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Hereditary Differential Systems with Constant Delays. I. General Case*

M. C. DELFOUR

Centre de Recherches Mathématiques, Université de Montréal, Montréal 101, Canada

AND

S. K. MITTER

Electronic Systems Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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This paper presents a discussion of the structure of hereditary differential systems defined on a Banach space with initial data in the space of p-integrable maps. Both finite and infinite time histories are allowed. A unified approach to Global and Local Cauchy problems on finite or infinite time intervals is presented. An existence theorem for Carathéodory systems and an existence and uniqueness theorem for Lipschitz systems are derived. In both cases continuity of a solution with respect to the initial data is established.

1. INTRODUCTION

This work is concerned with the study of hereditary differential systems with initial data which are not necessarily continuous. It also introduces appropriate function spaces which constitute the basic framework for the study of these systems. Fundamental results on existence, uniqueness and the continuity of the solution with respect to the initial datum are presented. Although hereditary differential systems with an initial datum in the space of continuous functions have been extensively treated in the literature, a study on the lines of this paper has apparently not been made.

Let $N \geqslant 1$ be an integer, let R be the set of real numbers, let $a > 0$ and $-a = \theta_N < \cdots < \theta_1 < \theta_0 = 0$ be elements of R, let E be a Banach space

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and let $\mathscr{F}(-b, 0; E)$ be a vector space of maps $I(-b, 0) \to E$, where $I(\alpha, \beta) = [\alpha, \beta] \cap R$ for $\alpha < \beta$ in $[-\infty, \infty]$. We denote by m the (complete) Lebesgue measure on R and by f an arbitrary map

$$[t_0, t_1[\times E^{N+1} \times \mathscr{F}(-b, 0; E) \to E,$$

where $t_0 \in R$ and $t_1 \in [t_0 + a, +\infty]$. The *global Cauchy problem* with initial datum $h \in \mathscr{F}(-b, 0; E)$ consists of finding a map x: $[t_0, t_1[\to E$ for which $x(t_0) = h(0)$ and the map

$$s \mapsto \tilde{x}(s) = \begin{Bmatrix} h(s - t_0) & \text{for } I(t_0 - b, t_0) \\ x(s) & \text{for }]t_0, t_1[\end{Bmatrix} : I(t_0 - b, t_0) \cup [t_0, t_1[\to E \tag{1.1}$$

satisfies the equation

$$\text{(S)} \qquad (d\tilde{x}/dt)(t) = f(t, \tilde{x}(t + \theta_N), \ldots, \tilde{x}(t + \theta_1), \tilde{x}(t), \tilde{x}_t) \tag{1.2}$$

almost everywhere in $[t_0, t_1[$, where for each $t \in [t_0, t_1[$, $\tilde{x}_t \in \mathscr{F}(-b, 0; E)$ is defined by

$$\theta \mapsto \tilde{x}_t(\theta) = \tilde{x}(t + \theta): I(-b, 0) \to E. \tag{1.3}$$

(More precise definitions and statements will be given in Section 3.) The *local* and *global* aspects of the Cauchy problem will be unified by a slight modification of the concept of local solution in Section 3.1. The Cauchy problem corresponding to (S) and for $\mathscr{F}(-b, 0; E) = C(-b, 0; E)$ (the space of bounded continuous maps $I(-b, 0) \to E$) has been extensively studied in the literature. An account of this can be found in R. Bellman and K. L. Cooke [1], C. Corduneanu [6, 7] and J. K. Hale and C. Imaz [14]. To compile a complete and meaningful bibiliography is in itself a formidable task. For this reason we shall limit ourselves to a list of references. It is interesting to note that when $\mathscr{F}(-b, 0; E)$ is $C(-b, 0; E)$, (S) is a special case [15, 17] of the differential system

$$(d\tilde{x}/dt)(t) = f(t, \tilde{x}_t), \qquad \text{a.e.} \quad [t_0, t_1[, \tag{1.4}$$

associated with the map f: $[t_0, t_1[\times C(-b, 0; E) \to E$ and the initial datum $h \in C(-b, 0; E)$, since the map

$$h \mapsto (h(\theta_N), \ldots, h(\theta_0)): C(-b, 0; E) \to E^{N+1} \tag{1.5}$$

is linear and continuous.

