STABILIZATION OF AN UNSTABLE SYSTEM

BY MEANS OF FINITE MEMORY FEEDBACK

by

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ABSTRACT

The problem of stabilizing by feedback an unstable system is considered within the framework of stationary linear systems. The concept of a simple feedback scheme is introduced and the situation is considered where a simple feedback scheme fails to stabilize the system. In this case a more elaborate feedback scheme, with finite memory, can be used to achieve stability. A study in this direction was done by Krasovskii and his paper is reviewed. Then a simple case of an oscillator is considered and it is proved that a finite memory feedback scheme can stabilize it. Some considerations follow on the problem of determining whether a simple feedback scheme is successful or not. Then the general case is considered and the finite memory feedback is analyzed as a perturbation of the system stabilized by reconstructing the state by means of an observer. It is proved that in this case too the finite memory feedback scheme is successful provided an additional assumption is made. Comments and suggestions for further research conclude the study.

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CONTENTS

1. INTRODUCTION

In this thesis we consider linear, autonomous dynamical systems described in state-space form by the equations

$$(1.1)\qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, and $y \in R^\ell$ is the output vector. The matrix A is $n \times n$, the matrix B is $n \times m$, and the matrix C is $\ell \times n$. It is also assumed that A, B, C are constant matrices.

We want to study the problem of stabilizing an unstable system by means of finite memory feedback. For the purpose of this discussion, we will call a feedback scheme simple if it can be represented by the product of a constant matrix and the output vector:

$$(1.2)\qquad u = Ky$$

We are interested in particular in dynamical systems that cannot be stabilized by a simple feedback scheme and we will consider a more general feedback scheme that will insure their stability.

By way of illustration let us consider the sysyem

$$(1.3) \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{cases}$$

It is easy to verify that this system is both controllable and observable. Let us try to stabilize system (1.3) by means of a simple feedback scheme:

$$(1.4) \qquad u = ky = kx_1$$

We obtain:

$$(1.5) \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (k-1)x_1 \end{cases}$$

The characteristic equation in this case is

$$(1.6) \qquad \lambda^2 + (k-1) = 0$$

and the system is unstable for all values of k . The behaviour of system (1.3) can be compared with the behaviour of the following two systems, which differ from it in the choice of the matrices B or C :

$$(1.7) \quad \begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -x_1 \\ y = x_1 \end{cases}$$

$$(1.8) \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_2 \end{cases}$$

A simple feedback scheme applied to system (1.7) or (1.8) gives the characteristic equation:

$$(1.9) \qquad \lambda^2 - k\lambda + 1 = 0$$

For proper choice of k this equation has roots with negative real parts and the system is stable.

This example indicates that there exist systems which cannot be stabilized by a simple feedback scheme. It also gives a hint that the failure of the simple feedback scheme is associated with the unavailability of a derivative, or more generally of part of the state, for control purposes. This fact is immediately obvious if we consider the scalar differential equations equivalent to system (1.3):

$$(1.10) \qquad \begin{cases} \ddot{x} + x = u \\ y = x \end{cases}$$

to system (1.7):

$$(1.11) \qquad \begin{cases} \ddot{x} + x = \dot{u} \\ y = x \end{cases}$$

and to system (1.8):

$$(1.12) \qquad \begin{cases} \ddot{x} + x = u \\ y = \dot{x} \end{cases} .$$

More generally we can consider the problem of stabilizing system (1.1) when A is not a stability matrix (i. e. a matrix whose eigenvalues have negative real parts) and when B and C are matrices that prevent a simple feedback scheme from being effective. We can expect that the failure of the simple feedback scheme will somehow be related to the unavailability of some key state components for control purpose, and that it will be necessary to reconstruct these components if we want to stabilize the system.

There are two basic issues at hand. First to characterize a system in such a way that it will be clear whether a simple scheme will work or not. Second, given that a simple feedback scheme does not work, to devise a more complicated scheme that will work. In this thesis we will consider only the case of linear autonomous systems. The first problem is at present unsolved, in the sense that a general test, easy to apply, that will indicate whether the simple feedback scheme is suitable or not, is not available. The second problem was considered by Krasovskii (1963), who obtained some general results. Additional results are reported here.

We will consider first the results of Krasovskii in a brief review of his paper; then we will consider in turn two different methods of solving the problem for system (1.3) and for the general system (1.1). Specifically we will consider a feedback of the type

$$u(t) = \int_{-\tau}^{0} K(\vartheta)\, y(t+\vartheta)\, d\vartheta \tag{1.13}$$

and prove that it is possible to choose $K(\vartheta)$ in such a way that systems (1.3) and (1.1) are exponentially stable. In both cases we will consider the effect of the feedback (1.13) as a perturbation of a properly defined, exponentially stable linear autonomous system. Detailed mathematical developments are carried out in the appendices.

2. KRASOVSKII PAPER

Krasovskii considers the general problem of stabilizing by means of feedback a system described by the vector differential equations

$$(2.1)\qquad \begin{cases} \dot{z} = f[t, z, u] \\ w = \varphi[t, z] \end{cases}$$

around an unstable trajectory $z^{\circ}(t)$. He constructs therefore the perturbed equations of motion around $z^{\circ}(t)$:

$$(2.2)\qquad \begin{cases} \dot{x} = p[t, x, u] \\ y = q[t, x] \end{cases}$$

and seeks a feedback of type:

$$(2.3)\qquad \dot{u} = U[t, y(t+\vartheta), u(t+\vartheta)] \qquad (-\tau \leq \vartheta \leq 0, \tau = \text{const} > 0)$$

(where U is a vector whose components are functionals) which will make $x = 0$, $u^{(m)} = \ldots = u = 0$ stable, subject to (2.2), while at the same time minimizing locally an appropriate functional J of the perturbed motion.

Under suitable differentiability conditions for p and q, system (2.2) can be approximated in the neighborhood of the origin by a time varying linear system:

$$(2.4)\qquad \begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x \end{cases}$$

Therefore it is convenient to choose the functionals U_j linear, and the performance functional J quadratic. At this point Krasovskii splits the problem into two separate subproblems. The first one is the usual linear-quadratic problem with state feedback, a solution of which will automatically insure stability. The second one is the reconstruction of the state from the observation of the input and the output.

We will not consider here the first problem, whose solution is nowdays well known (Brockett, 1970; Athans and Falb, 1966); we will just mention that Krasovskii enlarges the state to include the equation $\dot{u}=\zeta$ and introduces appropriate controllability conditions over an interval of length T uniformly in t to insure that a solution of this problem exists. More interesting for our purposes is the problem of the reconstruction of the state. Here again Krasovskii assumes suitable observability conditions over an interval of length τ uniformly in t and states the auxiliary problem of finding a linear operator $P[t, y(\vartheta), u(\vartheta)]$ such that:

$$(2.5) \quad x(t) = P[t, y(t+\vartheta), u(t+\vartheta)] \qquad \begin{matrix}(-\tau \leqslant \vartheta \leqslant 0) \\ (t \geqslant \tau)\end{matrix}$$

Clearly if $x(t)$ can be reconstructed exactly from $y(t)$, $u(t)$ in $[t-\tau, t]$, the original problem is solved.

The key result in Krasovskii's paper is his lemma 4.2 which states that if system (2.4) is observable over an interval of length τ uniformly in t then there exists a linear operator P for which (2.5) is valid. Its form is as follows:

$$(2.6) \quad P[t, y(\vartheta), u(\vartheta)] = \int_{-\tau}^{0} \{L(t,\vartheta) y(\vartheta) + K(t,\vartheta) u(\vartheta)\} \, d\vartheta$$

where all the elements of L and K are continuous and bounded for $t \geqslant \tau$. In order to prove this lemma, Krasovskii considers the auxiliary problem of finding an $n \times \ell$ matrix $V(t,\vartheta)$ defined for $t \geqslant \vartheta$ such that for all vectors x_o in R^n

$$(2.7) \quad x_o = \int_{-\tau}^{0} V(t,\vartheta) C(t+\vartheta) \, \phi(t+\vartheta, t) \, x_o \, d\vartheta$$

where $\phi(t, t_o)$ is the transition matrix of (2.4).

In other words $V(t,\vartheta)$ is the solution of the matrix equation

(2.8) $$\int_{-\tau}^{0} V(t,\vartheta)\, C(t+\vartheta)\, \phi(t+\vartheta,t)\, d\vartheta = I$$

where I is the n-dimensional identity matrix. he assumes the form of $V(t,\vartheta)$ to be

(2.9) $$V(t,\vartheta) = \Lambda\, \phi^{T}(t+\vartheta,t)\, C^{T}(t+\vartheta)$$

with Λ a constant unknown $n \times n$ diagonal matrix. Then (2.8) becomes

(2.10) $$\Lambda \int_{-\tau}^{0} \phi^{T}(t+\vartheta,t)\, C^{T}(t+\vartheta)\, C(t+\vartheta)\, \phi(t+\vartheta,t)\, d\vartheta = I$$

and the assumed observability condition insures that a solution for Λ exists. It should be noted that equation (2.10) would nowdays be written as

(2.11) $$\Lambda\, M(t-\tau,t) = I$$

where $M(t-\tau,t)$ is the observability gramian over the interval $[t-\tau,t]$. Once Λ is found, it is very easy to obtain the operator P in the form (2.6) by means of a few manipulations which are carried out in detail in the paper. The solution so obtained is not unique, and Krasovskii suggests a couple of different ways for obtaining the matrix $V(t,\vartheta)$ as a matrix with piecewise constant or impulsive elements. Taking into account the solution of the linear quadratic problem and the expression for the operator P, the closed loop system, stabilized by this technique, assumes the following form:

$$
(2.12)\quad \begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x \\ \dot{u} = D(t)u + \int_{-\tau}^{0} \left[K_1(t,\vartheta)y(t+\vartheta) + K_2(t,\vartheta)u(t+\vartheta) \right] d\vartheta \end{cases}
$$

The paper then concludes with applications of this result to the study of system (2.1) and the specialization to the autonomous case.

