_{ij}(-s) n_{im}(s)] \tag{3.18}$$

where

$$(i,j) \in S_{ij} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\} \tag{3.19}$$

and m-k rows of the form (from equation (3.12))

$$[0 \ \ldots \ 0 \quad d_j(s) d_j(-s) \quad 0 \ \ldots \ 0]$$

$$= d_j(s) d_j(-s) \, [0 \ \ldots \ 0 \ \ 1 \ \ 0 \ \ldots \ 0] \tag{3.20}$$

where

$$j \notin S_j = \{j_1, j_2, \ldots, j_k\} \qquad 1 \le j \le m \tag{3.21}$$

and the nonzero element is on the main diagonal of $\underline{V}$.

The first type of row, equation (3.18), has the common factor $q_i p_j$ and the second type of row, equation (3.20), has the common factor $d_j(s) d_j(-s)$. These can be factored out of $\det(\underline{V})$ to yield

$$\det(\underline{V}) = \prod_{(i,j) \in S_{ij}} q_i p_j \prod_{\substack{l \notin S_j \\ 1 \le l \le m}} d_l(s) d_l(-s) \det(\underline{\tilde{V}}) \tag{3.22}$$

where the sets S_{ij} and S_j are as defined by equations (3.19) and (3.21) and $\underline{\tilde{V}}$ is what remains of $\underline{V}$ after the common factors are removed. $\underline{\tilde{V}}$ contains m-k rows consisting of zeros except for a single "1" on the main diagonal which result from the rows of $\underline{V}$ given by equation (3.20). The rows and columns corresponding to these 1's can be deleted without affecting the value of the determinant, so

$$(3.23) \qquad \det(\underline{V}) = \prod_{(i,j)\in S_{ij}} q_i p_j \prod_{\substack{l \notin S_j \\ 1\le l\le m}} d_l(s) d_l(-s) \det(\underline{N}(S_{ij}))$$

where

$$(3.24) \qquad N(S_{ij}) =$$

$$\begin{bmatrix} n_{i_1 j_1}(s) n_{i_1 j_1}(-s) & n_{i_1 j_2}(s) n_{i_1 j_1}(-s) & \cdots & n_{i_1 j_k}(s) n_{i_1 j_1}(-s) \\ n_{i_2 j_1}(s) n_{i_2 j_2}(-s) & n_{i_2 j_2}(s) n_{i_2 j_2}(-s) & \cdots & n_{i_2 j_k}(s) n_{i_2 j_2}(-s) \\ \vdots & \vdots & & \vdots \\ n_{i_k j_1}(s) n_{i_k j_k}(-s) & n_{i_k j_2}(s) n_{i_k j_k}(-s) & \cdots & n_{i_k j_k}(s) n_{i_k j_k}(-s) \end{bmatrix}$$

$$(3.25) \qquad S_{ij} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$$

$$(3.26) \qquad S_j = \{j : j \in S_{ij}\} = \{j_1, j_2, \ldots, j_k\} .$$

Now that $\det(\underline{V})$ has been expanded to factor out the p's and q's , it can be substituted into equation (3.17) to give

$$(3.27) \qquad \det(\underline{H}(s)) = d(s)\, d(-s) + \sum_{k=1}^{m} \sum_{S_{ij}} \prod_{(i,j)\in S_{ij}} q_i p_j \prod_{\substack{l \notin S_j \\ 1\le l\le m}} d_l(s)\, d_l(-s) \det(\underline{N}(S_{ij}))$$

where the inner summation is over all sets S_{ij} satisfying

$1 \le j_1 < j_2 < \ldots < j_k \le m$

$(i_1, i_2, \ldots, i_k)$ distinct with $1 \le i \le m$.

The above conditions on the summation sets S_{ij} can be simplified and made more symmetrical by noting that the factor

$$\prod_{(i,j)\in S_{ij}} q_i \; p_j$$

depends only on the i's and j's in S_{ij} and not the way they are paired. For a given set of k i's and k j's, only one ordering of the j's is valid but the i's can be permuted k! ways, yielding a total of k! pairings. Combining this fact with equations (3.10) and (3.27) gives the final result

$$(3.28) \qquad (-1)^n\, m(s) = d(s)\, d(-s) + \sum_{k=1}^{m} \sum_{S_i} \sum_{S_j} \prod_{i\in S_i} q_i \prod_{j\in S_j} p_j \prod_{\substack{l\notin S_j \\ 1\le l\le m}} d_l(s)\, d_l(-s) \sum_{r=1}^{k!} \det(\underline{M}(\sigma_r(S_i);S_j))$$

where the S_i and S_j summations are over all sets

$$S_i = \{i_1, i_2, \ldots, i_k\} \quad \text{with } 1\le i_1 < i_2 < \ldots < i_k \le m$$

$$S_j = \{j_1, j_2, \ldots, j_k\} \quad \text{with } 1\le j_1 < j_2 < \ldots < j_k \le m$$

and

$\sigma_r(S_i)$ = r^{th} permutation of the elements of S_i

$\underline{M}(\sigma_r(S_i);S_j)$ = equation (3.24) with the "i" subscripts permuted .

C. DISCUSSION

Equation (3.28) is in a form that can be easily programmed on a computer. The summation sets S_i and S_j can be generated using a sequence of logic type programming statements.

Each term in the summation can be factored into two terms, one involving only the variable weighting matrix elements, i.e.

(3.29) $$\prod_{i \in S_i} q_i \prod_{j \in S_j} p_j$$

and the other depending only on the fixed plant, i.e.

(3.30) $$\prod_{\substack{l \notin S_j \\ 1 \leq l \leq m}} d_l(s)\, d_l(-s) \sum_{r=1}^{k!} \det(\underline{N}(\sigma_r(S_i); S_j)).$$

The terms given by (3.30) can be calculated using polynomial arithmetic routines and the definitions (3.3), (3.5) and (3.24). The terms given by (3.29) can be either explicitly written out in the program or program generated by writing, for example

$$\prod q_i = (q_1)^{\alpha_1} (q_2)^{\alpha_2} \ldots (q_m)^{\alpha_m}$$

and setting $\alpha_i = 1$ if q_i belongs in the product and $\alpha_i = 0$ otherwise.

Chapter IV

AN ILLUSTRATIVE EXAMPLE

A. INTRODUCTION

In this chapter the results of the previous chapters are illustrated, and additional results derived, within the framework of a numerical example. The example was chosen to be simple enough to allow hand calculation, but a higher order system would present nothing essentially different anyway.

B. PROBLEM STATEMENT

Suppose we have the 2-input, 2-output, completely controllable, completely observable "block companion" plant

$$d\underline{x}/dt = \underline{A}\ \underline{x} + \underline{B}\ \underline{u}$$

$$\underline{y} = \underline{C}\ \underline{x}$$

with

$$\underline{A} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{array}\right], \quad \underline{B} = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$$\underline{C} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$$

and want to minimize a cost functional of the form

$$J = \int_0^\infty (\underline{u}'\ \underline{R}\ \underline{u} + \underline{y}'\ \underline{Q}\ \underline{y})\ dt$$

with $$\underline{R} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 1/p_1 & 0 \\ 0 & 1/p_2 \end{bmatrix} \quad \text{positive definite}$$

and $$\underline{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad \text{positive semi-definite} .$$

How do the eigenvalues of the optimal closed loop system, for the given plant and cost functional, depend on the elements of the cost weighting matrices $\underline{R}$ and $\underline{Q}$?

C. DISCUSSION OF PROBLEM

The problem can be given a physical interpretation by defining

$$W_i = \int_0^\infty u_i^2 \, dt \quad i=1,2$$

as the "control energy" expended by the i^{th} input and

$$E_i = \int_0^\infty y_i^2 \, dt \quad i=1,2$$

as the "integral-squared error" of the i^{th} output. Then the cost becomes

$$\begin{aligned} J &= \int_0^\infty (\underline{u}' \, \underline{R} \, \underline{u} + \underline{y}' \, \underline{Q} \, \underline{y}) \, dt \\ &= r_1 \int_0^\infty u_1^2 \, dt + r_2 \int_0^\infty u_2^2 \, dt + q_1 \int_0^\infty y_1^2 \, dt + q_2 \int_0^\infty y_2^2 \, dt \\ &= r_1 W_1 + r_2 W_2 + q_1 E_1 + q_2 E_2 \end{aligned}$$

so the cost weighting matrix elements r_i and q_i are a measure of the relative penalty being assigned to "control energy" and "integral-squared error". In addition, the choice of $\underline{R}$ and $\underline{Q}$ affects the dynamic response of the regulator by changing the eigenvalues of the optimal closed loop system. The design of the regulator therefore involves three interrelated and possibly conflicting goals: low "control energy" expenditure, low "integral-squared error", and "satisfactory" dynamic response characteristics. The method developed here provides a systematic procedure for exploring the trade-offs involved in trying to simultaneously satisfy these goals.

The problem stated in section B differs from the original statement of the optimal regulator problem in that the matrix $\underline{Q}$ is now only assumed to be positive semi-definite. The following theorem provides a sufficient condition under which the solution of the optimal regulator problem is still valid for $\underline{Q}$ positive semi-definite. It can be readily verified that the given problem satisfies this condition.

Theorem 3

The solution of the optimal regulator problem is still valid with $\underline{Q}$ only positive semi-definite as long as the pair $(\underline{A}\ ,\ \underline{Q}^{\frac{1}{2}}\ \underline{C})$ is completely observable, i.e. if the matrix

$$\left[(\underline{Q}^{\frac{1}{2}}\underline{C})' \;\vdots\; \underline{A}'(\underline{Q}^{\frac{1}{2}}\underline{C})' \;\vdots\; \cdots \;\vdots\; (\underline{A}')^{n-1}(\underline{Q}^{\frac{1}{2}}\underline{C})'\right]$$

has rank n .

Proof:

See Kleinman[6], pp. 4,5 .