If $t \in [t_0, t_1[$ is interpreted as *time*, the right side of the differential equation (S) depends on the time t and the *past history* of x corresponding to the time interval $I(t - b, t)$. We shall use the terminology "*hereditary differential* *systems with constant delays*" for such differential equations. These systems have "*finite* (resp., *infinite*) *history*" when $b = a$ (resp., $b = +\infty$).

Our objective is twofold. First the space of initial data will be enlarged from $C(-b, 0; E)$ to the space of p-integrable maps, $\mathscr{L}^p(-b, 0; E)$, $1 \leqslant p < \infty$ (not to be confused with the space of equivalence classes of such maps). In doing this it is no longer obvious that the Cauchy problem for system (S) is a particular case of the corresponding Cauchy problem for system (1.4). Secondly, the hypothesis which requires that f be continuous in its arguments will be relaxed in favor of hypotheses of the Carathéodory type, namely, measurability with respect to t and continuity with respect to the other arguments. In order to obtain global existence (and uniqueness) theorems as well as the continuity of the solution with respect to its initial datum (Theorems 3.3, 3.5 and 3.7) we use two function spaces: one, $M^p(-b, 0; E)$, in which the initial datum will be picked and another, $AC^p(t_0, t; E)$, in which solutions will be sought. The spaces $M^p(-b, 0; E)$ are obtained when one considers a partition of the spaces $\mathscr{L}^p(-b, 0; E)$ which is different from the one leading to the quotient space L^p. Notice that the pointwise character of the initial datum h is only used to obtain $h(0)$ which fixes the value of x at time t_0. The remaining part of h can be treated as an element of the space $L^p(-b, 0; E)$ since f need only be defined almost everywhere for integration with respect to t. This very naturally leads us to introduce the seminorm

$$\| h \| = \left[| h(0) |_E^p + \int_{-b}^{0} | h(\theta) |_E^p \, d\theta \right]^{1/p} \tag{1.6}$$

on the space $\mathscr{L}^p(-b, 0; E)$ and to construct the quotient of $\mathscr{L}^p(-b, 0; E)$ by its linear subspace consisting of all elements with zero seminorm. This space is denoted by $M^p(-b, 0; E)$.

It is felt that the basic contribution of this paper is the fact that if we look at the right-hand side of Eq. (S) as a map defined on $[t_0, t_1[\times E^N \times M^p(-b, 0; E)$ then we can make sense of Eq. (S) and we can also obtain the appropriate existence, uniqueness and well-posedness results with the enlarged space of initial data. The main results have been announced in [8].

The particular framework adopted here is not arbitrary. It represents an attempt to regularize the theory of hereditary differential systems with specific applications to optimal control and stability theories in mind. In particular, when E is a Hilbert space the spaces $M^2(-b, 0; E)$ and $AC^2(t_0, t; E)$ are also Hilbert spaces. Then the theory of optimal control for partial differential equations as developed by J. L. Lions [19] can be effectively used for the study of linear optimal control problems for hereditary differential systems. This will be done in a forthcoming series of papers [10, 11].

Notations and Terminology

Let B be a Banach space and let B^* denote its *topological dual* space. We define the symbol $\langle x, x^* \rangle_B$ by $\langle x, x^* \rangle_B = x^*(x)$, where the right-hand side is the value of the linear form x^* at the point x. The map $(x, x^*) \mapsto \langle x, x^* \rangle_B$ is a bilinear form on $B \times B^*$.

Let $f: E \to F$ be a map between two real topological vector spaces E and F. The map f is an *isomorphism* if it is bijective, linear and bicontinuous.