For the purpose of this thesis there are two points which should be emphasized. The first is the form of the control law which results from (2.12). In (2.12) either $u(\cdot)$ is assumed differentiable or the functional equation for $u(\cdot)$ has to be interpreted in the integral equation sense; the control $u(t)$ is given implicitly as the solution of a functional differential equation. In this thesis the control $u(t)$ is given explicitly by the equation

$$(2.13) \quad u(t) = \int_{-\tau}^{0} K(\vartheta)\, y(t+\vartheta)\, d\vartheta$$

which is different from (2.12) though very similar to it. The key issue is that both (2.12) and (2.13) use the whole output from $t-\tau$ to t to produce the control signal. The second point is the technique used to obtain the result. While Krasovskii obtains (2.12) from (2.5) which gives an exact reconstruction of the state at time t, this thesis uses a technique which does not reconstruct the state $x(t)$ exactly from the output $y(t)$, but reconstructs it approximately, giving in addition a perturbing term.

3. THE OSCILLATOR PROBLEM

Let us start by considering the simple two dimensional system:

$$(3.1)\quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{cases}$$

This system is controllable, observable and unstable. We want to stabilize it by means of feedback, but a simple feedback scheme does not work, as was pointed out in the introduction. On the basis of Krasovskii's results we can try a solution of the type:

$$(3.2)\quad u(t) = \int_{-\tau}^{0} k(\vartheta)\, y(t+\vartheta)\, d\vartheta \quad ,$$

consider the existence of k and τ which will solve the problem, and investigate ways of specifying them.

Let us expand $y(t+\vartheta) = x_1(t+\vartheta)$ in t by means of Taylor's theorem:

$$(3.3)\quad x_1(t+\vartheta) = x_1(t) + \dot{x}_1(t)\vartheta + \ddot{x}_1(t+\eta)\frac{\vartheta^2}{2}$$

where η is a point in the interval $[t, t+\vartheta]$. We note that (3.1) allows us to express $\dot{x}_1$ and $\ddot{x}_1$ in terms of x_2 , so that (3.3) becomes:

$$(3.4) \quad x_1(t+\vartheta) = x_1(t) + x_2(t)\vartheta + \dot{x}_2(t+\eta)\frac{\vartheta^2}{2}$$

We can see now that (3.4) allows us to introduce in the feedback a term in $x_2(t)$ so that we have somehow reconstructed the state for feedback purpose. However we have the additional term $\dot{x}_2(t+\eta)$ which can be considered a perturbation. The feedback term then becomes:

$$(3.5) \quad u(t) = x_1(t)\int_{-\tau}^{0} k(\vartheta)\,d\vartheta + x_2(t)\int_{-\tau}^{0} \vartheta k(\vartheta)\,d\vartheta + \int_{-\tau}^{0} \frac{\vartheta^2}{2} k(\vartheta)\,\dot{x}_2(t+\eta)\,d\vartheta$$

Let us define:

$$(3.6) \quad \begin{cases} k_0 \triangleq \int_{-\tau}^{0} k(\vartheta)\,d\vartheta \\ k_1 \triangleq \int_{-\tau}^{0} \vartheta k(\vartheta)\,d\vartheta \end{cases}$$

Then we can write (3.1) as:

$$(3.7)\qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (k_0 - 1)x_1 + k_1 x_2 + \int_{-\tau}^{0} \frac{\vartheta^2}{2} k(\vartheta)\dot{x}_2(t+\eta)\,d\vartheta \end{cases}$$

We can treat (3.7) as a nonhomogeneous system with associated homogeneous system

$$(3.8)\qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (k_0 - 1)x_1 + k_1 x_2 \end{cases}$$

and forcing term or perturbation $\int_{-\tau}^{0} \frac{\vartheta^2}{2} k(\vartheta)\dot{x}_2(t+\eta)\,d\vartheta$ The coefficients k_0 and k_1 depend on the choice of the function $k(\vartheta)$. Let us assume for the time being that it is possible to choose k_0 and k_1 so that (3.8) is exponentially stable. It will be proved later that this is in fact possible. Then we are led to consider the stability of a perturbed system, with a perturbation in the form of a functional. To system (3.8) we can apply theorem 4.6 of Halanay (1966) which is repeated here for convenience.

<u>Theorem</u>: Let us consider the system

$$(3.9)\qquad \dot{x}(t) = A(t, x(t+s)) + f(t, x(t+s))$$

where $A(t, x(t+s))$ is for every t a vector whose components are linear functionals in the space of the functions continuous in $[-\tau, 0]$ (with norms bounded as functions of t), and the components of vector f are for every t continuous functionals in the same space, with the property that

$$(3.10) \quad |f(t, x(t+s))| < \gamma \| x(t+s) \|$$

γ being sufficiently small for $\| x(t+s) \| \leqslant H$.
If the trivial solution of the first-approximation linear system

$$(3.11) \quad \dot{y}(t) = A(t, y(t+s))$$

is uniformly asymptotically stable, then the trivial solution of system (3.9) is likewise uniformly asymptotically stable. ■
Note: $| \quad |$ means euclidean norm, $\| \quad \|$ means uniform norm in Halanay, 1966.
In our case the conditions on system (3.11) are verified by assumption. If we can prove that (3.10) is valid, then we will be insured that our system will be uniformly asymptotically stable, in fact exponentially stable due to linearity and stationarity.

Let us define

$$(3.12) \qquad k_2 \triangleq \int_{-\tau}^{0} \left| \frac{\vartheta^2}{2} k(\vartheta) \right| d\vartheta$$

It is proved in appendix 1 that our perturbing term satisfies (3.10) provided we redefine the functional on $[t-2\tau, t]$ by setting $k(\vartheta) \equiv 0$ on $[-2\tau, -\tau]$. This does not change the problem and allows application of the theorem. The constant γ is given by the expression

$$(3.13) \qquad \gamma = (1 + k_0^*) k_2$$

with

$$(3.14) \qquad k_0^* \triangleq \int_{-\tau}^{0} |k(\vartheta)| d\vartheta$$

The problem is therefore completely solved if, chosen a priori three numbers k_0, k_1, k_2^*, we can find a function $k(\vartheta)$ satisfying (3.6) and such that

$$(3.15) \qquad k_2 \leqslant k_2^*$$

with k_2^* small enough to insure a satisfactory γ. Let us now choose $k(\vartheta)$ in the form of a piecewise constant function. Let us divide the interval $[-\tau, 0]$ in N equal subintervals and let $k(\vartheta)$ be constant in each one of them. It is proved in appendix 2 that given a priori three numbers k_0, k_1, k_2^*, and the number of intervals $N \geqslant 2$, it is always possible to choose the memory interval τ and the values $k^{(j)}$ that $k(\vartheta)$ assumes on the j-th subinterval in such a way that (3.6) and (3.15) are satisfied. The problem is thus completely solved.

We can remark that in order to apply theorem 4.6 of Halanay, we must insure that k_2 is sufficiently small, in particular we might see what happens for $k_2 \to 0$. Let us assume in addition that we choose $k_0 = 0$, $N = 2$. Then $k(\vartheta)$ must have positive and negative values, so that the area under it is zero, and will assume the shape of a doublet (two pulses of opposite polarity side by side). In order for k_2 to tend to zero, we must have the two pulses as close as possible to the origin, so that the interval over which the doublet is different from zero tends to zero. We see therefore that $k(\vartheta)$ tends to a multiple of the derivative of the unit impulse distribution. This we should expect, since

$$(3.16) \qquad \int_{-\tau}^{0} x(t+\vartheta)\,\delta'(\vartheta)\,d\vartheta = \dot{x}(t)$$

and we are just trying to reconstruct the derivative of $x(t)$ in the differential equation

$$(3.17)\qquad \ddot{x} + x = u$$

which is equivalent to system (3.1).

The theory just developed substantiates the expectations of intuitive reasoning and insures that an imperfect realization of a differentiator, within limits, will not impair stability. It also gives a basis for considering tradeoffs in the choice of k_0, k_1, k_2, τ, since we must choose a large enough $|k_1|$ and a small enough k_2 to insure stability. It is clear that k_2 will be kept small if τ is small, while the maximum value of $|k(\vartheta)|$ must increase as τ tends to 0 in order to keep k_1 constant. More precisely the functional dependence of k_0, k_1, k_2 on the values $k^{(j)}$ is of the type

$$(3.18)\qquad \begin{cases} k_0 = \left(\frac{\tau}{N}\right) \sum_j a_{0j} k^{(j)} \\ k_1 = -\frac{1}{2}\left(\frac{\tau}{N}\right)^2 \sum_j a_{1j} k^{(j)} \end{cases}$$

$$(3.19)\qquad k_2 = \frac{1}{3}\left(\frac{\tau}{N}\right)^3 \sum_j a_{2j} |k^{(j)}|$$

where a_{0j}, a_{1j}, a_{2j} are appropriate constants. The values $k^{(j)}$ as determined by (3.18) are linear in k_0, k_1 so they are of first and second order in $\left(\frac{\tau}{N}\right)$ while (3.19) is of third order in $\left(\frac{\tau}{N}\right)$. So it is possible to have $k_2 \to 0$ while k_0, k_1 remain constant, by choosing shorter memory intervals. At the same time equations (3.18) can be written as

$$(3.20) \quad \begin{cases} \frac{N}{\tau} k_0 = \sum_j a_{0j} k^{(j)} \\ -2\left(\frac{N}{\tau}\right)^2 k_1 = \sum_j a_{1j} k^{(j)} \end{cases}$$

so, clearly, for constant k_0, k_1, and τ tending to zero the solution $k^{(j)}$ must have components increasing in absolute value.

4. THE GENERAL PROBLEM

Let us turn now to the general case of system

$$(4.1) \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where A, B, C are constant matrices and the system is controllable and observable. The first thing that should be settled is how to characterize systems for which the simple feedback scheme fails. As a first step in this direction let us transform system (4.1) so as to reduce A to a canonical form. Brockett (1970) states in his theorem 4 of section 12 that it is always possible to reduce A to a block-diagonal form where the blocks corresponding to realeigenvalues have the usual Jordan form, while blocks corresponding to complex eigenvalues have the following form:

$$(4.2) \qquad \begin{bmatrix} S_i & I & 0 & \dots & 0 \\ 0 & S_i & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & S_i \end{bmatrix}$$

with

$$(4.3) \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(4.4) \qquad S_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$$

Brockett refers to Gantmacher (1959) for the proof, however Gantmacher does not have a proof for the part relative to complex eigenvalues, the reduction to canonical form being considered over the complex field. In view of the interest of the canonical form claimed by Brockett for the characterization of linear systems, a proof of it is given in appendix 3, which moreover is applicable to the more general situation of a reduction to canonical form over a field more restricted than the reals.