D. CALCULATION OF THE CHARACTERISTIC-SQUARED EQUATION

The example is in the form considered in Chapter III with $m=n_1=n_2=2$, so by inspection from equation (3.3)

$$d_1(s) = 2 + 2\,s + s^2 \quad \text{(open loop poles at } s=-1\pm j)$$

$$d_2(s) = 6 + 5\,s + s^2 \quad \text{(open loop poles at } s=-2,\ s=-3)$$

and from equation (3.5)

$$n_{11}(s) = 1 \qquad \text{(open loop zero at } s=\infty)$$

$$n_{12}(s) = s \qquad \text{(open loop zero at } s=0)$$

$$n_{21}(s) = 1 + s \qquad \text{(open loop zero at } s=-1)$$

$$n_{22}(s) = 1 \qquad \text{(open loop zero at } s=\infty) \; .$$

Equation (3.28) then gives

$$(-1)^4 \, m(s) = d(s)\,d(-s) + \sum_{k=1}^{2} \sum_{S_i} \sum_{S_j} \prod_{i \in S_i} q_i \prod_{j \in S_j} p_j \prod_{\substack{l \notin S_j \\ 1 \le l \le 2}} d_l(s)\,d_l(-s) \sum_{r=1}^{k!} \det(\underline{N}(\sigma_r(S_i);S_j))$$

where the summation sets S_i and S_j must satisfy

$$S_i = \{i_1, \; \ldots \; , i_k\} \quad 1 \le i_1 < \ldots < i_k \le 2$$

$$S_j = \{j_1, \; \ldots \; , j_k\} \quad 1 \le j_1 < \ldots < j_k \le 2 \quad .$$

Therefore, for k=1

$$S_i = \{1\} \text{ or } \{2\}$$

$$S_j = \{1\} \text{ or } \{2\}$$

and for k=2

$$S_i = \{1,2\}$$

$$S_j = \{1,2\}$$

so m(s) can be expanded to give

$$m(s) = d(s)d(-s) + q_1 p_1 d_2(s) d_2(-s) \det(\underline{N}(1;1)) + q_1 p_2 d_1(s) d_1(-s) \det(\underline{N}(1;2)) + q_2 p_1 d_2(s) d_2(-s) \det(\underline{N}(2;1)) + q_2 p_2 d_1(s) d_1(-s) \det(\underline{N}(2;2)) + q_1 q_2 p_1 p_2 \left(\det(\underline{N}(1,2;1,2)) + \det(\underline{N}(2,1;1,2)) \right).$$

Also, from equation (3.24)

$$\det(\underline{N}(1;1)) = n_{11}(s)n_{11}(-s) = 1$$

$$\det(\underline{N}(1;2)) = n_{12}(s)n_{12}(-s) = -s^2$$

$$\det(\underline{N}(2;1)) = n_{21}(s)n_{21}(-s) = -s^2 + 1$$

$$\det(\underline{N}(2;2)) = n_{22}(s)n_{22}(-s) = 1$$

$$\det(\underline{N}(1,2;1,2)) = \det\begin{bmatrix} n_{11}(s)n_{11}(-s) & n_{12}(s)n_{11}(-s) \\ n_{21}(s)n_{22}(-s) & n_{22}(s)n_{22}(-s) \end{bmatrix}$$

$$= -s^2 - s + 1$$

$$\det(\underline{N}(2,1;1,2)) = \det\begin{bmatrix} n_{21}(s)n_{21}(-s) & n_{22}(s)n_{21}(-s) \\ n_{11}(s)n_{12}(-s) & n_{12}(s)n_{12}(-s) \end{bmatrix}$$

$$= s^4 - 2\,s^2 + s \quad .$$

These results can now be combined to yield

$$\begin{aligned} m(s) = {} & (s^4+4)(s^4-13\,s^2+36) + q_1p_1(s^4-13\,s^2+36) \\ & -q_1p_2\,s^2(s^4+4) + q_2p_1(-s^2+1)(s^4-13\,s^2+36) \\ & +q_2p_2(s^4+4) + q_1q_2p_1p_2(s^4-3\,s^2+1) \end{aligned}$$

or letting $z=s^2$

$$\begin{aligned} m(z) = {} & (z^2+4)(z^2-13\,z+36) + q_1p_1(z^2-13\,z+36) \\ & -q_1p_2\,z(z^2+4) + q_2p_1(-z+1)(z^2-13\,z+36) \\ & +q_2p_2(z^2+4) + q_1q_2p_1p_2(z^2-3\,z+1). \end{aligned}$$

E. APPLICATION OF ROOT-SQUARE LOCUS

For any given set of weighting matrix elements, $m(z)=0$ is a fourth degree polynomial equation and can be solved for z numerically. Note, however, that each weighting matrix element enters into $m(z)$ only to the first power.

Therefore, if one of the p's or q's is varied, with the other three held constant, a "root-square locus" of the optimal system eigenvalues can be drawn in the $z=s^2$ plane.

Letting k stand for the variable weighting matrix element, write m(z) as follows

$$m(z) = (\text{terms independent of } k) + (\text{terms involving } k)$$
$$= (\text{terms independent of } k) + k\ (\text{polynomial})$$
$$= d'(z) + k\ n'(z)$$

and then

$$m(z) = 0 = d'(z) + k\ n'(z)$$

implies

$$1 + k\ (n'(z)/d'(z)) = 0\ .$$

This is a conventional root locus equation, except for being in the $z=s^2$ plane, and conventional root locus techniques [8,9] can be used. In particular, the root-square locus starts at the roots of $d'(z)=0$ for $k=0$ and terminates at infinity or at the roots of $n'(z)=0$ for $k=\infty$.

Once the root-square locus is plotted, the optimal system root locus is found by determining the left half plane square roots of a sufficient number of points on the root-square locus. Figure 1 illustrates the nature of the required mapping. The mapping is unique except for points on the negative real z axis. However, the strict stability of the optimal system assures that this ambiguous case will not arise. The mapping can be easily done graphically using, for example, a Spirule for polar coordinate measurements.

To cover all possible cases, an infinite set of loci would be required. Instead, consider four "typical" cases.

<u>Case 1:</u> vary q_1 $0<q_1<\infty$ with $r_1=r_2=1$, $q_2=0$.

The root-square locus equation is

$$0 = 1 + (-q_1)(z^3-z^2+17\ z-36)/(z^2+4)(z^2-13\ z+36)$$

so the roots of $d'(z)=0$ are at

$$z=\pm 2j \quad , \quad z=4 \quad , \quad z=9$$

and the roots of $n'(z)=0$ are at

$$z=1.92 \quad , \quad z=-.46 \pm .433j \quad .$$

Figure 2 shows the root-square locus and Figure 3 the corresponding optimal system eigenvalue plot for this case. Note that for $q_1=0$ the optimal system eigenvalues are the open loop poles.

<u>Case 2:</u> vary q_2 $0<q_2<\infty$ with $r_1=r_2=1$, $q_1=0$.

The root-square locus equation is

$$0 = 1 + (-q_2)(z^3-15\ z^2+49\ z-40)/(z^2+4)(z^2-13\ z+36)$$

so the roots of $d'(z)=0$ are at

$$z=\pm 2j \quad , \quad z=4 \quad , \quad z=9$$

and the roots of $n'(z)=0$ are at

$$z=1.25 \quad , \quad z=2.95 \quad , \quad z=10.8 \quad .$$

Figure 4 shows the root-square locus and Figure 5 shows the corresponding optimal system eigenvalue plot. This plot again starts at the open loop poles but the eigenvalue motion is in a generally different direction compared to case 1 . By appropriate choice of non-zero q_1 and q_2, it would therefore seem possible to achieve eigenvalue locations between the plots of case 1 and case 2.

<u>Case 3:</u> vary $p_1 = 1/r_1 \quad 0 < r_1 < \infty$ with $q_1 = q_2 = r_2 = 1$.

The root-square locus equation is

$$0 = 1 + (-p_1)(z^3 - 16\ z^2 + 65\ z - 73)/(z^2+4)(z^2 - 14\ z + 37)$$

so the roots of $d'(z)=0$ are at

$$z = \pm 2j \quad , \ z = 3.536 \quad , \ z = 10.464$$

and the roots of $n'(z)=0$ are at

$$z = 1.93 \quad , \ z = 3.62 \quad , \ z = 10.447 \quad .$$

The root-square locus is plotted in Figure 6 and the locus of optimal system eigenvalues shown in Figure 7. The plots are shown as a function of $r_1 = 1/p_1$, so the interpretation of start and end points are interchanged and the complex conjugate open loop pole pair exists for $r_1 = \infty$. The separation between the real axis segments is greatly exaggerated for clarity. It is interesting that two of the optimal system eigenvalues for $r_1 = 0$ are almost the same as two for $r_1 = \infty$.

<u>Case 4:</u> vary $p_2=1/r_2$ $0<r_2<\infty$ with $q_1=q_2=r_1=1$.

This time the root-square locus equation is

$$0 = 1 + (-p_2)(z^3-2\ z^2+7\ z-5)/(z^2-z+6)(z^2-13\ z+36)$$

so the roots of $d'(z)=0$ are at

$$z=4\ ,\ z=9\ ,\ z=.5\pm 2.398j$$

and the roots of $n'(z)=0$ are at

$$z=.83\ ,\ z=.58\pm 2.38j\ .$$

Figure 8 shows the root-square locus for this case and Figure 9 is the corresponding optimal system eigenvalue plot. Now the real axis open loop poles are present for $r_2=\infty$ and the complex conjugate eigenvalues are almost unaffected by the value of r_2.

In the single-input, single-output case the optimal eigenvalues approach the open loop zeros or infinity as k approaches infinity. However, in the present case, three of the optimal eigenvalues remain in the finite part of the s plane although there are only two finite open loop zeros. This difference can be partly explained by the fact that the open loop zeros are altered by the optimal feedback, as is shown in the next section.

F. ZEROS OF THE OPTIMAL SYSTEM

The transfer matrix of the optimal system is

$$\underline{G}^*(s) = \underline{C}(s\underline{I} - \underline{A} + \underline{B}\,\underline{R}^{-1}\underline{B}'\underline{K})^{-1}\underline{B} = (1/p(s))\underline{N}^*(s)$$

where $\underline{K}$ is the solution of the Ricatti equation, $p(s)$ is the optimal system characteristic polynomial, and $\underline{N}^*(s)$ is a matrix of polynomials related to the zeros of the optimal system. Theorem 2 provides a method for determining $p(s)$ without having to solve the Ricatti equation, but no corresponding general result for $\underline{N}^*(s)$ has been found.