2. Function Spaces for the Study of Hereditary Differential Systems

This section contains the basic material relevant to the study of hereditary differential systems. We shall use integration theory and L^p spaces with values in a Banach space E as developed by S. Lang [18, Chapters X and XI]. We shall also use his terminology and definitions. $C(\alpha, \beta; E)$ is the Banach space of all bounded continuous maps $I(\alpha, \beta) \to E$ endowed with the usual sup norm $\| \ \|_C$. $C_c(\alpha, \beta; E)$ is the vector space of all continuous maps $I(\alpha, \beta) \to E$ with compact support in $I(\alpha, \beta)$; $\mathscr{L}^p(\alpha, \beta; E)$ is the vector space of all m-measurable (m, the complete Lebesgue measure on R) maps $I(\alpha, \beta) \to E$ which are p integrable, $1 \leqslant p < \infty$, or essentially bounded, $p = \infty$; the natural Banach space associated with $\mathscr{L}^p(\alpha, \beta; E)$ is denoted by $L^p(\alpha, \beta; E)$ and the L^p norm by $\| \ \|_p$.

Let $b \in]0, \infty]$. Consider the following seminorm defined on $\mathscr{L}^p(-b, 0; E)$:

$$\alpha_p(f) = \begin{cases} (|f(0)|^p + \|f\|_p^p)^{1/p} & \text{for } 1 \leqslant p < \infty \\ \max\{|f(0)|, \|f\|_\infty\} & \text{for } p = \infty \end{cases}. \tag{2.1}$$

Let $M^p(-b, 0; E)$ denote the quotient space of $\mathscr{L}^p(-b, 0; E)$ by its linear subspace $S^p = \{f \in \mathscr{L}^p \mid \alpha_p(f) = 0\}$. It is a Banach space with norm α_p and the following facts are easily verified.

PROPOSITION 2.1. (i) *The map*

$$f \mapsto \kappa(f) = (f(0), [f]_{L^p}) : M^p(-b, 0; E) \to E \times L^p(-b, 0; E)$$

is a norm preserving isomorphism when the product space is endowed with the norm

$$\|(x, f)\|_{E \times L^p} = \begin{cases} (|x|^p + \|f\|_p^p)^{1/p}, & 1 \leqslant p < \infty \\ \max(|x|, \|f\|_\infty), & p = \infty. \end{cases} \tag{2.2}$$

(ii) *For all $f \in M^p(-b, 0; E)$, $1 \leqslant p < \infty$, there exists an α_p sequence $\{f_n\}$ of maps in $C_c(-b, 0; E)$ converging to f and such that $f_n(0) = f(0)$ for all n.*

(iii) *For* $1 \leqslant p < \infty$ *the linear subspace* $\{(f(0), [f])_{L^p} \mid f \in C_c(-b, 0; E)\}$ *of* $E \times L^p(-b, 0; E)$ *is dense.*

(iv) *If* E *is a reflexive Banach space and* $1 \leqslant p < \infty$, *the topological dual* $M^p(-b, 0; E)^*$ *of* $M^p(-b, 0; E)$ *is isometrically isomorphic to* $M^q(-b, 0; E^*)$, $q^{-1} + p^{-1} = 1$. *Each continuous linear functional* Λ *on* $M^p(-b, 0; E)$ *has the following representation in terms of a unique element* g *in* $M^q(-b, 0; E^*)$:

$$\Lambda f = \langle f(0), g(0)\rangle_E + \int_{-b}^{0} \langle f(\theta), g(\theta)\rangle_E \, d\theta, \qquad \forall f \in M^p. \quad \blacksquare$$

Remarks. The use of the spaces M^p is not without precedent in the literature of functional differential equations. Indeed motivated by problems in continuum mechanics Coleman and Mizel [4, 5] have introduced spaces analogous to M^p. More specifically they considered a Banach space $\mathscr{B}$ of maps $\phi:]-\infty, 0] \to E$ with norm

$$\| \phi \|_{\mathscr{B}} = | \phi(0)|_E + \int_{-\infty}^{0} k(\theta) \, | \phi(\theta)|_E \, d\theta,$$

where $k(\theta) > 0$, $\int_{-\infty}^{0} k(\theta) \, d\theta < \infty$, $dk/d\theta \geqslant 0$.