Let us assume then that system (4.1) has been transformed to canonical form

$$(4.5) \qquad \begin{cases} \dot{x} = \hat{A}x + \hat{B}u \\ y = \hat{C}x \end{cases}$$

with $\hat{A}$ in block-diagonal form according to Brockett.

We can now investigate the relation between controllability/observability and the problem of stabilization by a simple feedback scheme. To this end, let us partition conformally $\hat{A}$, $\hat{B}$, $\hat{C}$ into blocks. The problem then is reduced to that of finding a matrix K such that each one of the blocks $[\hat{A}_j + \hat{B}_j K \hat{C}_j]$ is a stability matrix. If we can find such a matrix K that stabilizes each block, then the problem is solved. If such a matrix does not exist we will have to have recourse to more complicated feedback schemes, in particular we might try a feedback of type

$$(4.6) \qquad u(t) = \int_{-\tau}^{0} K(\vartheta)\, y(t+\vartheta)\, d\vartheta$$

We first note that A , B , C controllable/observable implies that each $\hat{A}_j$, $\hat{B}_j$, $\hat{C}_j$ is controllable and observable (see appendix 6). We can consider the elementary divisors of A (Gantmacher, 1959) and we can distinguish two situations. First: the elementary divisor is a linear monic polynomial over the reals: $(\lambda - \lambda_j)^{n_j}$. Then the block $\hat{A}_j$ assumes the form

$$(4.7) \qquad \begin{bmatrix} \lambda_j & 1 & 0 & \dots & 0 \\ 0 & \lambda_j & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_j \end{bmatrix}$$

Its powers up to the n_j-th are upper triangular matrices, all linearly independent. The products $\hat{A}_j^k \hat{B}_j$ are linear combinations of the columns of $\hat{A}_j^k$ by means of coefficients of $\hat{B}_j$. In order to have the last row of

$$(4.8) \qquad \left[\hat{B}_j , \hat{A}_j \hat{B}_j , \ldots \hat{A}_j^{n_j - 1} \hat{B}_j \right]$$

different from zero it is necessary that $\hat{B}_j$ has a nonzero entry in its last row. The independence of the first n_j powers of $\hat{A}_j$ then insures that the condition is also sufficient for controllability. Similarly, for observability it is necessary and sufficient that $\hat{C}_j$ has a nonzero entry in its first column. In order to stabilize $\hat{A}_j$ we must consider the matrix

$$(4.9) \qquad \hat{A}_j + \hat{B}_j K \hat{C}_j$$

Let us assume that $\hat{B}_j$ and $\hat{C}_j$ satisfy the minimum requirements for controllability/observability i. e. just one nonzero entry in the appropriate column and row. Then $\hat{B}_j K \hat{C}_j$ will have just one element determined by the row of $\hat{B}_j$ and column of $\hat{C}_j$ with a nonzero entry. Since $\hat{A}_j$ is upper-triangular, in order to change its eigenvalues by addition of $\hat{B}_j K C_j$ it is necessary that the nonzero element of $\hat{B}_j K \hat{C}_j$ be a subdiagonal element. Moreover it must be impossible to decompose $\hat{A}_j + \hat{B}_j K \hat{C}_j$ in blocks so that it is block-upper-triangular with some blocks upper-triangular themselves, since these will still have the same eigenvalues. Since this impossibility depends on which row of $\hat{B}_j$ and column of $\hat{C}_j$ have the nonzero entry, we can construct examples of blocks that cannot be stabilized by a simple feedback scheme while being controllable and observable. Second: the elementary divisor is a quadratic monic polynomial over the reals:

$$(4.10) \qquad (\lambda^2 - 2\sigma_j \lambda + \sigma_j^2 + \omega_j^2)^{n_j}$$

Then the block $\hat{A}_j$ assumes the form:

$$(4.11) \qquad \begin{bmatrix} \sigma_j & \omega_j & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ -\omega_j & \sigma_j & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \sigma_j & \omega_j & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & -\omega_j & \sigma_j & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \sigma_j & \omega_j \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & -\omega_j & \sigma_j \end{bmatrix}$$

This is block-upper-triangular and we can repeat for it the discussion done before, working with blocks instead of matrix elements. Controllability/observability give again conditions on the last block-row of $\hat{B}_j$ and first block-column of $\hat{C}_j$. If $\hat{B}_j K \hat{C}_j$ has only one nonzero block, we can construct again examples of controllable and observable blocks for which the simple feedback scheme fails to stabilize.

It is unfortunate that our knowledge of this problem is at present very poor. There is no general theory available, in particular there is no simple test for the feasibility of a simple feedback scheme. Considering the system in block form, as is done here, helps to visualize the mechanism of stabilization from an algebraic point of view, and might give suggestions for theorems or methods of proof, but it represents only a starting point and a lot of work still remains to be done.

5. RELATIONS BETWEEN THE FINITE MEMORY FEEDBACK AND THE OBSERVER

Let us consider now the problem of stabilizing system

$$(5.1) \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with A , B , C constant matrices, under the assumption of controllability and observability. Let us consider the observer (Mitter and Willems, 1971), described by the equations

$$(5.2) \qquad \dot{\xi} = A\xi + Bu + HC(x - \xi)$$

$$(5.3) \qquad \dot{\xi} = (A - HC)\xi + Bu + HCx$$

The observer is a system of same dimension as system (5.1) and is coupled to system (5.1) through the term $HCx = Hy$ We assume that x is not available, but ξ is, for control purpose. Let us close the loop coupling system (5.1) to the observer (5.3) by means of

$$(5.4) \qquad u = \hat{K}\xi$$

We obtain this way a complete system, composed of the original system (5.1), the observer (5.3) and the feedback (5.4). Defining the error between system (5.1) and observer (5.3) as

$$(5.5) \qquad e \triangleq x - \xi$$

we obtain the following description

$$(5.6) \qquad \begin{cases} \dot{x} = (A + B\hat{K})x - B\hat{K}e \\ \dot{\xi} = (A + B\hat{K})\xi + HCe \\ \dot{e} = (A - HC)e \end{cases}$$

This can be interpreted as two nonhomogeneous equations with forcing terms determined by the third, homogeneous equation. Let us consider then the homogeneous system

$$(5.7) \qquad \begin{cases} \dot{x} = (A + B\hat{K})x \\ \dot{\xi} = (A + B\hat{K})\xi \\ \dot{e} = (A - HC)e \end{cases}$$

The controllability/observability hypothesis insures that it is possible to choose $\hat{K}$ and H so that (5.6) is exponentially stable (Mitter, Willems, 1971), moreover its eigenvalues can be chosen at will.

System (5.6) can also be written as

$$(5.8)\quad \begin{cases} \dot{x} = Ax + B\hat{K}\xi \\ \dot{\xi} = (A + B\hat{K} - HC)\xi + HCx \end{cases}$$

We can impose the condition that $\xi(0) = 0$. Then we obtain from the variation of constants formula

$$(5.9)\quad \xi(t) = \int_0^t \Phi_{A+B\hat{K}-HC}(t-s)\, HC\, x(s)\, ds$$

Let us change variables by setting

$$(5.10)\quad \vartheta = s - t$$

which implies

$$(5.11)\quad d\vartheta = ds$$

so that

(5.12) $$\xi(t) = \int_{-t}^{0} \Phi_{A+B\hat{K}-HC}(-\vartheta)\, HC\, x(t+\vartheta)\, d\vartheta$$

Defining now

(5.13) $$\tilde{K}(\vartheta) = \hat{K}\, \Phi_{A+BK-HC}(-\vartheta)\, H$$

we obtain

(5.14) $$\hat{K}\, \xi(t) = \int_{-t}^{0} \tilde{K}(\vartheta)\, y(t+\vartheta)\, d\vartheta$$

Let us compare now this expression with the proposed finite memory feedback scheme

(5.15) $$u = \int_{-\tau}^{0} K(\vartheta)\, y(t+\vartheta)\, d\vartheta$$

and let us choose $K(\vartheta) = \tilde{K}(\vartheta)$ in $[-\tau, 0]$.
In (5.14) we can split the integral from $-t$ to 0 in two integrals: one from $-t$ to $-\tau$, the other from $-\tau$ to 0 . The second one will give the same contribution to (5.8) as the finite memory feedback, which is therefore equivalent to the observer except for the missing contribution

(5.16) $$\int_{-t}^{-\tau} \tilde{K}(\vartheta)\, y(t+\vartheta)\, d\vartheta$$

This last term can be considered as a perturbation on the observer stabilization scheme. More precisely we can say that the finite memory feedback scheme is equivalent to perturbing the observer stabilization scheme by means of the perturbation

$$(5.17) \quad \eta(t) = \begin{cases} \int_{-t}^{-\tau} \tilde{K}(\vartheta)\, y(t+\vartheta)\, d\vartheta & t > \tau \\ 0 & t \leq \tau \end{cases}$$

This equivalence allows us to represent the finite memory scheme, with the choice (3.13) for $K(\vartheta)$, by means of the equation

$$(5.18) \quad \dot{x} = (A + B\hat{K})x - B\hat{K}e - \eta$$

which is derived from (5.6) by adding the perturbing term η. The problem is then reduced to the study of perturbations on an exponentially stable system. Appendix 4 has the details of the proof that the perturbed system will be stable under appropriate conditions if we make the assumption that the matrix $A + B\hat{K} - HC$ is itself a stability matrix. Then the perturbed system will be exponentially stable if

(5.19) $$\frac{\beta\mu}{\nu-\alpha}\, e^{(\alpha-\nu)\tau} - \alpha < 0$$

where α , β , μ , ν are obtained from bounds on the norms of transition matrices:

(5.20) $$\| \phi_{A+B\hat{K}}(t-s) \| \leq \beta e^{-\alpha(t-s)} \qquad \alpha,\beta > 0$$

(5.21) $$\| \phi_{A+B\hat{K}-HC}(-t) \| \leq \mu e^{\nu t} \qquad \mu,\nu > 0$$

and α is chosen so that in addition $\nu > \alpha$. It is clear that the exponential factor in (5.19) with $(\nu-\alpha),\tau > 0$ will allow the inequality to be satisfied for proper choice of τ .