For single-input, single-output systems, Brockett[10] has shown that linear state variable feedback can cancel, but not move, the open loop zeros. For multi-input, multi-output systems the invariance of the open loop zeros no longer holds in general. In particular, for the type of block companion systems and cost functionals considered in section B, the zeros of the optimal system can be made equal to the open loop zeros only for certain $\underline{C}$ matrices, and then only by restricting the $\underline{Q}$ matrix.

Instead of using the numerical values of section B, consider the more general $\underline{A}$ and $\underline{C}$ matrices

$$\underline{A} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -a_0 & -a_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -b_0 & -b_1 \end{array}\right]$$

$$\underline{C} = \left[\begin{array}{cc|cc} c_{11} & c_{12} & c_{13} & c_{14} \\ \hline c_{21} & c_{22} & c_{23} & c_{24} \end{array}\right]$$

The other matrices remain as before. Let the unknown positive definite symmetric solution of the Ricatti equation be

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix}$$

then

$$(s\underline{I} - \underline{A} + \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{K})^{-1}\underline{B} = (1/p(s)) \left[\begin{array}{c|c} h_2(s) & -m_1(s) \\ s\ h_2(s) & -s\ m_1(s) \\ \hline -m_2(s) & h_1(s) \\ -s\ m_2(s) & s\ h_1(s) \end{array}\right]$$

where

$$p(s) = h_1(s)\ h_2(s) - m_1(s)\ m_2(s)$$

$$h_1(s) = s^2 + (p_1\ k_{22} + a_1)\ s + (p_1\ k_{12} + a_0)$$

$$h_2(s) = s^2 + (p_2\ k_{44} + b_1)\ s + (p_2\ k_{34} + b_0)$$

$$m_1(s) = p_1\ (k_{24}\ s + k_{23})$$

$$m_2(s) = p_2\ (k_{24}\ s + k_{14})\ .$$

<u>If</u> the Ricatti equation could be satisfied with $k_{14}=k_{23}=k_{24}=0$, then $m_1(s)=m_2(s)=0$ and the above result would simplify to

$$(s\underline{I} - \underline{A} + \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{K})^{-1}\underline{B} = \left[\begin{array}{c|c} 1/h_1(s) & 0 \\ s/h_2(s) & 0 \\ \hline 0 & 1/h_2(s) \\ 0 & s/h_2(s) \end{array}\right]$$

and the optimal system would have the same zeros as the open loop system, but different poles. This is not generally possible however.

Let

$$(4.1) \qquad \underline{E} = \underline{C}'\ \underline{Q}\ \underline{C} + \underline{K}\ \underline{A} + \underline{A}'\ \underline{K} - \underline{K}\ \underline{B}\ \underline{R}^{-1}\ \underline{B}'\ \underline{K}$$

so that $\underline{E} = \underline{0}$ if and only if $\underline{K}$ is a solution of the Ricatti equation. Assuming $k_{14}=k_{23}=k_{24}=0$, the matrix equation $\underline{E} = \underline{0}$ consists of ten distinct scalar equations in the seven remaining unknown k's. Since the number of equations exceeds the number of unknowns, in general no solution will exist. However, the assumption that $\underline{R}$ is diagonal allows the ten equations to be divided into two groups: one consisting of four equations involving only k_{13}, the other consisting of six equations involving only the other six unknown k's. Only the equations in the first group are needed and they are

$$(4.2) \qquad e_{13} = q_1\ c_{11}\ c_{13} + q_2\ c_{21}\ c_{23}$$

$$(4.3) \qquad e_{14} = q_1\ c_{11}\ c_{14} + q_2\ c_{21}\ c_{24} + k_{13}$$

$$(4.4) \qquad e_{23} = q_1\ c_{12}\ c_{13} + q_2\ c_{22}\ c_{23} + k_{13}$$

$$(4.5) \qquad e_{24} = q_1\ c_{12}\ c_{14} + q_2\ c_{22}\ c_{24}\ .$$

A necessary condition for $\underline{K}$ with $k_{14}=k_{23}=k_{24}=0$ to satisfy the Ricatti equation is therefore to have $e_{13}=e_{14}=e_{23}=e_{24}=0$ in equations (4.2) to (4.5). If it is required that the zeros remain invariant for <u>all</u> positive definite diagonal $\underline{Q}$ matrices, then equations (4.2) to (4.5) imply that

$$c_{11} = c_{12} = 0 \quad \text{or} \quad c_{13} = c_{14} = 0$$

and

$$c_{21} = c_{22} = 0 \quad \text{or} \quad c_{23} = c_{24} = 0 \; .$$

That is, $\underline{C}$ must be in one of the four forms

$$\underline{C}_1 = \left[\begin{array}{cc|cc} 0 & 0 & c_{13} & c_{14} \\ \hline 0 & 0 & c_{23} & c_{24} \end{array}\right]$$

$$\underline{C}_2 = \left[\begin{array}{cc|cc} 0 & 0 & c_{13} & c_{14} \\ \hline c_{21} & c_{22} & 0 & 0 \end{array}\right]$$

$$\underline{C}_3 = \left[\begin{array}{cc|cc} c_{11} & c_{12} & 0 & 0 \\ \hline 0 & 0 & c_{23} & c_{24} \end{array}\right]$$

$$\underline{C}_4 = \left[\begin{array}{cc|cc} c_{11} & c_{12} & 0 & 0 \\ \hline c_{21} & c_{22} & 0 & 0 \end{array}\right]$$

$\underline{C}_1$ and $\underline{C}_4$ result in unobservable systems, so they do not even satisfy the assumptions of the problem. $\underline{C}_2$ and $\underline{C}_3$ are valid, but result in the system decomposing into two uncoupled single-input, single-output subsystems. Therefore the zeros of the transfer matrix can be kept invariant for arbitrary diagonal positive definite $\underline{R}$ and $\underline{Q}$ matrices only in the trivial case of two completely uncoupled subsystems.

If we no longer insist on retaining complete freedom in choosing $\underline{Q}$, a solution may or may not exist, depending on $\underline{C}$. For example, using the $\underline{C}$ matrix in the example of section B, i.e.

$$(4.6) \qquad \underline{C} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$$

in equations (4.2) to (4.5) gives

$$e_{13} = q_2$$
$$e_{14} = q_1 + k_{13}$$
$$e_{23} = q_2 + k_{13}$$
$$e_{24} = 0$$

so $\underline{E} = \underline{0}$ implies $q_1 = q_2 = k_{13} = 0$. Thus no valid diagonal $\underline{Q}$ matrix exists that keeps the transfer matrix zeros invariant. On the other hand, if $\underline{C}$ is changed to

(4.7) $$\underline{C} = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$$

then

$$e_{13} = -q_1 + q_2$$
$$e_{14} = q_1 + k_{13}$$
$$e_{23} = q_2 + k_{13}$$
$$e_{24} = 0$$

so now $\underline{E} = \underline{0}$ implies $q_1 = q_2 = -k_{13}$. Therefore the zeros are invariant in this case only if $q_1 = q_2$.

Letting the off-diagonal element of $\underline{Q}$ be nonzero, i.e.

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \quad \text{with} \quad q_{11}\, q_{22} - q_{12}^2 > 0$$

in equation (4.1) changes equations (4.2) to (4.5) to

(4.8) $$e_{13} = q_{11}c_{11}c_{13} + q_{12}(c_{11}c_{23} + c_{13}c_{21}) + q_{22}c_{21}c_{23}$$

(4.9) $$e_{14} = q_{11}c_{11}c_{14} + q_{12}(c_{11}c_{24} + c_{14}c_{21}) + q_{22}c_{21}c_{24} + k_{13}$$

(4.10) $$e_{23} = q_{11}c_{12}c_{13} + q_{12}(c_{12}c_{23} + c_{13}c_{22}) + q_{22}c_{22}c_{23} + k_{13}$$

(4.11) $$e_{24} = q_{11}c_{12}c_{14} + q_{12}(c_{12}c_{24} + c_{14}c_{22}) + q_{22}c_{22}c_{24} \; .$$

This does not necessarily provide greater freedom in keeping the zeros invariant. Using the C matrix given by equation (4.6) in equations (4.8) to (4.11) gives

$$e_{13} = q_{12} + q_{22}$$
$$e_{14} = q_{11} + q_{12} + k_{13}$$
$$e_{23} = q_{22} + k_{13}$$
$$e_{24} = q_{12}$$

so $\underline{E} = \underline{0}$ implies $q_{11}=q_{12}=q_{22}=k_{13}=0$, as before. Likewise, the C matrix given by equation (4.7) yields

$$e_{13} = -q_{11} + q_{22}$$
$$e_{14} = q_{11} + q_{12} + k_{13}$$
$$e_{23} = -q_{12} + q_{22} + k_{13}$$
$$e_{24} = q_{12}$$

but $e_{24}=0$ implies $q_{12}=0$, so nothing is gained in this case by allowing Q to be non-diagonal.

Allowing R to be non-diagonal vastly increases the complexity of the problem, so this case will not be considered here.

Chapter V

CONCLUSIONS

Theorem 2 provides explicit relationships between the optimal system eigenvalues, the open loop plant, and the cost weighting matrices $\underline{R}$ and $\underline{Q}$. Whereas the optimal regulator problem was stated in the time domain, Theorem 2 involves only frequency domain quantities and the cost weighting matrices. This allows general results, independent of the coordinate system of the plant, to be derived. For example, from equation (2.11) it is clear that as $\underline{R}^{-1}$ approaches $\underline{0}$ the optimal system eigenvalues approach the open loop poles, if they are in the left half plane, or the "mirror images" of the open loop poles that are in the right half plane.

For practical application as a design tool, however, Theorem 2 presents some computational difficulties. Expansion of a determinant involving literals is not well suited for either hand or machine calculation but is required if the weighting matrix elements are to be varied. The special case considered in Chapter III is one of the simplest non-trivial multi-input, multi-output cases possible, but the resulting expansion formula, equation (3.28), is nonetheless fairly complicated. This is partially offset by the fact that in general the order of the determinant can be made equal to either the number of plant inputs (r) or outputs (m), whichever is smaller, and this is often much less than the number of plant states (n). Also, once the characteristic-squared equation is determined the computational effort required to find the optimal eigenvalues for any given R and Q is negligible compared to resolving the Ricatti equation. This suggests that if the optimal eigenvalues are required for a large number of choices of R and Q, expanding the determinant in Theorem 2, despite its difficulty, may still be computationally efficient.