For problems of stability where we would like to consider equilibrium solutions other than the zero solution only, it is clearly advantageous to use the norm $\| \, \|_{\mathscr{B}}$ [the nonzero constant functions do not belong to $M^p(-\infty, 0; E)$]. Most of the results presented in this paper can be carried through when we use the space $\mathscr{B}$ as defined above instead of M^p.

On the other hand we were motivated by problems of optimal control and with this in mind we have developed our theory using M^p as the space of initial data.

Let $\alpha \in R$. Let $AC^p(\alpha, \beta, E)$ be the vector space of all maps $f: I(\alpha, \beta) \to E$ which are differentiable almost everywhere in $I(\alpha, \beta)$ with derivative in $\mathscr{L}^p(\alpha, \beta; E)$ and such that

$$f(t) = f(\alpha) + \int_{\alpha}^{t} \frac{df}{ds}(s) \, ds, \qquad t \in I(\alpha, \beta). \tag{2.3}$$

The properties of the spaces AC^p are summarized in the following proposition.

PROPOSITION 2.2. (i) *The functional*

$$n_p(f) = \begin{cases} [| f(\alpha)|^p + \| df/dt \|_p^p]^{1/p}, & 1 \leqslant p < \infty \\ \max\{| f(\alpha)|, \| df/dt \|_\infty\}, & p = \infty \end{cases} \tag{2.4}$$

is a norm on $AC^p(\alpha, \beta; E)$.

(ii) *$AC^p(\alpha, \beta; E)$ is a Banach space isometrically isomorphic to $E \times L^p(\alpha, \beta; E)$.*

(iii) *If E is a Banach space and $1 \leqslant p < \infty$, the topological dual $AC^p(\alpha, \beta; E)^*$ of $AC^p(\alpha, \beta; E)$ is isometrically isomorphic to $AC^q(\alpha, \beta; E^*)$, $q^{-1} + p^{-1} = 1$. Each continuous linear functional Λ on $AC^p(\alpha, \beta; E)$ has the following representation in terms of a unique element g in $AC^q(\alpha, \beta; E^*)$:*

$$\Lambda f = \langle f(\alpha), g(\alpha) \rangle_E + \int_\alpha^\beta \left\langle \frac{df}{dt}(t), \frac{dg}{dt}(t) \right\rangle \, dt, \qquad f \in AC^p. \quad \blacksquare$$

We shall also need certain Fréchet spaces to study the global Cauchy problem. A systematic study of such spaces can be found in [9]. Let t_0 be an element of R and t_1 an element of $[t_0, \infty]$. Denote by $\pi_t(f)$ the restriction of a map $f: [t_0, t_1[\to E$ to the interval $[t_0, t]$, $t \in]t_0, t_1[$. Let $L^p_{\text{loc}}(t_0, t_1; E)$, $AC^p_{\text{loc}}(t_0, t_1; E)$ and $C_{\text{loc}}(t_0, t_1; E)$ be the vector spaces of all maps $x: [t_0, t_1[\to E$ such that for all t in $]t_0, t_1[$, $\pi_t(x)$ is an element of $L^p(t_0, t_1; E)$, $AC^p(t_0, t_1; E)$ and $C(t_0, t_1; E)$, respectively. They are Fréchet spaces when their topology is defined by the saturated family of seminorms $q_t(x) = \| \pi_t(x) \|_F$, t in $]t_0, t_1[$, where F is L^p, AC^p or C, respectively. It is easily seen that the spaces C_{loc}, L^p_{loc} and AC^p_{loc} all have the same structure. In particular the topology generated by the family of seminorms $\{q_t\}$ is equivalent to the initial topology generated by the family of maps $\{\pi_t\}$.

3. Formulation of the Cauchy Problem and Main Theorems

Let E, N, a, b, $I(-b, 0)$, t_0, t_1, $\{\theta_i\}_{i=0}^N$ be as defined in Section 1. In this section we formulate the Cauchy problem in appropriate function spaces. This leads to a theorem on global existence and uniqueness for *Lipschitz systems* and to a theorem on global existence for *Carathéodory systems.* We also establish the continuity of a solution with respect to its initial datum.