At this point we can also consider the problem of perturbing $\tilde{K}(\vartheta)$ in the interval $[-\tau, 0]$. That is let us assume that the implementation of the ideal $K(\vartheta)$ has some errors. Consider first the system

(5.22) $$\dot{x} = (A + B\hat{K})x - B\hat{K}e + \varepsilon$$

corresponding to the case in which there is no truncation error but only imperfect realization of $\tilde{K}(\vartheta)$ in the interval $[-\tau, 0]$. Then it is proved in appendix 5 that the system (5.22) is exponentially stable if

$$\varphi_0 < \frac{\alpha - \sigma}{\beta} e^{(\sigma - \alpha)\tau} \tag{5.23}$$

where

$$\sigma \triangleq \frac{\beta\mu}{|\nu - \alpha|} e^{(\alpha - \nu)\tau} \tag{5.24}$$

and

$$\varphi_0 \triangleq \int_{-\tau}^{0} \| \varphi(\vartheta) \| \, d\vartheta \tag{5.25}$$

is a measure of the error in implementing $K(\vartheta)$. As long as the error is small enough, the system (5.22) is exponentially stable. Then we can consider again the effect of truncation as was done before. The only change due to the imperfect realization of $K(\vartheta)$ is in the numbers used to bound exponentially $\| x \|$ so it amounts to just a redefinition of α and β and the same arguments carry through.

We have been able then to prove that for an appropriate choice of the kernel, a finite memory feedback is equivalent to introducing perturbations in a system stabilized by an observer. There are two kinds of perturbations, one due to imperfect realization of $K(\vartheta)$ in

$[-\tau, 0]$, the other due to truncation. Neither one will impair stability for an appropriate choice of the parameters in the problem provided $A + B\hat{K} - HC$ is a stability matrix. This additional assumption is satisfied by a class of control systems. Relations (5.19), (5.23), (5.24), and (5.25) are the basic equations to be used for determining the memory length and the margin of error in the implementation of $K(\theta)$.

6. SPECIAL PROBLEMS AND SUGGESTIONS FOR FURTHER RESEARCH

We will consider now several problems related to this thesis which could be the objects of further research.

We have seen how one can choose a matrix $K(\vartheta)$ and scalar τ so that the system can be stabilized. The choice however is not unique, so that it is possible to investigate optimality criteria for the choice and their implications. The problem requires the selection of appropriate optimality criteria and then the solution of the optimization problems thus generated. The task is not easy since one would work in the context of nonlinear functional differential equations, $K(\vartheta)$ being a multiplier of the state x. There could be two different ways of attacking the problem, corresponding to the approaches used here for the oscillator and the observer. In particular it would be very fruitful to find an optimization scheme of recursive type, which would improve at each step an appropriate performance index, while always giving stable solutions.

One could also examine the following suboptimal problem: consider the optimal solution of the linear-quadratic problem and find the optimum cost under the assumption $C = I$. Choose $\hat{K}$ on the basis of this solution and compare the optimum cost with the cost obtained with the finite memory feedback for C as given. Then see if there are possible tradeoffs in the choice of τ and $K(\vartheta)$.

A deeper understanding of the requirements for the feasibility of a simple feedback scheme could also be a useful subject for research. The starting point could conceivably be one of the canonical forms for matrix A and the type of result sought would be a test for feasibility to be done on the original matrix as given.

A related problem of independent interest is the study of the effect on the spectrum of a change of a subdiagonal element in an upper-triangular matrix, in particular a Jordan matrix.

In the course of this investigation the author has been confronted at times with problems where the unknown is a matrix X which appears implicitly in some matrix equation of the type $f(X, A, B, C) = 0$ where A , B , C are known matrices. Quite often f is linear in X , however the matrices are not necessarily square. An extensive research, even if not deep, of the existing literature on matrices has failed to reveal any systematic tretment of this type of problems, which are of great practical interest in the study of dynamical systems. This is another area of more fundamental mathematical character, that would be worth exploring.

A related field of research is the study of the geometrical properties of the set of stability matrices in matrix space. It is shown in appendix 4 that this set contains a convex cone of dimension $n \times n$. An interesting question then is what type of set we obtain by transforming this cone with all possible similarity transformations. It is easy to see that the diagonal stability matrices with equal eigenvalues are invariant under similarity and belong to a line in $R^{n \times n}$. Then $R^{n \times n}$ can be decomposed into the direct sum of 2 invariant subspaces, one of which is this line, and the other is a hyperplane. This decomposition can be utilized to study properties of the set of stability matrices under similarity transformations. The study of the precise structure of the set of stability matrices in $R^{n \times n}$ is an interesting topological problem, whose solution is likely to shed light over many areas of control theory.

In the same vein it is possible to consider more general problems associated with stability matrices, in particular the effects on the spectra of the operations defined for matrices. Very few results are available at present along these lines. More generally one should consider elementary divisors rather than eigenvalues, and see if it is possible to extend to the elementary divisors the results known on eigenvalue assignment and the effects of addition and multiplication on the elementary divisors. The problems can then be complicated by introducing an inner product in $R^{n \times n}$. In fact, since matrices form not only a vector space, but rather an algebra, the resulting structure must be very rich and the possibility of variation of the problem very great.

7. CONCLUSION

We have considered the problem of stabilizing an unstable system by means of finite memory feedback in the event that a simple feedback scheme fails to achieve this goal. We obtained a solution for a two dimensional example (oscillator). We considered the characterization of general systems with respect to the feasibility of stabilization by means of a simple feedback scheme. At present this characterization is not satisfactory and it should be improved. We obtained a solution of the stabilization problem in the general case under an additional assumption, using the theory of the observer. Finally we have given suggestions for further research in this area, specially with respect to the problems that are not fully understood at present. Some of this suggested research has independent interest from a strictly mathematical point of view.

A.1. APPENDIX 1

Given the system

$$(A.1.1)\quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \int_{-\tau}^{0} k(\vartheta)\, x_1(t+\vartheta)\, d\vartheta \end{cases}$$

which can also be written as

$$(A.1.2)\quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (k_0 - 1)x_1 + k_1 x_2 + \frac{1}{2}\int_{-\tau}^{0} \vartheta^2 k(\vartheta)\, \dot{x}_2(t+\eta)\, d\vartheta \end{cases}$$

with $\vartheta \leq \eta \leq 0$ we want to prove that

$$(A.1.3)\quad \frac{1}{2}\int_{-\tau}^{0} \vartheta^2 k(\vartheta)\, \dot{x}_2(t+\eta)\, d\vartheta < \gamma \sup_{s\in[-2\tau,0]} |x(t+s)|$$

with γ arbitrarily small

Let us substitute in (A.1.3) the expression (A.1.1) for $\dot{x}_2$

We obtain

$$(A.1.4)\quad \frac{1}{2}\int_{-\tau}^{0} \vartheta^2 k(\vartheta)\, \dot{x}(t+\eta)\, d\vartheta = \frac{1}{2}\int_{-\tau}^{0} \vartheta^2 k(\vartheta)\Big[-x_1(t+\eta) + \int_{-\tau}^{0} k(\lambda)\, x_1(t+\eta+\lambda)\, d\lambda\Big] d\vartheta$$

$$= -\frac{1}{2}\int_{-\tau}^{0}\vartheta^2 k(\vartheta)x_1(t+\eta)d\vartheta + \frac{1}{2}\int_{-\tau}^{0}\vartheta^2 k(\vartheta)\left[\int_{-\tau}^{0}k(\lambda)x_1(t+\eta+\lambda)d\lambda\right]d\vartheta$$

The terms on the right hand side can be bounded as follows:

(A.1.5) $$\frac{1}{2}\int_{-\tau}^{0}\vartheta^2 k(\vartheta)x_1(t+\eta)d\vartheta \leq \sup_{s\in[-\tau,0]}\frac{1}{2}\|x(t+s)\|\int_{-\tau}^{0}\vartheta^2|k(\vartheta)|d\vartheta =$$

$$= k_2 \sup_{s\in[-\tau,0]}\|x(t+s)\|$$

(A.1.6) $$\frac{1}{2}\int_{-\tau}^{0}\vartheta^2 k(\vartheta)\left[\int_{-\tau}^{0}k(\lambda)x_1(t+\eta+\lambda)d\lambda\right]d\vartheta \leq \sup_{s\in[-2\tau,0]}\frac{1}{2}\|x(t+s)\|\int_{-\tau}^{0}\vartheta^2|k(\vartheta)|\left[\int_{-\tau}^{0}|k(\lambda)|d\lambda\right]d\vartheta$$

with k_2 defined by (3.12).

Define

(A.1.7) $$k_o^* \triangleq \int_{-\tau}^{0}|k(\lambda)|d\lambda$$

Then we obtain

(A.1.8) $$\frac{1}{2}\int_{-\tau}^{0}\vartheta^2 k(\vartheta)x_1(t+\eta)d\vartheta \leq (1+k_o^*)k_2 \sup_{s\in[-2\tau,0]}\|x(t+s)\|$$

This is identical to (A.1.3) with

(A.1.9) $$\gamma = (1+k_o^*)k_2$$

It is pointed out in appendix 2 that k_0 , and therefore also k_0^*, is of first order in τ while k_2 is of third order in τ . Therefore, for small enough τ , γ can be made arbitrarily small.

A.2. APPENDIX 2

Given numbers k_0 , k_1 , k_2^*, we want to find a piecewise constant function $k(\vartheta)$ such that

$$
\text{(A.2.1)} \qquad
\begin{aligned}
k_0 &= \int_{-\tau}^{0} k(\vartheta)\, d\vartheta \\
k_1 &= \int_{-\tau}^{0} \vartheta\, k(\vartheta)\, d\vartheta \\
k_2 &= \int_{-\tau}^{0} \vartheta^2 |k(\vartheta)|\, d\vartheta \leq k_2^*
\end{aligned}
$$

Let us ignore for the time being the inequality determined by k_2^* and let us consider more generally the problem of finding $k(\vartheta)$ when

$$
\text{(A.2.2)} \qquad k_\ell = \int_{-\tau}^{0} \vartheta^\ell k(\vartheta)\, d\vartheta
$$

is given for $\ell = 0, 1, \dots L$.