Probably the most notable result of the numerical example of Chapter IV is the severe restriction optimality, in terms of the given cost functional form, places on the location of the optimal system eigenvalues. The restriction is due to optimality, and not an inherent property of the plant, as it can be shown using modal control theory[11] that linear state variable feedback is capable of moving the eigenvalues of the given plant anywhere in the complex plane (as long as complex eigenvalues are in complex conjugate pairs). Furthermore, Simon and Mitter[12] have shown that for block companion plants with distinct open loop poles the freedom to move the eigenvalues is retained even if the zeros of the transfer matrix are required to remain fixed. As it was also demonstrated in Chapter IV that the zeros of the optimal system must move, except in some special cases, the following result is obtained: Modal control of a block companion plant that leaves the transfer matrix zeros invariant is not, in general, optimal with respect to any quadratic cost functional with diagonal weighting matrices. Besides, except in the trivial case of uncoupled subsystems, invariance of the zeros, if possible at all, requires that the elements of the $\underline{Q}$ matrix be proportional. Allowing $\underline{Q}$ to be non-diagonal does not necessarily alter the situation. The case with a non-diagonal $\underline{R}$ matrix is more involved and no result has been obtained.

Many of the results presented here can be applied, with little modification, to the dual problem of designing a minimum variance linear estimator (Kalman-Bucy filter)[13, 14] for a plant subject to input and measurement disturbances. In this case $\underline{R}$ and $\underline{Q}$ are interpreted as noise covariance matrices, which may not be known exactly a priori. Schweppe[15] discusses some situations in which the present results may be useful in such an estimation context.

Chapter VI

RECOMMENDATIONS FOR FUTURE WORK

The practical application of the results of this investigation would require a general, computationally effective method of expanding the determinant in Theorem 2. One possible approach would be to derive a general expansion formula along the lines of Chapter III, but the result is likely to be too complicated to be very useful. Therefore, despite the awkwardness of manipulating literals on a computer, a direct determinant expansion algorithm may be easier to implement.

If neither of the above approaches proves feasable, a possible alternative to Theorem 2 is the parameter-imbedded Ricatti equation[16]. In this method the Ricatti equation is first solved for a particular choice of the $\underline{R}$ and $\underline{Q}$ matrices and then an imbedding equation is solved numerically for each element of $\underline{R}$ and $\underline{Q}$ that is to be varied. Each imbedding equation yields a function $\underline{K}(k)$ where $\underline{K}$ is the solution of the Ricatti equation and k is the variable weighting element. $\underline{K}(k)$ can then be used in equation (2.10) to yield the optimal system eigenvalues. Various combinations of $\underline{R}$ and $\underline{Q}$ can be considered by using the solution of one imbedding equation as the initial condition for another imbedding equation. As the amount of computational effort required to solve an imbedding equation is of the same order as for a Ricatti equation, the computational advantage of this method over the use of Theorem 2 is not clear at present.

In any case, the ideal method of presenting the eigenvalue plots would be on a computer operated CRT display. This would allow the designer to immediately see the effect of the choice of cost functional on the optimal system eigenvalues.

The location of the zeros of the optimal system was considered only briefly for a special case as a general result has yet to be derived. Further investigation of this point may prove interesting.