Our first objective is to give a precise formulation of the Cauchy problem in appropriate function spaces. We start with $\mathscr{L}^p(-b, 0; E)$ as the space of initial data and with an arbitrary map

$$f: [t_0, t_1[\times E^{N+1} \times \mathscr{L}^p(-b, 0; E) \to E. \tag{3.1}$$

Definition 3.1. (i) The *global Cauchy problem* on $[t_0, t_1[$ associated with the map $f: [t_0, t_1[\times E^{N+1} \times \mathscr{L}^p(-b, 0; E) \to E$ and with the initial datum $h \in \mathscr{L}^p(-b, 0; E)$ at time t_0 consists of finding an element $x \in AC^1_{\text{loc}}(t_0, t_1; E)$ for which

$$x(t_0) = h(0) \tag{3.2}$$

and the map

$$s \mapsto \tilde{x}(s) = \begin{cases} h(s - t_0), & s \in I(t_0 - b, t_0) \\ x(s), & s \in]t_0, t_1[\end{cases} \tag{3.3}$$

satisfies the differential equation

$$\text{(S)} \qquad \frac{d\tilde{x}}{dt} = f(t, \tilde{x}(t + \theta_N), ..., \tilde{x}(t + \theta_1), \tilde{x}(t), \tilde{x}_t), \qquad \text{a.e. in } [t_0, t_1[, \tag{3.4}$$

where $\theta \mapsto \tilde{x}_t(\theta) = \tilde{x}(t + \theta): I(-b, 0) \to E$. The map x will be termed a *global solution* to the Cauchy problem in $[t_0, t_1[$ with initial datum h at time t_0.

(ii) The *local Cauchy problem* on $[t_0, t_1[$ associated with the map f [Eq. (3.1)] and with initial datum $h \in \mathscr{L}^p(-b, 0; E)$ at time t_0 consists of finding a real number α, $0 < \alpha < t_1 - t_0$, for which the global Cauchy problem on $[t_0, t_0 + \alpha]$ has a global solution in $AC^1(t_0, t_0 + \alpha; E)$. This global solution on $[t_0, t_0 + \alpha]$ is called a *local solution* to the Cauchy problem on $[t_0, t_1[$ with initial datum h at time t_0. ∎

We shall now transform our differential equation in a form which has advantages from a technical point of view. For this purpose we introduce the product spaces $B^p(-b, 0; E) = E^N \times M^p(-b, 0; E)$ endowed with the norm

$$\| y \|_{B^p} = \left| \sum_{j=1}^{N} | y_j |^p + \| y_0 \|_{M^p}^p \right|^{1/p}, \tag{3.5}$$

where $y = (y_N, ..., y_1, y_0) \in B^p(-b, 0; E)$. It is a Banach space isomorphic to the product $E^{N+1} \times L^p(-b, 0; E)$ when it is endowed with the norm (3.5) since $M^p(-b, 0; E)$ is isomorphic to $E \times L^p(-b, 0; E)$. The special form of the differential equation (3.4) makes it possible to start with a map $f: [t_0, t_1[\times B^p(-b, 0; E) \to E$.

It is also technically advantageous to introduce the concept of a *memory map*. We abbreviate the spaces $M^p(-b, 0; E)$ and $C_{\text{loc}}(t_0, t_1; E)$ with the symbols M^p and C_{loc}, respectively. Consider the closed linear subspace of all (h, x) in $M^p \times C_{\text{loc}}$ for which $x(t_0) = h(0)$ and denote it $M^p \circ C_{\text{loc}}$.