Let us divide the interval $[-\tau, 0]$ in $N > L$ equal subintervals and let us consider $k(\vartheta)$ as a piecewise constant function in each subinterval, i.e.

$$
\text{(A.2.3)} \qquad k(\vartheta) = k^{(j)} \qquad \text{for} \quad \vartheta \in \left(-j\frac{\tau}{N}, -(j-1)\frac{\tau}{N}\right]
$$

Then

$$(A.2.4)\quad k_\ell = \int_{-\tau}^{0} \vartheta^\ell k(\vartheta)\,d\vartheta = \sum_{j=1}^{N} \int_{-j\frac{\tau}{N}}^{-(j-1)\frac{\tau}{N}} k^{(j)} \vartheta^\ell d\vartheta =$$

$$= \frac{(-1)^{\ell+1}}{\ell+1}\left(\frac{\tau}{N}\right)^{\ell+1} \sum_{j=1}^{N} \left[(j-1)^{\ell+1} - j^{\ell+1}\right] k^{(j)}$$

$$\ell = 0, 1, \ldots L$$

This system of $L+1$ equations in $N \geqslant L$ unknowns can be written

$$(A.2.5)\quad \sum_{j=1}^{N} \Delta_{\ell j}\, k^{(j)} = \tilde{k}_\ell$$

with

$$(A.2.6)\quad \tilde{k}_\ell \triangleq (-1)^{\ell+1}(\ell+1)\left(\frac{N}{\tau}\right)^{\ell+1} k_\ell$$

$$(A.2.7)\quad \Delta_{\ell j} \triangleq \left[(j-1)^{\ell+1} - j^{\ell+1}\right]$$

Let Δ be the matrix with elements $\Delta_{\ell j}$
System (A.2.5) has solution if rank $\Delta = L+1$.
Let us consider the first $L+1$ columns of Δ and form with them the $(L+1)\times(L+1)$ square matrix $\tilde{\Delta}$. We will prove that

$$(A.2.8)\quad \det \tilde{\Delta} = 2^L \cdot 3^{L-1} \ldots L^2 \cdot (L+1) \neq 0$$

so that $\operatorname{rank} \Delta = L+1$ and a solution exists. In particular if $N > L+1$ there will be infinite solutions so that it might be possible to define an optimality criterion that will give a unique optimal solution; this problem however is not considered further.

Let us consider the matrix $\tilde{\Delta}$. It can be expressed as the product of 2 matrices

$$\Delta = \Gamma \Theta \tag{A.2.9}$$

with

$$\Theta \triangleq \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 \\ -1 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -1 & 1 \end{bmatrix} \tag{A.2.10}$$

$$\Gamma \triangleq \begin{bmatrix} 1 & 2 & 3 & . & . & . & (L+1) \\ 1^2 & 2^2 & 3^2 & . & . & . & (L+1)^2 \\ . & . & . & . & . & . & . \\ 1^{(L+1)} & 2^{(L+1)} & 3^{(L+1)} & . & . & . & (L+1)^{(L+1)} \end{bmatrix} \tag{A.2.11}$$

Since Θ is lower triangular its determinant is the product of the diagonal elements, hence

(A.2.12) $$\det \tilde{\Delta} = \det \Theta \det \Gamma = \det \Gamma$$

The matrix Γ is of the type

(A.2.13) $$\begin{bmatrix} a & b & c & \cdot & \cdot & \cdot & \ell \\ a^2 & b^2 & c^2 & \cdot & \cdot & \cdot & \ell^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a^{(L+1)} & b^{(L+1)} & c^{(L+1)} & \cdot & \cdot & \cdot & \ell^{(L+1)} \end{bmatrix}$$

Let us reduce (A.2.13) to upper triangular form by row operations (Frame, 1964). Let us first subtract in turn from each row the preceding one multiplied by a, and get the new matrix

(A.2.14) $$\Gamma_{(1)} = \begin{bmatrix} a & b & c & \cdot & \cdot & \cdot & \ell \\ 0 & b(b-a) & c(c-a) & \cdot & \cdot & & \ell(\ell-a) \\ 0 & b^2(b-a) & c^2(c-a) & \cdot & \cdot & & \ell^2(\ell-a) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & b^L(b-a) & c^L(c-a) & \cdot & \cdot & & \ell^L(\ell-a) \end{bmatrix}$$

Then

(A.2.15) $$\Gamma = \Gamma_{(1)} H_{(1)}$$

with

(A.2.16) $$H_{(1)} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 \\ -a & 1 & 0 & . & . & . & 0 & 0 \\ 0 & -a & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -a & 1 \end{bmatrix}$$

so that $\det H_{(1)} = 1$ and $\det \Gamma = \det \Gamma_{(1)}$.
We can continue this procedure by multiplying successively by the matrices

(A.2.17) $$H_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & -b & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & -b & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & -b & 1 \end{bmatrix}$$

$$(A.2.18)\quad H_{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & -c & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & -c & 1 \end{bmatrix}$$

and so on, each one with determinant equal to 1. We obtain eventually an upper-triangular matrix which has the upper triangular form:

$$(A.2.19)\quad \Gamma_{(L)} = \begin{bmatrix} a & b & c & d & \cdots \\ 0 & b(b-a) & c(c-a) & d(d-a) & \cdots \\ 0 & 0 & c(c-b)(c-a) & d(d-b)(d-a) & \cdots \\ 0 & 0 & 0 & d(d-c)(d-b)(d-a) & \cdots \\ . & . & . & . & . \end{bmatrix}$$

and such that

$$(A.2.20)\quad \det \tilde{\Delta} = \det \Gamma = \det \Gamma_{(L)}$$

Therefore

$$(A.2.21)\quad \det \Gamma = \det \Gamma_{(L)} = a \cdot b(b-a) \cdot c(c-b)(c-a) \cdot \ldots .$$

In particular for $a = 1$, $b = 2$, $c = 3$, . . .
we have

$$\text{(A.2.22)} \quad \det \Gamma = 1^{(L+1)} \cdot 2^{L} \cdot 3^{(L-1)} \cdot \; . \; . \; . \; L^2(L+1) \neq 0$$

Therefore it is possible to solve (A.2.2) for any arbitrary set of numbers k_ℓ .

Suppose we have specified k_0 and k_1 and found a solution $k(\vartheta)$. We still have to satisfy the requirement that

$$\text{(A.2.23)} \quad k_2^* \geq k_2 = \int_{-\tau}^{0} \vartheta^2 |k(\vartheta)| \, d\vartheta \, .$$

Since equations (A.2.5) are linear we see that $k^{(j)}$ are at most of second order in $\frac{N}{\tau}$ on account of (A.2.6) for fixed values of k_0 and k_1 . On the other hand

$$\text{(A.2.24)} \quad k_2 = -\frac{1}{3}\left(\frac{\tau}{N}\right)^3 \sum_{j=1}^{N} \left[(j-1)^{\ell+1} - j^{\ell+1} \right] |k^{(j)}|$$

$$k_2 \leq \frac{1}{3}\left(\frac{\tau}{N}\right)^3 \left| \sum_{j=1}^{N} \left[(j-1)^{\ell+1} - j^{\ell+1} \right] \right| \max |k^{(j)}|$$

But $\max |k^{(j)}| = O\left[(N/\tau)^2 \right]$ at most, so that for fixed N , k_0 and k_1 , $k_2 = O(\tau)$ and it will always be possible to satisfy (A.2.23) by choosing τ small enough. More generally we should choose among the infinite solutions of (A.2.5) a solution that will have large values of $k^{(j)}$ only near zero in order to keep k_2 small. This consideration should be kept in mind in the choice of the optimality criterion if such criterion is used to specify $k(\vartheta)$.

A.3. APPENDIX 3

We want to prove that in the case of complex conjugate eigenvalues, the matrix A is similar to a block-diagonal matrix where blocks corresponding to real eigenvalues have the usual Jordan form, while blocks corresponding to complex conjugate eigenvalues of multiplicity k_i have the $k_i \times k_i$ block-structure

$$\text{(A.3.1)} \quad \hat{A}_i = \begin{bmatrix} S_i & I & O & . & . & . & O \\ O & S_i & I & . & . & . & O \\ . & . & . & . & . & . & . \\ O & O & O & . & . & . & I \end{bmatrix}$$

In (A.3.1) $\hat{A}_i$ is in block form, I is the 2×2 identity matrix O is the 2×2 zero matrix, and

$$\text{(A.3.2)} \quad S_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$$

Similar matrices have the same elementary divisors (Gantmacher, 1959) so it is necessary and sufficient to show that matrix $\hat{A}_i$ corresponds to the elementary divisors

(A.3.3) $(\lambda - \lambda_i)^{k_i}(\lambda - \bar{\lambda}_i)^{k_i}$

where λ_i , $\bar{\lambda}_i$ are the i-th complex conjugate eigenvalues and k_i is the degree of the elementary divisors corresponding to them. (Note: we label with a different index eigenvalues associated with separate elementary divisors even though they might be numerically equal). In other words we must prove that (A.3.3) is the minimal polynomial of $\hat{A}_i$. Let us consider a factorization of the minimal polynomial in polynomials irreducible over the reals. Then to the polynomial (A.3.3) will correspond the polynomial with real coefficients

(A.3.4) $\left[\lambda^2 - (\lambda_i + \bar{\lambda}_i)\lambda + \lambda_i\bar{\lambda}_i\right]^{k_i}$

More generally we might consider a polynomial

(A.3.5) $\left[\psi_i(\lambda)\right]^{k_i}$

irreducible over a more restricted field than the reals. The problem is to construct a matrix that will have (A.3.5) as its minimal polynomial. Therefore we require that

(A.3.6) $\left[\psi_i(\hat{A}_i)\right]^{k_i} = 0$

while

$$(\text{A.3.7}) \qquad [\psi_i(\hat{A}_i)]^k \neq 0$$

for all $k < k_i$.