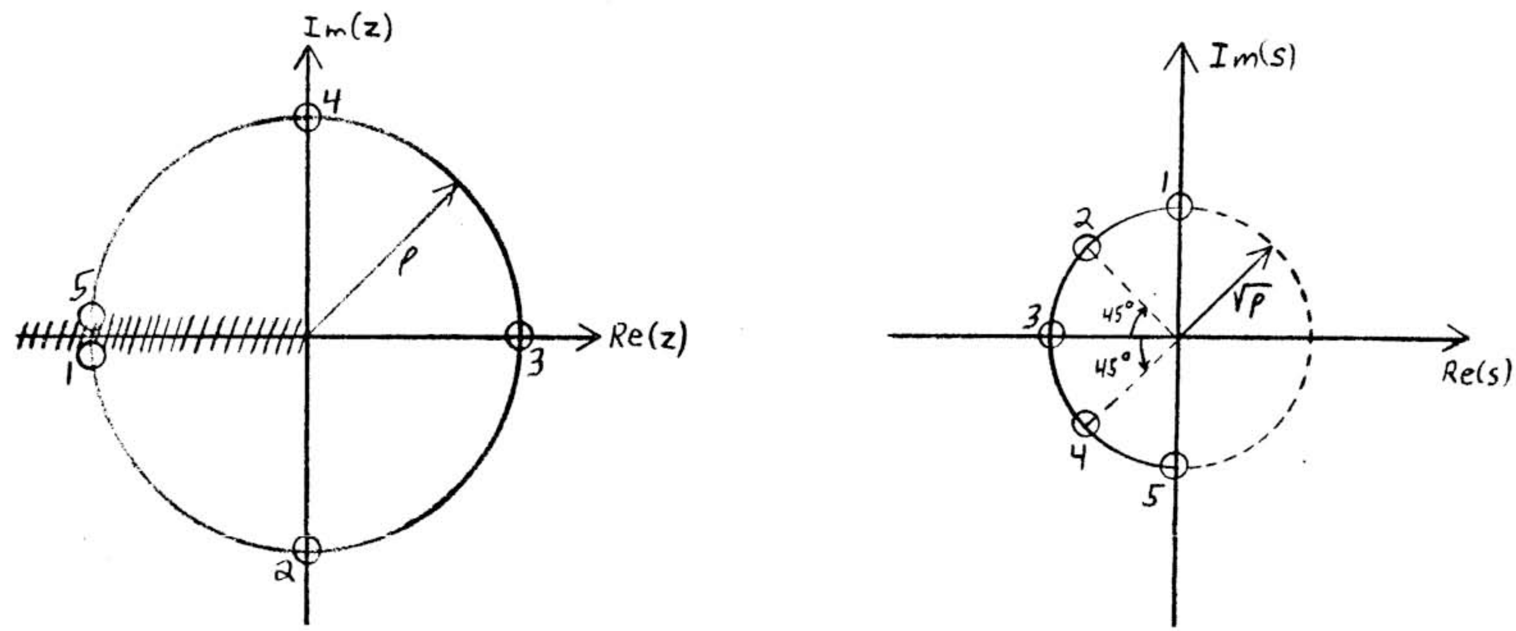

Figure 1 -- Mapping from z-plane to s-plane

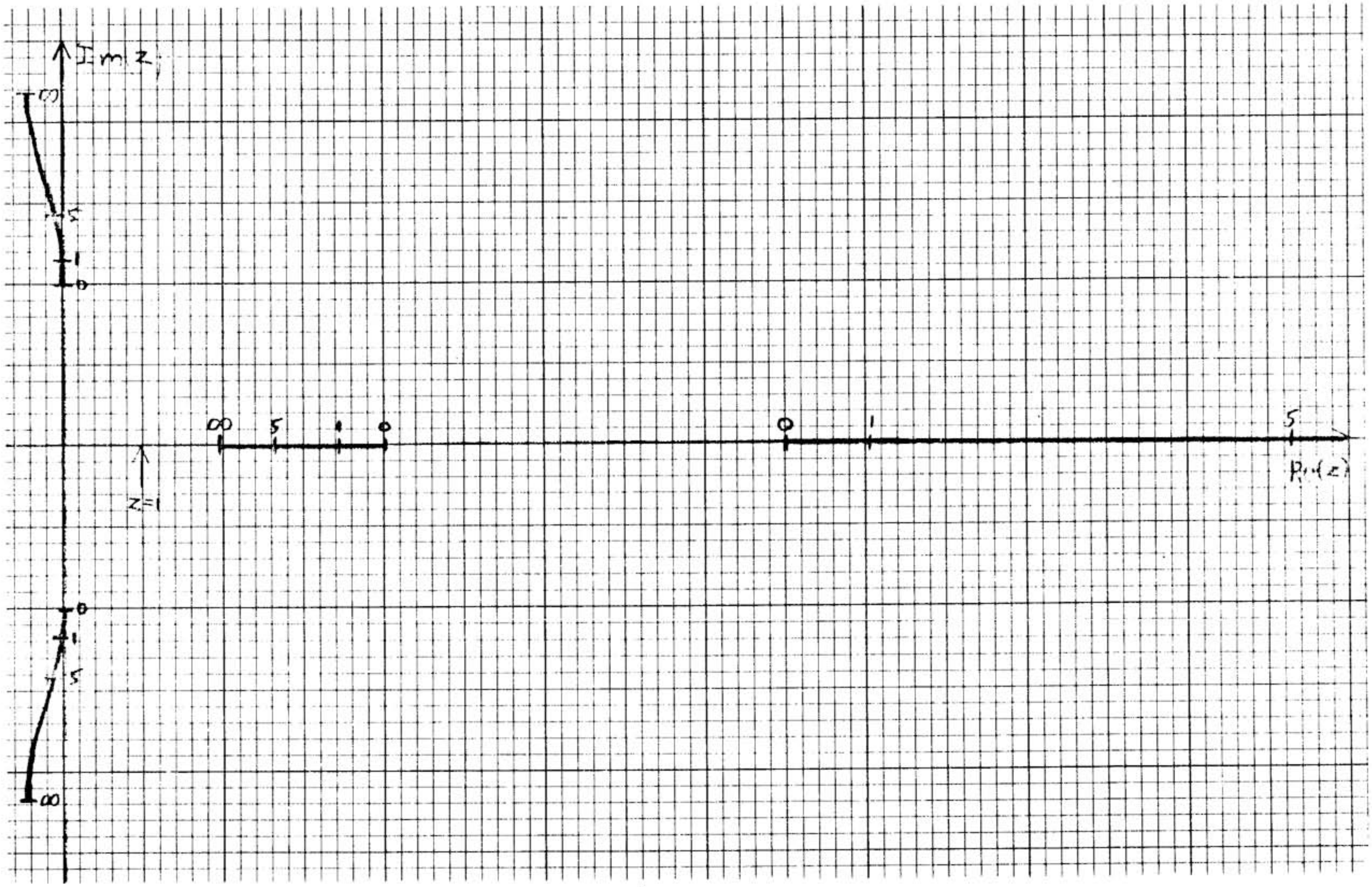

Figure 2 -- Root-square locus versus q_1

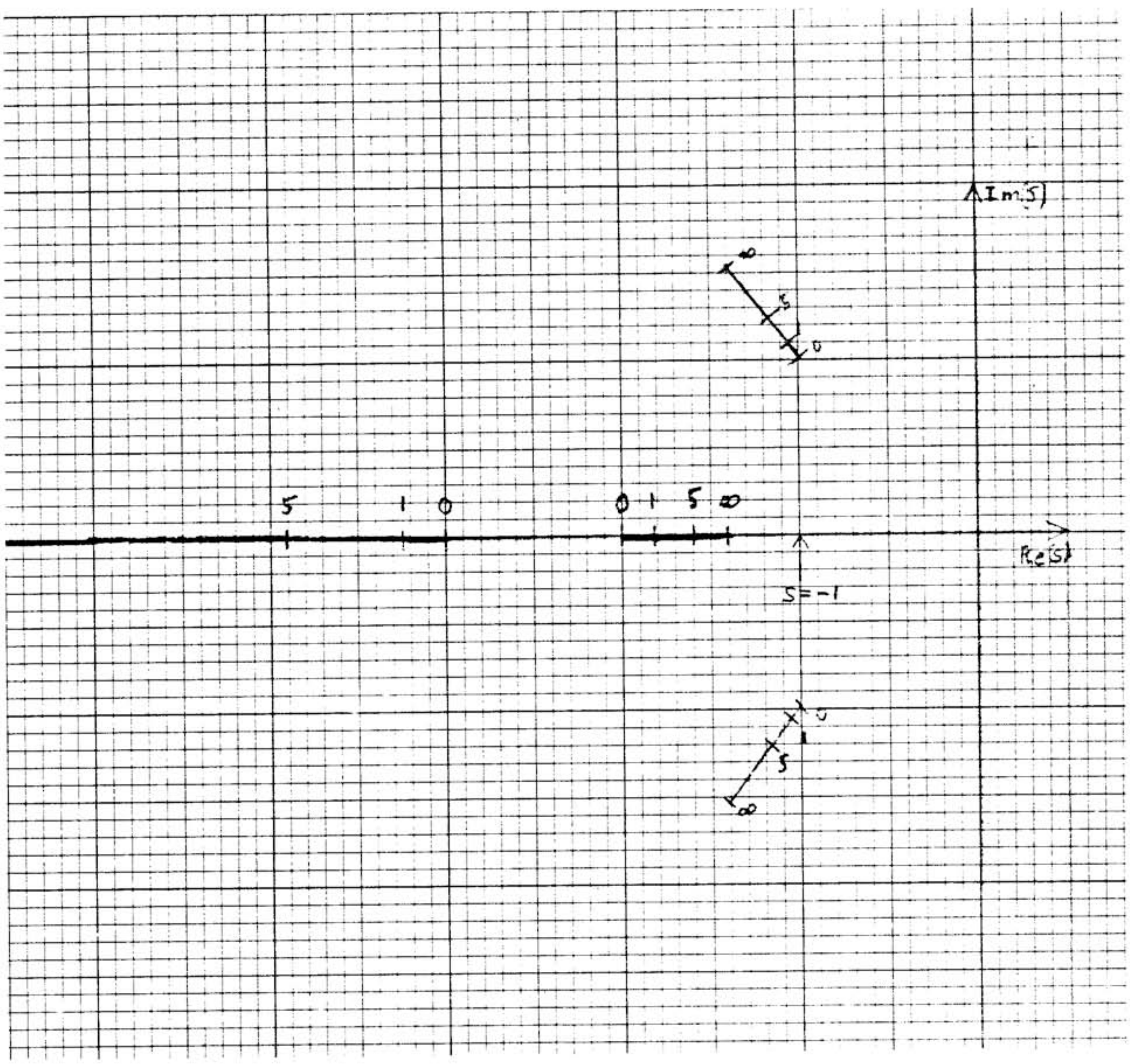

Figure 3 -- Optimal eigenvalues versus c_1

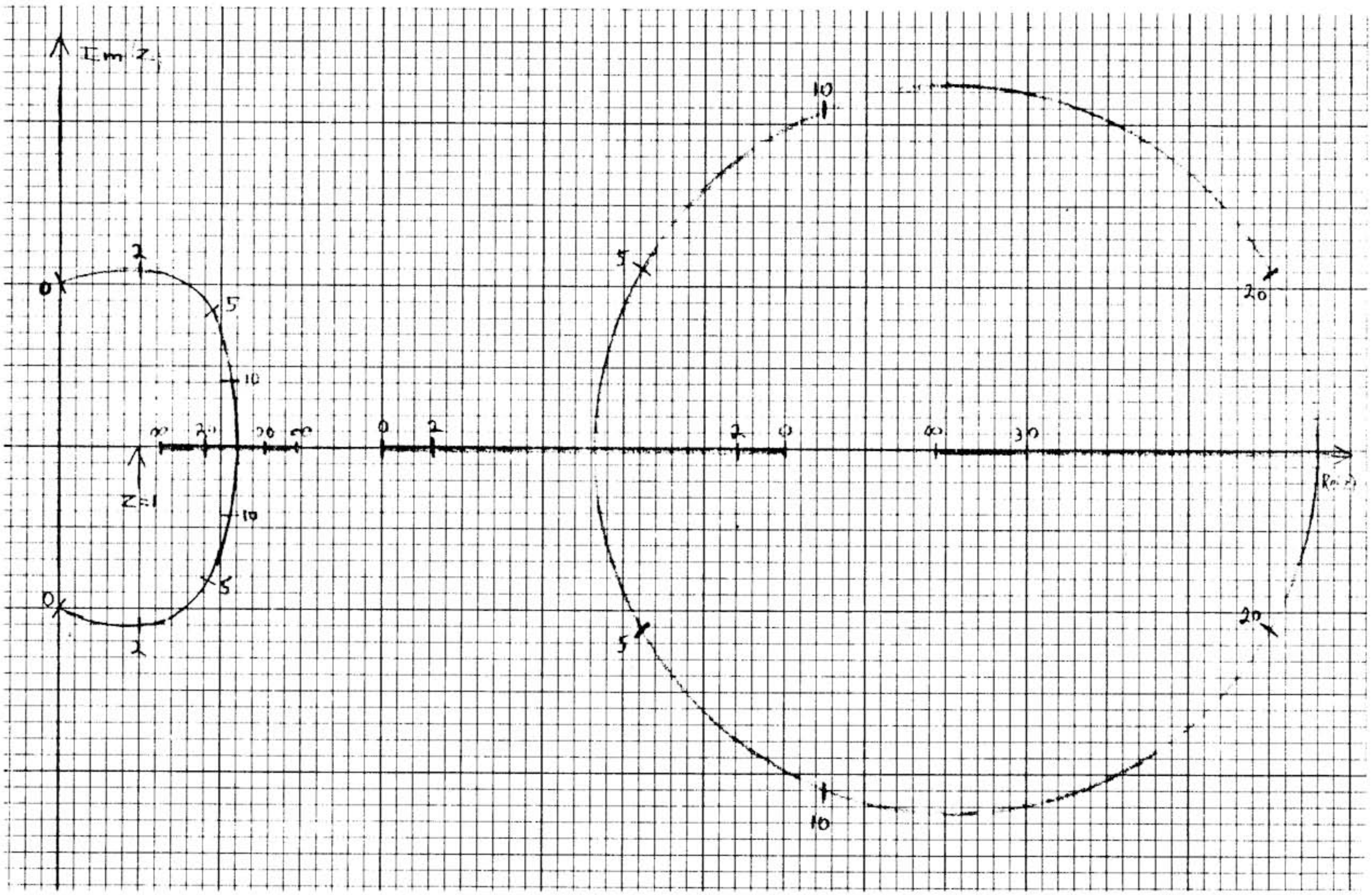

Figure 4 -- Root-square locus versus q_2

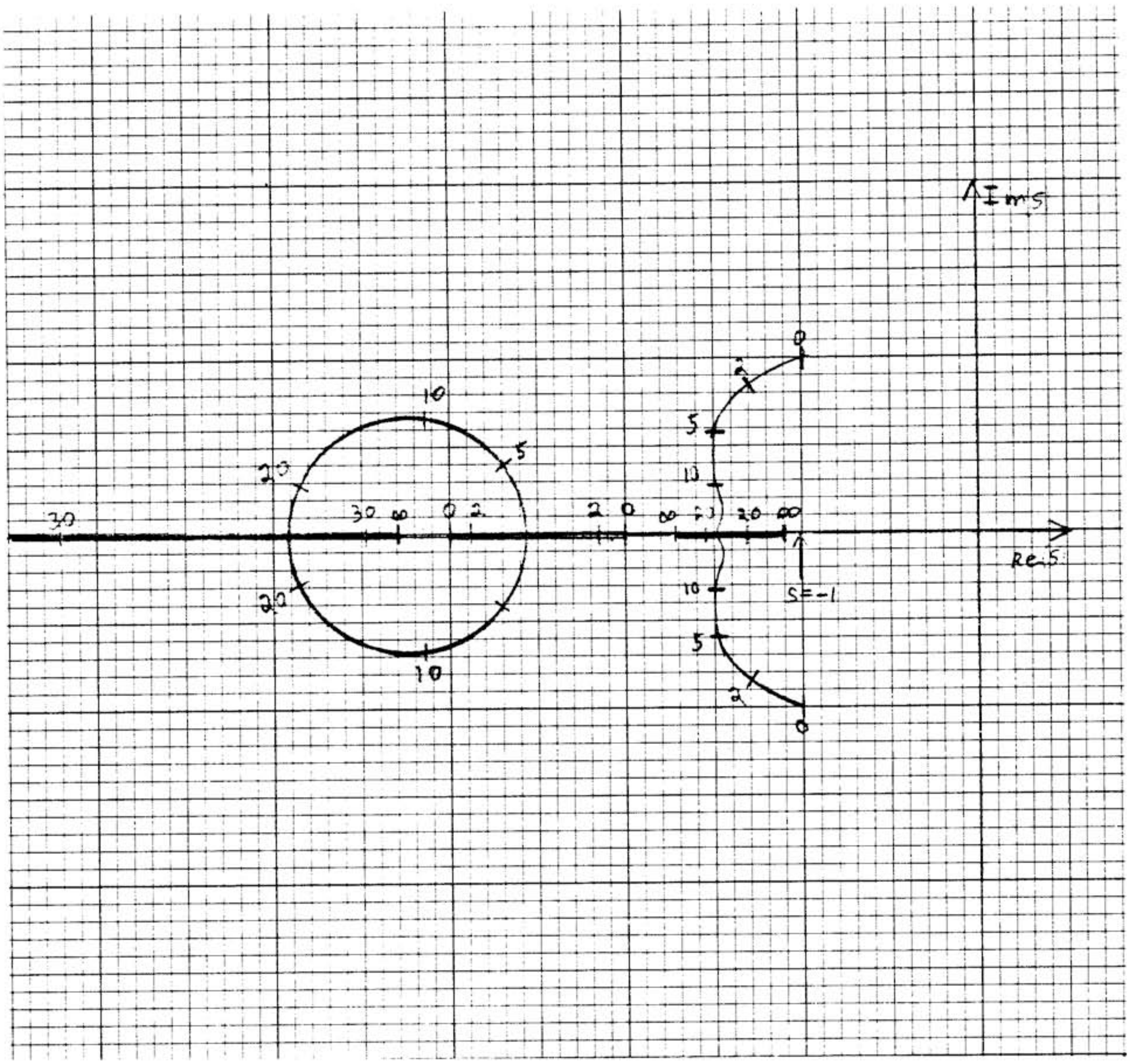

Figure 5 -- Optimal eigenvalues versus q_2

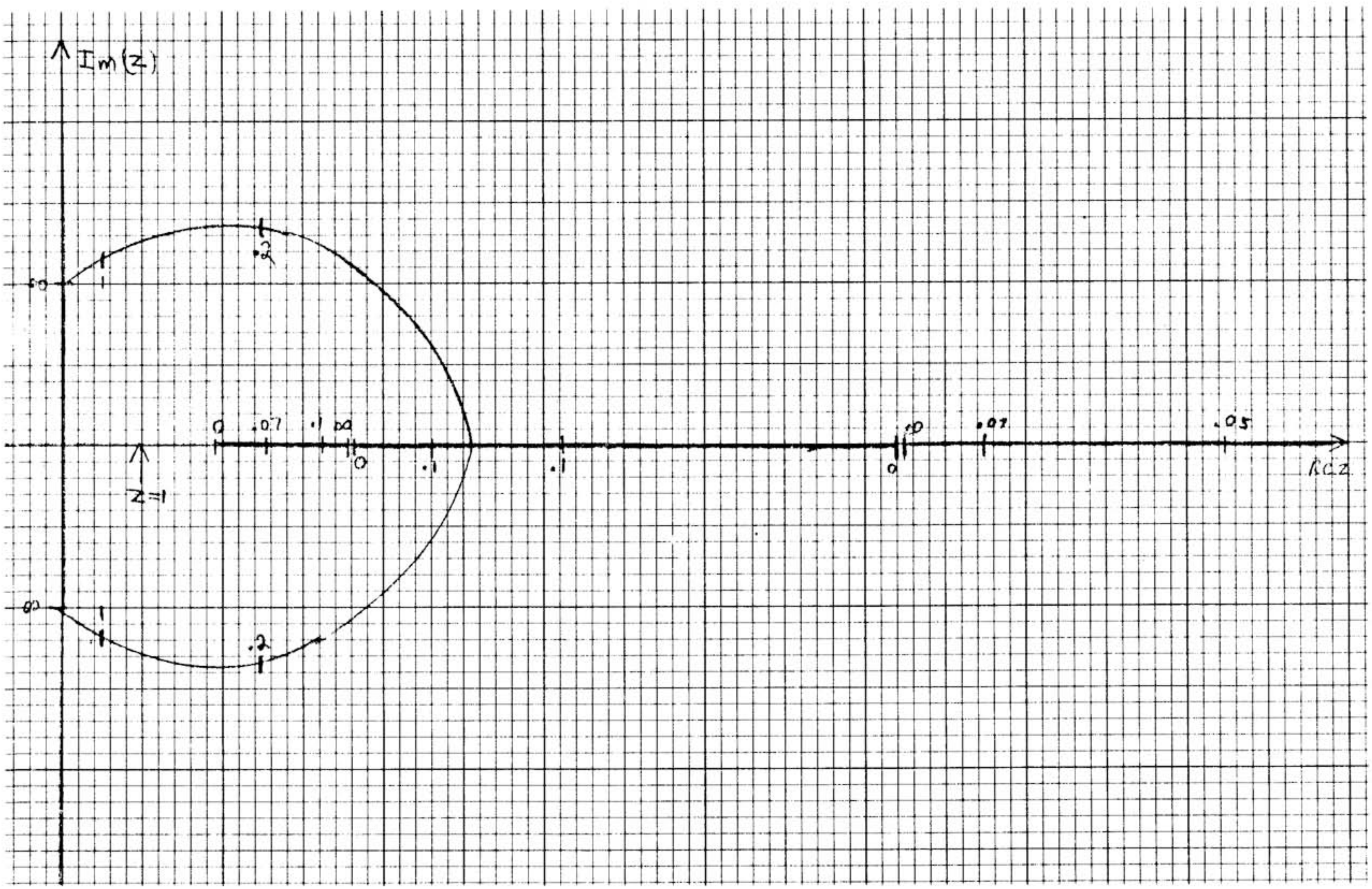

Figure 6 -- Root-square locus versus r_1

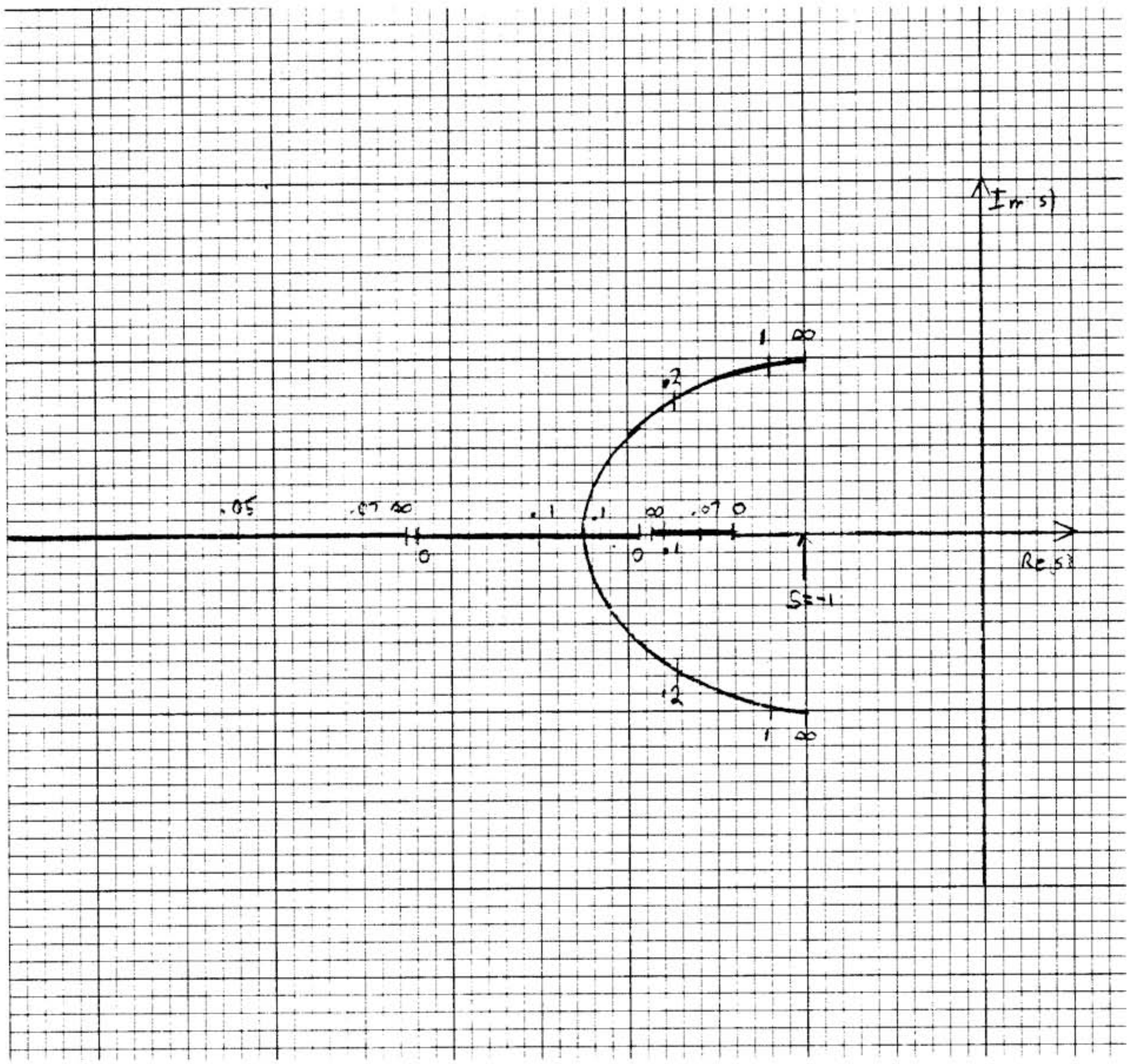

Figure 7 -- Optimal eigenvalues versus r_1

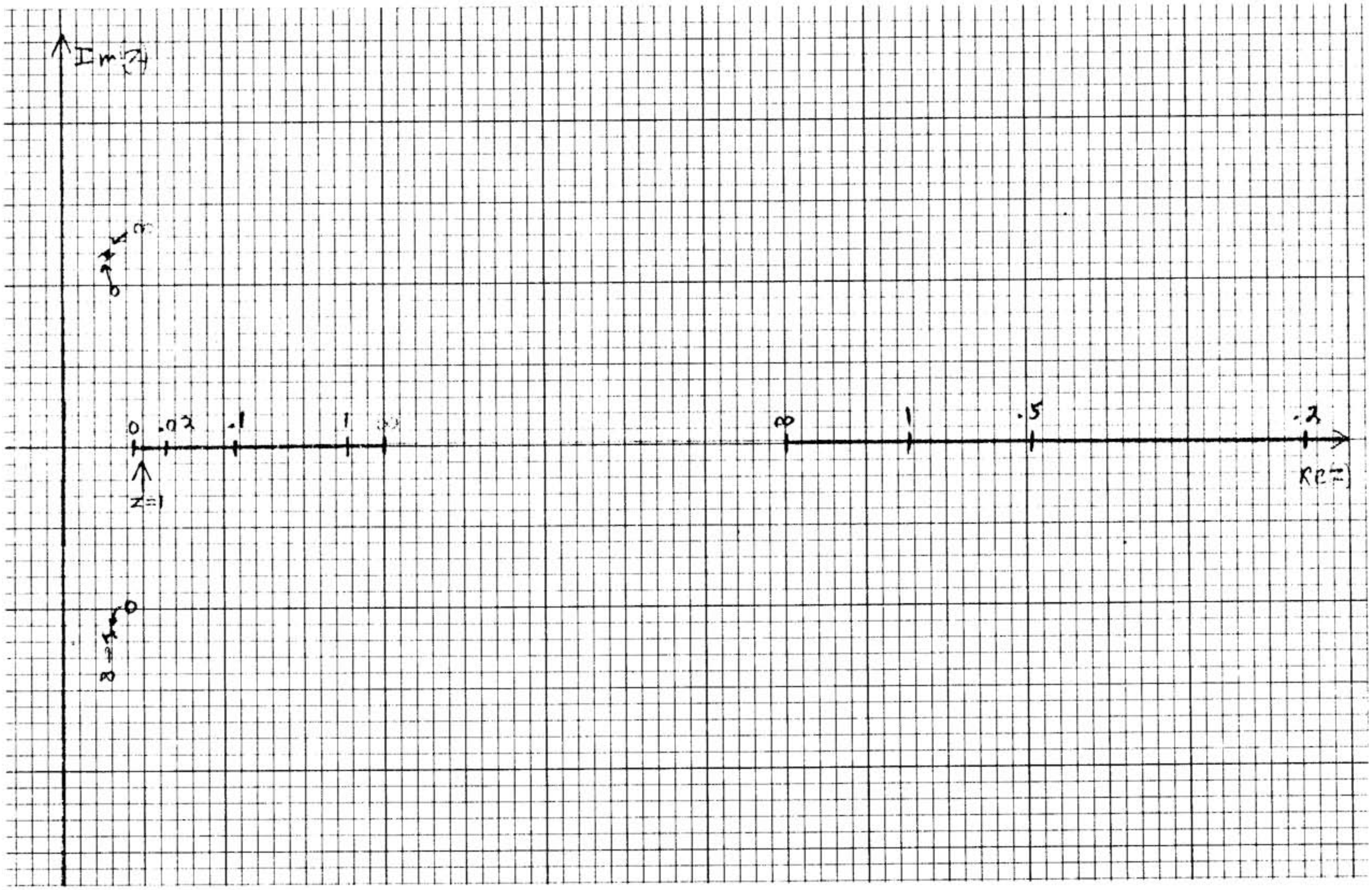

Figure 8 -- Root-square locus versus r_2

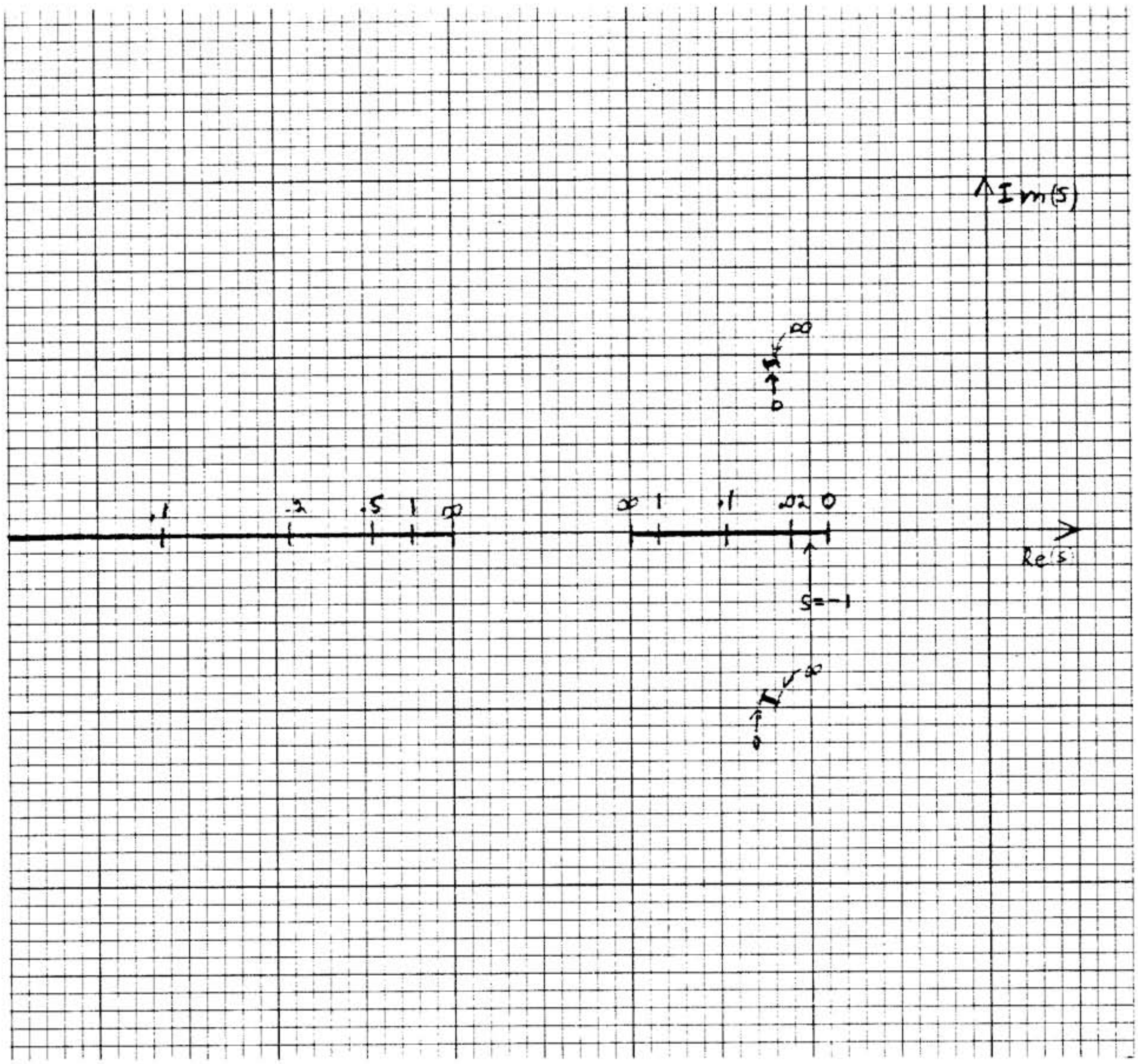

Figure 9 -- Optimal eigenvalues versus r_2

Appendix A

PROPERTIES OF DETERMINANTS

A. INTRODUCTION

This appendix collects some useful properties of determinants that are used throughout this work. The elementary properties listed are generally well known and require little comment. Proofs of the other properties can be found in the indicated references.

B. ELEMENTARY PROPERTIES

Property 1: If $\underline{A}$ is an n x n matrix, then $\det(\underline{A}') = \det(\underline{A})$.