DEFINITION 3.2. The map

$$s \mapsto (h \circ x)_s : [t_0, t_1[\to M^p, \tag{3.6}$$

where

$$(h \circ x)_s(\theta) = \begin{cases} x(s + \theta), & -(s - t_0) \leqslant \theta \leqslant 0 \\ h(s - t_0 + \theta), & \text{otherwise} \end{cases}, \tag{3.7}$$

is called the *memory map* of (h, x) in $M^p \circ C_{\text{loc}}$ and denoted $h \circ x$. ∎

With this last definition the global Cauchy problem of Definition 3.1 can be slightly reformulated: given f: $[t_0, t_1[\times B^p(-b, 0; E)$ find $x \in AC^1_{loc}(t_0, t_1; E)$ for which the memory map $h \circ x$ associated with $(h, x) \in M^p \circ C_{loc}$ satisfies the differential equation

$$(dx/dt)(t) = f(t, \sigma(h, x)(t)) \qquad \text{a.e. in} \quad [t_0, t_1[, \tag{3.8}$$

where the map

$$\sigma : M^p \circ C_{loc} \to L^p_{loc}(t_0, t_1; B^p(-b, 0; E)) \tag{3.9}$$

is defined by

$$\sigma(h, x)(t) = ((h \circ x)_t (\theta_N), \ldots, (h \circ x)_t (\theta_1), (h \circ x)_t). \tag{3.10}$$

(This makes sense by Proposition 4.3.) Section 4 will be devoted to the last delicate technical details associated with the maps

$$t \mapsto (h \circ x)_t (\theta_i): [t_0, t_1[\to E, \tag{3.11}$$

$i = 0, \ldots, N$, and

$$t \mapsto (h \circ x)_t : [t_0, t_1[\to M^p(-b, 0; E) \tag{3.12}$$

used in the construction of $\sigma(h, x)$.

We now state our main results, the proofs of which will be presented later in Section 5.

THEOREM 3.3. *Let the map* f: $[t_0, t_1[\times B^p(-b, 0; E) \to E$ *have the following properties*:

(CAR-1) *The map* $t \mapsto f(t, z)$: $[t_0, t_1[\to E$ *is* m *measurable for all* $z \in B^p(-b, 0; E)$.

(LIP) *There exists a nonnegative function* n *in* $L^q_{loc}(t_0, t_1; R)$, $p^{-1} + q^{-1} = 1$, *such that for all* z_1 *and* z_2 *in* $B^p(-b, 0; E)$

$$|f(t, z_1) - f(t, z_2)| \leqslant n(t) \| z_1 - z_2 \|_{B^p}, \text{ a.e. } [t_0, t_1[.$$

(BC) *The map* $t \mapsto f(t, 0)$: $[t_0, t_1[\to E$ *is an element of* $L^1_{loc}(t_0, t_1; E)$.

Then there exists a unique global solution $\phi(h)$ *in* $AC^1_{loc}(t_0, t_1; E)$ *to the Cauchy problem on* $[t_0, t_1[$ *with initial datum* $h \in M^p(-b, 0; E)$ *at time* t_0 *for the hereditary differential system* (3.8). *Moreover the map*

$$h \mapsto \phi(h) : M^p(-b, 0; E) \to AC^1_{loc}(t_0, t_1; E) \tag{3.13}$$

is continuous and for all $t \in]t_0, t_1[$

$$\| \pi_t(\phi(h) - \phi(k)) \|_{AC^1} \leqslant d(p, t - t_0) \| h - k \|_{M^p} \tag{3.14}$$

for some constant $d(p, t - t_0) > 0$.

COROLLARY 3.4. *In Theorem* 3.3 *if* $n \in L^q(t_0, t_1; R)$ *and the map* $t \mapsto f(t, 0): [t_0, t_1] \to E$ *is in* $L^1(t_0, t_1; E)$, *then* $\phi(h) \in AC^1(t_0, t_1; E)$ *and the map* $h \mapsto \phi(h): M^p(-b, 0; E) \to AC^1(t_0, t_1; E)$ *is continuous.*