Consider the $k_i \times k_i$ block-strictly-upper-triangular matrix

$$(\text{A.3.8}) \qquad H_i = \begin{bmatrix} 0 & I & L & . & . & . & M \\ 0 & 0 & J & . & . & . & N \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & K \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} .$$

Each of its powers has one more line of zeros parallel to the main diagonal, so that H_i satisfies the requirement that its k_i-th power is zero while its k-th power, for $k < k_i$, is not zero. It is therefore sufficient to choose $\hat{A}_i$ so that

$$(\text{A.3.9}) \qquad \psi_i(\hat{A}_i) = H_i$$

Choose now any $\tilde{A}_i$ with minimal polynomial $\psi_i(\lambda)$ and form the matrix

$$\text{(A.3.10)} \qquad \hat{A}_i = \begin{bmatrix} \tilde{A}_i & I & 0 & . & . & . & 0 \\ 0 & \tilde{A}_i & I & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \tilde{A}_i \end{bmatrix}$$

It is easy to verify that each power of $\hat{A}_i$ will be block-upper-triangular with diagonal entries the corresponding powers of $\hat{A}_i$, and therefore $\psi_i(\hat{A}_i)$ will be block-upper-triangular with diagonal entries $\psi_i(\tilde{A}_i)$. By construction $\psi_i(\lambda)$ is the minimal polynomial of $\tilde{A}_i$ so that by the Cayley-Hamilton theorem $\psi_i(\tilde{A}_i) = 0$ and $\psi_i(\hat{A}_i)$ will have the required property (A.3.9).

With this construction then one is insured that the block $\hat{A}_i$ will have the correct minimal polynomial and hence there exists a similarity transformation that will put the original matrix into the form claimed. In particular if the field considered is the real field, the irreducible polynomials will have degree at most 2 so the blocks $\tilde{A}_i$ will be 2×2 matrices and can be chosen in form (A.3.2) or, if preferred, in companion form.

A.4. APPENDIX 4

We want to prove that: given the system

$$(A.4.1)\quad \begin{cases} \dot{x} = (A + B\hat{K})x - B\hat{K}e - \eta \\ \dot{e} = (A - HC)e \end{cases}$$

with

$$(A.4.2)\quad \eta(t) = \begin{cases} \int_{-t}^{-\tau} \tilde{K}(\vartheta) C x(t+\vartheta)\, d\vartheta & t > \tau \\ 0 & t \leq \tau \end{cases}$$

and A , B , C a controllable and observable triplet, it is possible to choose τ , H , $\hat{K}$ so that the systems

$$(A.4.3)\quad \dot{x} = (A + B\hat{K})x$$

$$(A.4.4)\quad \dot{e} = (A - HC)e$$

are exponentially stable and the choice

$$(A.4.5)\quad \tilde{K}(\vartheta) = \hat{K}\,\Phi_{A+B\hat{K}-HC}(-\vartheta)H$$

makes (A.4.1) exponentially stable.

On account of (A.4.2) we can consider only the case $t > \tau$. We note that the controllability/observability hypothesis enables us to place at will the eigenvalues of (A.4.3), (A.4.4) (Mitter and Willems, 1971), in particular we can choose them with negative real parts. Equation (A.4.4) is homogeneous, hence we can bound $e(t)$ as follows

$$\|e(t)\| \leq e_0 e^{-\delta t} \tag{A.4.6}$$

with $e_0 = \|e(0)\|$, δ positive numbers. We can choose (A.4.3) exponentially stable so we can bound its transition matrix as follows:

$$\|\Phi_{A+B\hat{R}}(t-s)\| \leq \beta e^{-\alpha(t-s)} \tag{A.4.7}$$

with α, $\beta > 0$. The norms $\| \quad \|$ must be interpreted as follows: in R^n they indicate any convenient norm, in $R^{n \times n}$ they indicate norm induced by the R^n norm by means of

$$\|A\| \triangleq \max_{\|x\|=1} \|Ax\| \tag{A.4.8}$$

Because of linearity, the solution of (A.4.1) can be expressed as the sum of two terms x_1 and x_2 satisfying the equations

(A.4.9) $$\begin{cases} \dot{x}_1 = (A+B\hat{K})x_1 - B\hat{K}e \\ x_1(0) = x_0 \end{cases}$$

(A.4.10) $$\begin{cases} \dot{x}_2 = (A+B\hat{K})x_2 - \eta \\ x_2(0) = 0 \end{cases}$$

To these equations we can apply the variation of constants formula and obtain the bounds:

(A.4.11) $$\|x_1(t)\| \leq \beta\|x_0\|e^{-\alpha t} + \|B\hat{K}\| e_0 \beta \int_0^t e^{-\delta s} e^{-\alpha(t-s)} ds =$$

$$= \left[\beta\|x_0\| + \|B\hat{K}\| e_0 \beta \int_0^t e^{(\alpha-\delta)s} ds \right] e^{-\alpha t}$$

We can choose α and δ at will by placing the eigenvalues of $A + B\hat{K}$ and $A - HC$. If we choose $\delta > \alpha$ then

(A.4.12) $$\|x_1(t)\| \leq \lambda e^{-\alpha t}$$

where λ is a positive number of the order of $\|x_0\| + e_0$.

Next we want to bound the term x_2 . In order to do so we must find a bound for η , so we are led to consider the transition matrix used to define $\tilde{K}(\vartheta)$ i. e. the matrix $\Phi_{A+B\hat{K}-HC}(t)$. Because of linearity we can bound $\Phi_{A+B\hat{K}-HC}(t)$ as follows:

$$\text{(A.4.13)} \quad \| \Phi_{A+B\hat{K}-HC}(t) \| \leq \hat{\mu}\, e^{-\nu t}$$

with $\hat{\mu} > 0$. Let us assume first that $A + B\hat{K} - HC$ is a stability matrix so that $\nu > 0$. Then we can bound the kernel $\tilde{K}(\vartheta)C$ as follows:

$$\text{(A.4.14)} \quad \| \tilde{K}(\vartheta) C \| \leq \mu\, e^{\nu \vartheta}$$

with $\mu, \nu > 0$ and

$$\text{(A.4.15)} \quad \mu \triangleq \hat{\mu}\, \|\hat{K}\| \cdot \|H\| \cdot \|C\|$$

Now we can use (A.4.7), (A.4.12) and (A.4.14) to obtain a bound for $\eta(t)$. From the definition (A.4.2) of η we obtain

$$\text{(A.4.16)} \quad \| \eta(t) \| \leq \int_{-t}^{-\tau} \| \tilde{K}(\vartheta) C \| \cdot \| x(t+\vartheta) \| \, d\vartheta \leq$$

$$\leq \int_{-t}^{-\tau} \|\tilde{K}(\vartheta)C\| \cdot \|x_1(t+\vartheta)\| d\vartheta + \int_{-t}^{-\tau} \|\tilde{K}(\vartheta)C\| \cdot \|x_2(t+\vartheta)\| d\vartheta$$

We introduce (A.4.12) to bound $\|x_1(t)\|$, and the variation of constants formula for equation (A.4.10) to bound $\|x_2(t)\|$ and obtain:

$$\text{(A.4.17)} \quad \|\eta(t)\| \leq \lambda\mu \int_{-t}^{-\tau} e^{\nu\vartheta} e^{-\alpha(t+\vartheta)} d\vartheta + \beta\mu \int_{-t}^{-\tau} e^{\nu\vartheta} \left[\int_0^{t+\vartheta} e^{-\alpha(t+\vartheta-s)} \|\eta(s)\| ds \right] d\vartheta$$

$$= \frac{\lambda\mu}{\nu-\alpha} e^{-\alpha t} \left[e^{(\nu-\alpha)\vartheta} \right]_{-t}^{-\tau} + \beta\mu e^{-\alpha t} \int_{-t}^{-\tau} e^{(\nu-\alpha)\vartheta} \left[\int_0^{t+\vartheta} e^{\alpha s} \|\eta(s)\| ds \right] d\vartheta$$

Let us consider now the second term on the right hand side of this inequality, and let us change in it the order of integration keeping in mind that the domain of integration for the double integral is as shown in figure A.4.1. We obtain

$$\text{(A.4.18)} \quad \int_{-t}^{-\tau} e^{(\nu-\alpha)\vartheta} \left[\int_0^{t+\vartheta} e^{\alpha s} \|\eta(s)\| ds \right] d\vartheta =$$

$$= \int_0^{t-\tau} e^{\alpha s} \|\eta(s)\| \left[\int_{s-t}^{-\tau} e^{(\nu-\alpha)\vartheta} d\vartheta \right] ds$$

$$= \frac{1}{\nu-\alpha} \int_0^{t-\tau} e^{\alpha s} \|\eta(s)\| \left[e^{(\alpha-\nu)\tau} - e^{(\nu-\alpha)(s-t)} \right] ds$$

We can assume that $\nu > \alpha$ because (A.4.12) will remain valid if we replace α with $\alpha^* < \alpha$. Then we can write:

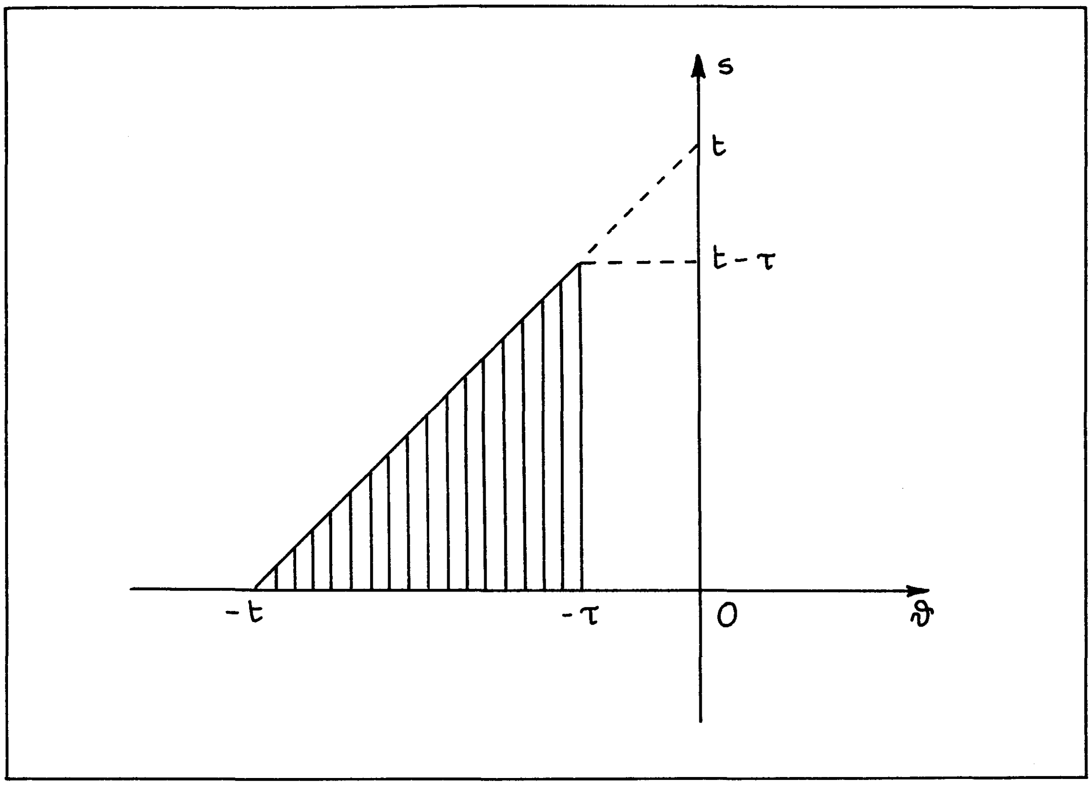

Figure A.4.1

(A.4.19) $$\|\eta(t)\| \leqslant \frac{\lambda\mu}{\nu-\alpha}\, e^{-\alpha t}\left[e^{(\alpha-\nu)\tau} - e^{(\alpha-\nu)t}\right] + $$
$$+ \frac{\beta\mu}{\nu-\alpha}\, e^{-\alpha t}\int_0^{t-\tau} e^{\alpha s}\|\eta(s)\| e^{(\alpha-\nu)\tau}\left[1 - e^{(\alpha-\nu)(t-s-\tau)}\right] ds$$