Property 2: If $\underline{A}$ is an n x n matrix, then

$\det(-\underline{A}) = (-1)^n \det(\underline{A})$.

Property 3: If $\underline{A}$ and $\underline{B}$ are n x n matrices, then

$\det(\underline{A}\ \underline{B}) = \det(\underline{A}) \det(\underline{B})$.

Property 4: If all the elements of the k^{th} row (or column) of an n x n matrix ($1 \leq k \leq n$) $\underline{A}$ are zero, then $\det(\underline{A}) = 0$.

Property 5: If two rows (or columns) of an n x n matrix $\underline{A}$ are identical, then $\det(\underline{A}) = 0$.

Property 6: If an n x n matrix $\underline{B}$ is obtained from an n x n matrix $\underline{A}$ by multiplying all elements of the k^{th} row (or column) of $\underline{A}$ by a scalar c, then

$\det(\underline{B}) = c \det(\underline{A})$.

Property 7: If each element of the k^{th} row of an n x n matrix

$\underline{A}$ is the sum of two terms, i.e. $a_{kj}=a^{*}_{kj}+a^{**}_{kj}$ 1 j n, then $\det(\underline{A}) = \det(\underline{A}^*) + \det(\underline{A}^{**})$ where $\underline{A}^*$ and $\underline{A}^{**}$ are obtained from $\underline{A}$ by replacing a_{kj} by a^{*}_{kj} and a^{**}_{kj}, respectively. The analagous result holds for the k^{th} column.

Property 8: The determinant of a block triangular matrix is the product of the determinants of the blocks on the main diagonal.

C. A PARTITIONED DETERMINANT THEOREM[17]

If $\underline{F}$ is a partitioned matrix

$$\underline{F} = \left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right]$$

with $\underline{A}$ and $\underline{D}$ square, then provided $\det(\underline{A}) \neq 0$

$$\det(\underline{F}) = \det(\underline{A}) \det(\underline{D} - \underline{C}\,\underline{A}^{-1}\,\underline{B})$$

and provided $\det(\underline{D}) \neq 0$

$$\det(\underline{F}) = \det(\underline{D}) \det(\underline{A} - \underline{B}\,\underline{D}^{-1}\,\underline{C}) .$$

D. A DETERMINANT ORDER THEOREM[18]

If $\underline{A}$ is a p x q matrix and $\underline{B}$ is a q x p matrix and s is a non-zero complex number, then

$$s^{q-p} \det(s\,\underline{I}_p - \underline{A}\,\underline{B}) = \det(s\,\underline{I}_q - \underline{B}\,\underline{A})$$

where $\underline{I}_p$ and $\underline{I}_q$ are, respectively, the p x p and q x q identity matrices.