Remark. Let $f: [t_0, t_1[\times B^p(-b, 0; E) \to E$ satisfy the hypotheses of Theorem 3.3. Denote by $\phi(t; s, h)$ the solution at time t of the differential system (3.8) with initial datum $h \in M^p(-b, 0; E)$ at time s, $t_0 \leqslant s \leqslant t < t_1$. This solution exists and is unique by Theorem 3.3. Denote by $\tilde{\phi}(t; s, h)$ the element of $M^p(-b, 0; E)$ defined by

$$\tilde{\phi}(t; s, h)(\theta) = \begin{cases} \phi(t+\theta; s, h), & -(t-s) \leqslant \theta \leqslant 0 \\ h(t+\theta) & \text{otherwise} \end{cases}; \tag{3.15}$$

$\tilde{\phi}(t; s, h)$ is the memory map of h and $\phi(\cdot; s, h)$ at time t with respect to time s. The following semigroup properties are easily verified:

(i) $\tilde{\phi}(t; r, h) = \tilde{\phi}(t; s, \tilde{\phi}(s; r, h))$, $t_0 \leqslant r \leqslant s \leqslant t < t_1$,

(ii) $\tilde{\phi}(t; t, h) = h$, $t_0 \leqslant t < t_1$,

(iii) $\| \tilde{\phi}(t; s, h) - \tilde{\phi}(t; s, k) \|_{M^p} \leqslant c(p, t-s) \| h - k \|_{M^p}$

for some positive constant $c(p, \alpha)$ which is monotonically increasing with $\alpha > 0$. ▮

THEOREM 3.5. *Let* $h \in M^p(-b, 0; E)$ *and let the map*

$$f: [t_0, t_1[\times B^p(-b, 0; E) \to E$$

satisfy the following properties:

(CAR-1) *The map* $t \mapsto f(t, z): [t_0, t_1[\to E$ *is* m *measurable for all* $z \in B^p(-b, 0; E)$.

(CAR-2) *The map* $z \mapsto f(t, z): B^p(-b, 0; E) \to E$ *is continuous for almost all* t *in* $[t_0, t_1[$.

(CAR-3) *Let* V *be a nonempty closed convex subset in* $C_{loc}(t_0, t_1; E)$, *and assume there exists a nonnegative map* m (*possibly dependent on* h) *in* $L^1_{loc}(t_0, t_1; R)$ *such that*

(a) *the set*

$$V_m = \left\{ x \in C_{loc}(t_0, t_1; E) \mid x(t_0) = h(0), \right.$$
$$\left. \max_{[t_0, t]} | x(s) - h(0) | \leqslant \int_{t_0}^{t} m(s)\, ds, \forall t \in]t_0, t_1[\right\} \tag{3.16}$$

is a subset of V,

(b) *and for all* $x \in V_0 = \{x \in V \mid x(t_0) = h(0)\}$,

$$| f(t, \sigma(h, x)(t))| \leqslant m(t), \qquad \text{a.e. } in \quad [t_0, t_1[. \tag{3.17}$$

Then there exists at least one global solution in $AC^1_{loc}(t_0, t_1; E)$ *to the Cauchy problem with initial datum* $h \in M^p(-b, 0; E)$ *at time* t_0.

COROLLARY 3.6. *If* $m \in L^1(t_0, t_1; R)$, *the global solution is in* $AC^1(t_0, t_1; E)$.

Remarks. (1) The set V can be defined pointwise. Let $\{V(t)\}_{t \in [t_0, t_1[}$ be a family of closed convex subsets of E. When it is not empty, the set $V = \{x \in C_{loc}(t_0, t_1; E) \mid x(t) \in V(t)\}$ is closed and convex in $C_{loc}(t_0, t_1; E)$. The converse is not true since the image of an arbitrary closed convex set V in $C_{loc}(t_0, t_1; E)$ under the map $x \mapsto x(t)$: $C_{loc}(t_0, t_1; E) \to E$ is convex but not necessarily closed. This alternative method of defining V was originally introduced by Carathéodory [3] in the context of ordinary differential equations. He chose $V(t) = \{z \in R^n \mid | z - x_0 | \leqslant b\}$ for some positive nonzero constant b and $x_0 \in R^n$.