By hypothesis $\alpha-\nu<0,\ t\geqslant\tau>0$ and $s\leqslant t-\tau$, so the expression (A.4.19) gives the simpler bound:

(A.4.20) $$\|\eta(t)\| \leqslant \frac{\lambda\mu}{\nu-\alpha}\, e^{-\alpha t} + \frac{\beta\mu}{\nu-\alpha}\, e^{(\alpha-\nu)\tau} e^{-\alpha t}\int_0^{t} e^{\alpha s}\|\eta(s)\|\, ds$$

Let us define now the quantities

$$\gamma(t) \triangleq \|\eta(t)\| e^{\alpha t} > 0 \tag{A.4.21}$$

$$\varrho \triangleq \frac{\lambda\mu}{\nu - \alpha} > 0 \tag{A.4.22}$$

$$\sigma \triangleq \frac{\beta\mu}{\nu - \alpha} e^{(\alpha-\nu)\tau} > 0 \tag{A.4.23}$$

Then (A.4.20) can be written as

$$\gamma(t) \leq \varrho + \sigma\int_0^t \gamma(s)\, ds \tag{A.4.24}$$

To this inequality we can apply the Gronwall-Bellman lemma (Halanay, 1966; Bellman, 1953) which is repeated here for convenience.

<u>Lemma</u>: If $u(t), v(t) \geqslant 0$, if C is a positive constant and if

$$u(t) \leq c + \int_0^t u(s)\, v(s)\, ds \tag{A.4.25}$$

then

$$u(t) \leq c \exp\left\{\int_0^t v(s)\, ds\right\} \tag{A.4.26}$$

In our case we obtain

$$\text{(A.4.27)} \quad \chi(t) \leq \varrho e^{\sigma t}$$

from which follows

$$\text{(A.4.28)} \quad \|\eta(t)\| \leq \varrho e^{(\sigma-\alpha)t}$$

We can use now (A.4.28) to obtain a bound for $\|x_2(t)\|$ by means of the variation of constants formula for equation (A.4.10). We obtain

$$\text{(A.4.29)} \quad \|x_2(t)\| \leq \beta \int_0^t e^{-\alpha(t-s)} \|\eta(s)\| ds \leq$$

$$\leq \beta \varrho \int_0^t e^{-\alpha(t-s)} e^{(\sigma-\alpha)s} ds =$$

$$= \beta \varrho e^{-\alpha t} \int_0^t e^{\sigma s} ds = \frac{\beta \varrho}{\sigma} e^{-\alpha t} [e^{\sigma t} - 1]$$

We can express ϱ and σ by means of (A.4.22) and (A.4.23) and obtain

$$\text{(A.4.30)} \quad \|x_2(t)\| \leq \lambda \left[e^{(\sigma-\alpha)t} - e^{-\alpha t} \right]$$

Taking (A.4.23) into account we see therefore that it is possible to choose τ so that $x_2(t)$ is exponentially stable.

Combining now (A.4.12) with (A.4.30) and noting that $\alpha > \alpha - \sigma$ since $\sigma > 0$ we conclude that

$$\text{(A.4.31)} \quad \|x(t)\| \leq 2\lambda\, e^{(\sigma-\alpha)t}$$

and system (A.4.1) is exponentially stable.

If we consider now the case when $\nu \geqslant 0$ the preceding approach fails to prove the stability of $x(t)$ since the bound obtained for $\eta(t)$ becomes a growing exponential. For lack of an alternate approach we will make the assumption that $A + B\hat{K} - HC$ is a stability matrix and show that the set $\mathcal{S}$ of matrices A, B, C such that there exist $\hat{K}$ and H which make all three matrices $A + B\hat{K}$, $A - HC$ and $A + B\hat{K} - HC$ stability matrices, is nonempty. Marcus and Minc (1964) contains the following theorem due to Gerŝgorin.

<u>Theorem</u>: The characteristic roots of an n-square complex matrix A lie in the closed region of the z-plane consisting of all the discs

$$|z - a_{ii}| \leq P_i \triangleq \left(\sum_{j=1}^{n} |a_{ij}|\right) - |a_{ii}| \quad \blacksquare$$

We can use this theorem to prove some properties of the set of stability matrices in $R^{n\times n}$. Let us consider in $R^{n\times n}$ the set of diagonal matrices with all eigenvalues equal. They form a linear variety of dimension one (line) in $R^{n \times n}$. Let us introduce in $R^{n \times n}$ the norm:

$$\text{(A.4.31)} \quad \| A \| = \sum_{i,j=1}^{n} | a_{ij} |$$

Gerŝgorin theorem then implies that given a diagonal stability matrix A and a matrix E with all elements different from zero, the matrix $A + E$ will be a stability matrix provided

$$\text{(A.4.32)} \quad \| E \| < \min \left\{ \left| Re\,(eigen\, A) \right| \right\}$$

In particular if A is a point in the half-line ϱ of diagonal stability matrices with all eigenvalues equal there is a ball of dimension n^2 centered on A and of radius equal to the eigenvalues of A whose interior is formed by stability matrices. On the other hand multiplying a matrix by a positive scalar has the effect of multiplying by the same scalar all its eigenvalues so that if A is a stability matrix then all matrices λA with $\lambda > 0$ are also stability matrices. We can therefore associate to the ball a cone σ of dimension $n \times n$ composed of stability matrices, which moreover is convex since the ball is a convex set. Let us consider now the expression $A + B\hat{K}$. The matrix A represent a point in $R^{n\times n}$ and $B\hat{K}$ represents a subspace of $R^{n\times n}$ if we consider $\hat{K}$ variable. Then $A + B\hat{K}$ represents a linear variety λ_B in $R^{n\times n}$ through the point A . Similarly $A - HC$ generates a linear variety λ_c through A as H varies. In order to have $A + B\hat{K}$, $A - HC$, $A + B\hat{K} - HC$ stability matrices it is sufficient that one of the sets $\lambda_B \cap \sigma$, $\lambda_c \cap \sigma$ be unbounded, or equivalently it is sufficient that there exist K^* or H^* such that for all positive scalars ε , $A + \varepsilon BK^*$ or $A - \varepsilon H^*C$ be contained in the convex hull of A and σ . This will be true if λ_B or λ_c have a nonempty intersection with the set $A \oplus \sigma$ where $\oplus$ denotes direct sum of sets. Since $A \oplus \sigma$ has dimension $n \times n$ it is clear that the set $\mathcal{S}$ is nonempty.

A.5. APPENDIX 5

We want to prove that: given the system

$$(A.5.1)\quad \begin{cases} \dot{e} = (A - HC)e \\ \dot{x} = (A + B\hat{K})x - B\hat{K}e + \varepsilon \end{cases}$$

with $A - HC$ and $A + B\hat{K}$ stability matrices and ε a perturbation due to the imperfect realization of $\tilde{K}(\vartheta)$ in the interval $[-\tau, 0]$, if the perturbation is small enough system (A.5.1) will be stable. (Note: in this case we assume no truncation error; in other words $\tilde{K}(\vartheta)$ is assumed to be in error only in the interval $[-\tau, 0]$ and to coincide with $\hat{K}\,\Phi_{A+B\hat{K}-HC}(-\vartheta)\,H$ in the interval $[-t, -\tau]$, so that its effect is taken into account by the form assumed for (A.5.1)).

Let us define a measure of the error as follows:

$$(A.5.2)\quad \varphi(\vartheta) \triangleq \left[\hat{K}\,\Phi_{A+B\hat{K}-HC}(-\vartheta)HC - \tilde{K}(\vartheta)C\right] \quad \vartheta\in[-\tau,0]$$

Then the term ε in (A.5.1) is

$$(A.5.3)\quad \varepsilon = \int_{-\tau}^{0} \varphi(\vartheta)\, x(t+\vartheta)\, d\vartheta$$

Linearity allows to express x as the sum of 3 contributions due to initial conditions, the input ℓ and the perturbation ε . We are interested in the effect of these 3 terms upon ε . The first two terms in x are exponentials decaying at a rate $\sigma-\alpha$, for $\delta>\alpha$ as seen in appendix 4. When multiplied by $\varphi(\vartheta)$ and integrated over $[-\tau,0]$ they give a total contribution to ε bounded in norm by $\varepsilon_0 e^{(\sigma-\alpha)t}$ where ε_0 is a certain positive number which can be made arbitrarily small by the choice of the initial conditions. Let us for convenience write this bound as $\varepsilon_0 e^{-\alpha t}$ by redefining α . This new value of α can also used in (A.4.7) since $\alpha-\sigma<\alpha$. We can express the third term in $x(t)$ by means of the variation of constants formula and obtain the bound

$$\text{(A.5.4)}\quad \|\varepsilon(t)\| \leq \varepsilon_0 e^{-\alpha t} + \int_{-\tau}^{0} \|\varphi(\vartheta)\|\cdot\|x_3(t+\vartheta)\|\, d\vartheta \leq$$

$$\leq \varepsilon_0 e^{-\alpha t} + \int_{-\tau}^{0} \|\varphi(\vartheta)\| \left[\int_0^{t+\vartheta} \beta e^{-\alpha(t+\vartheta-s)} \|\varepsilon(s)\| ds \right] d\vartheta =$$

$$= \varepsilon_0 e^{-\alpha t} + \beta e^{-\alpha t} \int_{-\tau}^{0} \|\varphi(\vartheta)\| e^{-\alpha\vartheta} \left[\int_0^{t+\vartheta} e^{\alpha s} \|\varepsilon(s)\| ds \right] d\vartheta$$