In particular, for s=1

$$\det(\underline{I}_p - \underline{A}\,\underline{B}) = \det(\underline{I}_q - \underline{B}\,\underline{A}) .$$

E. THE DETERMINANT OF A SUM OF MATRICES

Repeated application of Property 7 of section B yields the following result:[19]

If $\underline{A}$ is the sum of p n x n matrices, i.e.

$$\underline{A} = \underline{A}_1 + \underline{A}_2 + \dots + \underline{A}_p$$

then

$$\det(\underline{A}) = \sum_{\substack{i_1=0 \\ i_1+i_2+\dots+i_p=n}}^{n} \sum_{i_2=0}^{n} \dots \sum_{i_p=0}^{n} \Delta a_1(i_1)a_2(i_2)\dots a_p(i_p)$$

where $\Delta a_1(i_1)a_2(i_2)\dots a_p(i_p)$ denotes the sum of all possible determinants formed by choosing i_1 rows from $\underline{A}_1$, i_2 rows from $\underline{A}_2$, and so on, preserving the ordering of the rows. Since there are p choices for each of the n rows, the overall sum contains n^p determinants.

As an example, let n=p=2, i.e. $\underline{C}=\underline{A}+\underline{B}$ with $\underline{A}$, $\underline{B}$, and $\underline{C}$ 2 x 2 matrices. Direct application of Property 7 then gives

$$\det(\underline{C}) = \det(\underline{A} + \underline{B}) = \det \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \det \begin{bmatrix} b_{11} & b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} + \det \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$+ \det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} .$$

This agrees with the expansion formula as

$$\Delta a(2)b(0) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Delta a(1)b(1) = \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} + \det \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Delta a(0)b(2) = \det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

are the only terms that satisfy the constraint $i_a + i_b = 2$.

Appendix B

BLOCK COMPANION SYSTEMS

A. DEFINITION

A "block companion" system is defined as a dynamical system

$$d\underline{x}/dt = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$

$$\underline{y} = \underline{C}\,\underline{x}$$

in which $\underline{A}$ is a block diagonal matrix with each block on the main diagonal a companion matrix, $\underline{B}$ has a special form to be specified, and $\underline{C}$ is constrained only by the requirement that $(\underline{A},\underline{C})$ be completely observable. That is, $\underline{A}$ is an n x n matrix of the form

$$\underline{A} = \begin{bmatrix} \underline{A}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{A}_2 & \cdots & \underline{0} \\ \vdots & \vdots & & \vdots \\ \underline{0} & \underline{0} & \cdots & \underline{A}_m \end{bmatrix}$$

in which the k^{th} block is an $n_k \times n_k$ companion matrix

$$\underline{A}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{0,k} & -a_{1,k} & -a_{2,k} & \cdots & -a_{n_k-1,k} \end{bmatrix}$$

and $\sum_{k=1}^{m} n_k = n$.

$\underline{B}$ is an n x m matrix with a single "1" in each column, in the position corresponding to the last row of of each $\underline{A}_k$ block, i.e.

$$\underline{B} = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & & 1 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 & \underline{0} & & \underline{0} \\ \underline{0} & \underline{b}_2 & & \underline{0} \\ & & \ddots & \\ \underline{0} & \underline{0} & & \underline{b}_m \end{bmatrix} .$$

$\underline{C}$ is an m x n matrix with no particular structure

$$\underline{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} .$$

The complete controllability of $(\underline{A},\underline{B})$ is assured by the definition because each subsystem block $(\underline{A}_k,\underline{b}_k)$ is in controllable canonical form.

B. TRANSFER MATRIX

The transfer matrix is, by definition, $\underline{G}(s)$ where

$$\underline{G}(s) = \underline{C}(s\ \underline{I} - \underline{A})^{-1}\underline{B}.$$

Since, for a block companion system, the $\underline{A}$ matrix is block diagonal,

$$(s\underline{I}-\underline{A})^{-1} = \begin{bmatrix} (s\underline{I}-\underline{A}_1)^{-1} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & (s\underline{I}-\underline{A}_2)^{-1} & \cdots & \underline{0} \\ \vdots & \vdots & & \\ \underline{0} & \underline{0} & \cdots & (s\underline{I}-\underline{A}_m)^{-1} \end{bmatrix}$$

and each block is of the form

$$(s\underline{I}-\underline{A}_k)^{-1} = \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ 0 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_{0,k} & a_{1,k} & a_{2,k} & \cdots & s+a_{n_k-1,k} \end{bmatrix}^{-1} .$$

For any non-singular matrix $\underline{F}$

$$\underline{F}^{-1} = \text{adj}(\underline{F}) \;/\; \det(\underline{F})$$

where adj($\underline{F}$) is the transpose of the matrix of cofactors, i.e.

$$\text{adj}(\underline{F}) = \left[C_{ij}\right]'$$

where

$$C_{ij} = (-1)^{i+j}\, M_{ij}$$

and M_{ij} is the determinant of the submatrix obtained by deleting the i^{th} row and j^{th} column of $\underline{F}$.

As a consequence of the structure of the $\underline{B}$ matrix, only the last column of adj($s\underline{I}-\underline{A}_k$) is of significance, and this can be found by determining C_{ij} for $j=n_k$, i.e.

$$C_{n_k,j} = (-1)^{n_k+j}\, M_{n_k,j} \;.$$

Deleting the n_k^{th} row and j^{th} column of ($s\underline{I}-\underline{A}_k$) yields an upper triangular $n_k-1 \times n_k-1$ matrix with s in the first j-1 diagonal positions and -1 in the remaining n_k-j diagonal positions. Since the determinant of a triangular matrix is the product of the elements on the main diagonal,

$$M_{n_k,j} = s^{j-1} (-1)^{n_k-j}$$

and

$$\begin{aligned} C_{n_k,j} &= (-1)^{n_k+j} M_{n_k,j} \\ &= (-1)^{n_k+j} s^{j-1} (-1)^{n_k-j} \\ &= s^{j-1} . \end{aligned}$$

Therefore

$$\operatorname{adj}(s\underline{I}-\underline{A}_k) = \begin{bmatrix} x & x & \dots & 1 \\ x & x & \dots & s \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x & x & \dots & s^{n_k-1} \end{bmatrix}$$

where the "x" terms are irrelevant for present purposes. Also, using the previously determined cofactors, $\det(s\underline{I}-\underline{A}_k)=\det(\underline{F})$ can be written as

$$\begin{aligned} \det(s\underline{I}-\underline{A}_k) &= \sum_{j=1}^{n_k} f_{n_k,j}\, C_{n_k,j} \\ &= \sum_{j=1}^{n_k} f_{n_k,j}\, s^{j-1} \\ &= a_{0,k}\,(1) + a_{1,k}\, s + a_{2,k}\, s^2 + \dots \\ &\quad +(a_{n_k-1,k} + s)\, s^{n_k-1} \\ &= a_{0,k} + a_{1,k}\, s + a_{2,k}\, s^2 + \dots \\ &\quad + a_{n_k-1,k}\, s^{n_k-1} + s^{n_k} . \end{aligned}$$

Hence, letting $d_k(s)=\det(s\underline{I}-\underline{A}_k)=a_{0,k}+a_{1,k}\ s+\ldots+s^{n_k-1}$,

$$(s\underline{I}-\underline{A}_k)^{-1} = (1/d_k(s)) \begin{bmatrix} x & x & \ldots & 1 \\ x & x & \ldots & s \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x & x & \ldots & s^{n_k-1} \end{bmatrix}$$

and

$$d(s) = \det(s\underline{I}-\underline{A}) = \prod_{k=1}^{m} d_k(s) \ .$$

Multiplying $(s\underline{I}-\underline{A})^{-1}$ by $\underline{B}$ "picks off" the last column of each $(s\underline{I}-\underline{A}_k)^{-1}$ block to give

$$(s\underline{I}-\underline{A})^{-1}\underline{B} = \left[\begin{array}{c:c:c:c} 1/d_1(s) & & & \\ s/d_1(s) & \underline{0} & & \underline{0} \\ \vdots & & & \\ s^{n_1-1}/d_1(s) & & & \\ \hdashline & 1/d_2(s) & & \\ & s/d_2(s) & & \\ \underline{0} & \vdots & & \underline{0} \\ & s^{n_2-1}/d_2(s) & & \\ \hdashline & & \vdots & \\ \hdashline & & & 1/d_m(s) \\ \underline{0} & \underline{0} & & s/d_m(s) \\ & & & \vdots \\ & & & s^{n_m-1}/d_m(s) \end{array}\right] .$$

It is now convenient to define a block index l_j

$$l_j = 1 + \sum_{k=1}^{j-1} n_k \qquad 1\leq j\leq m$$

and partition the rows of $\underline{C}$ to correspond to the partition of $(s\underline{I}-\underline{A})^{-1}\underline{B}$

$$\underline{C} = \left[\begin{array}{ccc|c|ccc} c_{1,l_1} & \cdots & c_{1,l_1+n_1-1} & \cdots & c_{1,l_m} & \cdots & c_{1,l_m+n_m-1} \\ c_{2,l_1} & \cdots & c_{2,l_1+n_1-1} & \cdots & c_{2,l_m} & \cdots & c_{2,l_m+n_m-1} \\ & \vdots & & & & \vdots & \\ c_{m,l_1} & \cdots & c_{m,l_1+n_1-1} & \cdots & c_{m,l_m} & \cdots & c_{m,l_m+n_m-1} \end{array}\right].$$

With $\underline{C}$ written this way, it is easy to verify that

$$\begin{aligned} \underline{G}(s) &= \underline{C}(s\underline{I}-\underline{A})^{-1}\underline{B} \\ &= \begin{bmatrix} n_{11}(s)/d_1(s) & n_{12}(s)/d_2(s) & \cdots & n_{1m}(s)/d_m(s) \\ n_{21}(s)/d_1(s) & n_{22}(s)/d_2(s) & \cdots & n_{2m}(s)/d_m(s) \\ \vdots & \vdots & & \vdots \\ n_{m1}(s)/d_1(s) & n_{m2}(s)/d_2(s) & \cdots & n_{mm}(s)/d_m(s) \end{bmatrix} \end{aligned}$$

where

$$n_{ij}(s) = c_{i,l_j} + c_{i,l_j+1}\, s + \ldots + c_{i,l_j+n_j-1}\, s^{n_j-1}$$

$$l_j = 1 + \sum_{k=1}^{j-1} n_k$$

$$d_j = a_{0,j} + a_{1,j}\, s + \ldots + a_{n_j-1,j}\, s^{n_j-1} + s^{n_j} .$$

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