Let us interchange the order of integration taking into account the domain of integration which is shown in figure A.5.1

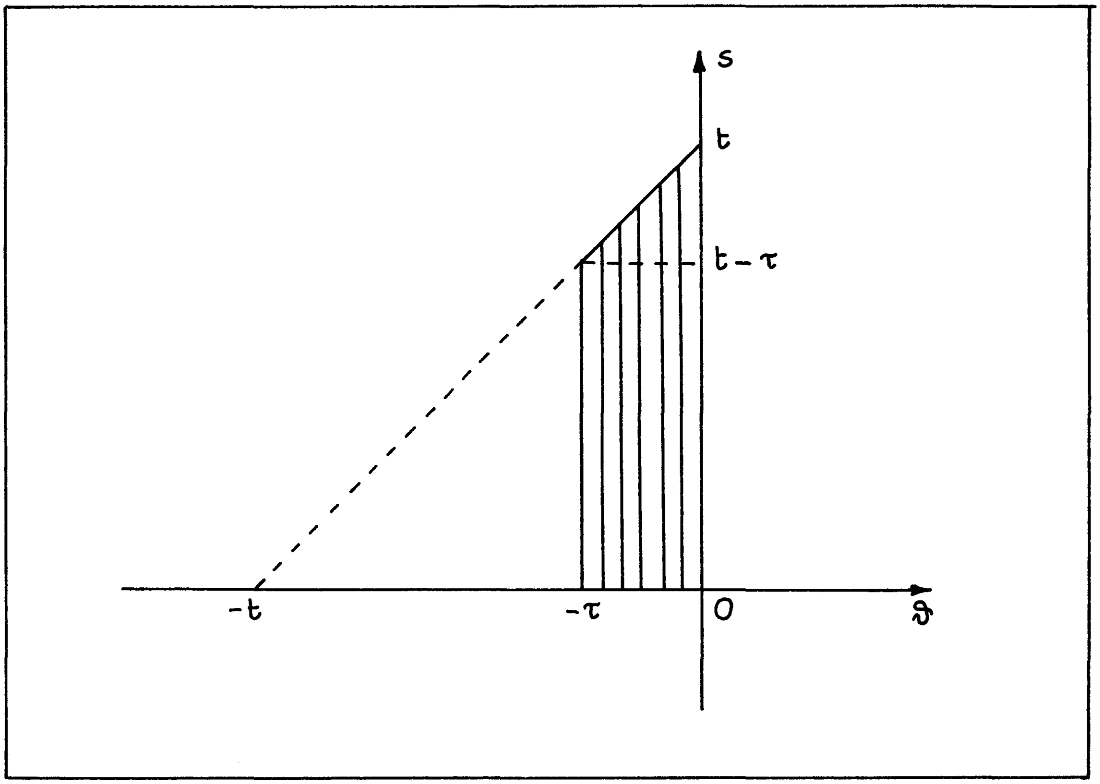

Figure A.5.1

We obtain

$$\text{(A.5.5)}\quad \|\varepsilon(t)\| \leqslant \varepsilon_0 e^{-\alpha t} + \beta e^{-\alpha t}\int_0^{t-\tau} e^{\alpha s}\|\varepsilon(s)\|\left[\int_{-\tau}^{0}\|\varphi(\vartheta)\| e^{-\alpha\vartheta} d\vartheta\right] ds +$$

$$+ \beta e^{-\alpha t}\int_{t-\tau}^{t} e^{\alpha s}\|\varepsilon(s)\|\left[\int_{s-t}^{0}\|\varphi(\vartheta)\| e^{-\alpha\vartheta} d\vartheta\right] ds \leqslant$$

$$\leqslant \varepsilon_0 e^{-\alpha t} + \beta e^{\alpha\tau} e^{-\alpha t}\int_0^{t-\tau} e^{\alpha s}\|\varepsilon(s)\|\left[\int_{-\tau}^{0}\|\varphi(\vartheta)\| d\vartheta\right] ds +$$

$$+ \beta e^{\alpha\tau} e^{-\alpha t}\int_{t-\tau}^{t} e^{\alpha s}\|\varepsilon(s)\|\left[\int_{-\tau}^{0}\|\varphi(\vartheta)\| d\vartheta\right] ds$$

Let us now define

$$\text{(A.5.6)} \quad \varphi_0 \triangleq \int_{-\tau}^{0} \|\varphi(\vartheta)\| \, d\vartheta$$

Then we can write

$$\text{(A.5.7)} \quad \|\varepsilon(t)\| \leq \varepsilon_0 e^{-\alpha t} + \beta e^{\alpha\tau} \varphi_0 e^{-\alpha t} \int_0^t e^{\alpha s} \|\varepsilon(s)\| \, ds$$

This can also be written as

$$\text{(A.5.8)} \quad e^{\alpha t} \|\varepsilon(t)\| \leq \varepsilon_0 + \beta e^{\alpha\tau} \varphi_0 \int_0^t e^{\alpha s} \|\varepsilon(s)\| \, ds$$

Let us define now a new function

$$\text{(A.5.8)} \quad \pi(t) \triangleq e^{\alpha t} \|\varepsilon(t)\| > 0$$

For $\pi(t)$ we obtain from (A.5.8)

$$\text{(A.5.9)} \quad \pi(t) \leq \varepsilon_0 + \beta \varphi_0 e^{\alpha\tau} \int_0^t \pi(s) \, ds$$

To this function we can apply Gronwall-Bellman Lemma (see appendix 4) and obtain

$$\text{(A.5.10)} \quad \pi(t) \leq \varepsilon_0 \exp\{\beta \varphi_0 e^{\alpha\tau} t\}$$

Therefore we deduce that

$$\|\varepsilon(t)\| = \pi(t)e^{-\alpha t} \leq \varepsilon_o \exp\left\{\left(\beta\varphi_o e^{\alpha\tau} - \alpha\right)t\right\} \tag{A.5.11}$$

If φ_o is sufficiently small the exponent will be negative and $\|\varepsilon(t)\|$ will tend to zero exponentially. We can now use bound (A.5.11) to obtain a bound for the third term in x

$$\|x_3(t)\| \leq \beta\int_0^t e^{-\alpha(t-s)}\varepsilon_o \exp\left\{\left(\beta\varphi_o e^{\alpha\tau} - \alpha\right)s\right\}ds = \tag{A.5.12}$$

$$= \beta\varepsilon_o e^{-\alpha t}\int_0^t \exp\left\{\beta\varphi_o e^{\alpha\tau}s\right\}ds = \frac{\varepsilon_o e^{-\alpha t}}{\varphi_o e^{\alpha\tau}}\left[e^{\beta\varphi_o e^{\alpha\tau}} - 1\right]$$

Since we have chosen already φ_o so that

$$\alpha > \beta\varphi_o e^{\alpha\tau} \tag{A.5.13}$$

we are insured that $x(t)$ is exponentially stable with rate of decay $\alpha - \beta\varphi_o e^{\alpha\tau}$. With the nomenclature of appendix 4 (A.5.13) is written as

$$\alpha - \sigma > \beta\varphi_o e^{(\alpha-\sigma)\tau} \tag{A.5.14}$$

and the rate of decay is

$$\alpha - \sigma - \beta\varphi_o e^{(\alpha-\sigma)\tau} \tag{A.5.15}$$

A.6. APPENDIX 6

We want to prove that if A , B , C is a controllable observable triplet, so are each of the triplets of conformal blocks $\hat{A}_j$, $\hat{B}_j$, $\hat{C}_j$ obtained by putting A in Jordan form.

Let us change frame of reference by setting

$$\xi = Sx \tag{A.6.1}$$

which implies

$$x = S^{-1}\xi \tag{A.6.2}$$

with S nonsingular. Then

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{A.6.3}$$

becomes

$$\begin{cases} \dot{\xi} = \hat{A}\xi + \hat{B}u \\ y = \hat{C}\xi \end{cases} \tag{A.6.4}$$

with

$$(\text{A.6.5}) \quad \begin{cases} \hat{A} = SAS^{-1} \\ \hat{B} = SB \\ \hat{C} = CS^{-1} \end{cases}$$

Then the conditions for controllability/observability become:

$$(\text{A.6.6}) \quad \operatorname{rank}\left[\hat{B}, \hat{A}\hat{B}, \ldots \hat{A}^{n-1}\hat{B}\right] = \operatorname{rank}\begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix} = n$$

Let us fix our attention on controllability. Substituting (A.6.5) in (A.6.6) we obtain

$$(\text{A.6.7}) \quad \operatorname{rank}\left[SB, SAB, \ldots SA^{n-1}B\right] = \operatorname{rank} S \cdot \left[B, AB, \ldots A^{n-1}B\right] = n$$

Since S is nonsingular and $[B, AB, \ldots A^{n-1}B]$ has rank n by hypothesis, we see that (A.6.6) is verified. Now $\hat{A}$ is in block-diagonal form so that its powers are also block-diagonal of the same type. Using the partition of $\hat{B}$ we obtain

$$(A.6.8) \quad \operatorname{rank}\left[\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\\ \vdots\\ \hat{B}_k\end{bmatrix}, \begin{bmatrix}\hat{A}_1\hat{B}_1\\ \hat{A}_2\hat{B}_2\\ \vdots\\ \hat{A}_k\hat{B}_k\end{bmatrix}, \ldots, \begin{bmatrix}\hat{A}_1^{n-1}\hat{B}_1\\ \hat{A}_2^{n-1}\hat{B}_2\\ \vdots\\ \hat{A}_k^{n-1}\hat{B}_k\end{bmatrix}\right] = n$$

This is verified if and only if all the rows are linearly independent. In particular the rows of the matrix

$$(A.6.9) \quad \left[\hat{B}_j, \hat{A}_j\hat{B}_j, \ldots \hat{A}_j^{n-1}\hat{B}_j\right]$$

must be independent, so that $\hat{A}_j$, $\hat{B}_j$ must be a controllable pair. An analogous proof is valid for observability.

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