(2) The local Cauchy problem arises when the set $V_m \not\subset V$. In such circumstances we seek an α, $t_1 - t_0 > \alpha > 0$, for which $\pi_{t_0+\alpha}(V_m) \subset \pi_{t_0+\alpha}(V)$ where the map $\pi_{t_0+\alpha} : C_{loc}(t_0, t_1; E) \to C(t_0, t_0 + \alpha; E)$ is the restriction of the elements of $C_{loc}(t_0, t_1; E)$ to the interval $[t_0, t_0 + \alpha]$.

(3) The introduction of the set V_m is due to C. Corduneanu [7].

The hypotheses (CAR-1), (CAR-2) and (CAR-3) are the classical Carathéodory hypotheses [3]; (LIP) is the Lipschitz hypothesis for uniqueness; and (BC) is the hypothesis first introduced by A. Bielecki [2] and C. Corduneanu [7] in the context of global differential systems for continuous maps f.

THEOREM 3.7. *Let the map* f: $[t_0, t_1[\times B^p(-b, 0; E) \to E$ *satisfy hypotheses* (CAR-1) *and* (CAR-2). *Let* $N(h)$ *be a neighborhood of* h *in* $M^p(-b, 0; E)$. *As in Theorem* 3.5 *we assume that there exist a nonempty closed convex subset* V *of* $C_{loc}(t_0, t_1; E)$ *and a nonnegative map* m *in* $L^1_{loc}(t_0, t_1; R)$ *such that hypotheses* (a) *and* (b) *be satisfied for all* $k \in N(h)$. *Finally assume that for* h *the solution* $\phi(h)$ *is unique in* $AC^1_{loc}(t_0, t_1; E)$. *Then for all sequences* $\{k_n\}$ *in* $N(h)$ *converging to* h *there exists a subsequence of solutions* $\{\phi(k_{n_k})\}$ *which converges to* $\phi(h)$ *in* $AC^1_{loc}(t_0, t_1; E)$.

COROLLARY 3.8. *Let the hypotheses of Theorem* 3.7 *hold. If* $m \in L^1(t_0, t_1; R)$, *the subsequence* $\{\phi(k_{n_k})\}$ *converges to* $\phi(h)$ *in* $AC^1(t_0, t_1; E)$.

Remark. Theorem 3.7 was first proved by Carathéodory [3] for ordinary differential equations.

Remark. The results of Theorem 3.3 up to Corollary 3.8 remain true with the space $C(-a, 0; E)$ in place of the spaces $B^p(-a, 0; E)$ and $M^p(-a, 0; E)$. This involves minor technical changes in the proofs.

The proofs of the theorems (Section 5) will proceed via several lemmas and propositions (Section 4).

4. Properties of the Memory Map and the Map σ

In the remainder of this paper the spaces $M^p(-b, 0; E)$, $B^p(-b, 0; E)$, $L^p(-b, 0; E)$ and $C_{\text{loc}}(t_0, t_1; E)$ will be abbreviated M^p, B^p, L^p and C_{loc}, respectively. Whatever be the local function space, $\pi_t(f)$ will denote the restriction of the map f defined on $[t_0, t_1[$ to the interval $[t_0, t]$, t in $]t_0, t_1[$. We shall also use the concept of a memory map for the product space $C(-b, 0; E) \times C(t_0, t; E)$, $C(-b, 0; E) \times C_{\text{loc}}(t_0, t_1; E)$ and $M^p(-b, 0; E) \times C_{\text{loc}}(t_0, t_1; E)$. In each instance we shall consider the closed linear subspace of all pairs (h, x) for which $x(t_0) = h(0)$.

Proposition 4.1. *Let* $1 \leqslant p < \infty$.

(i) *For all* t *in* $[t_0, t_1[$

$$\max_{s\in[t_0,t]} \|(h \circ x)_s\|_{M^p} \leqslant \|h\|_p + c(p, t - t_0)\| \pi_t(x)\|_{C(t_0,t;E)}, \tag{4.1}$$

where $c(p, t - t_0) = \max\{1, (t - t_0)^{1/p}\}$, *and*

$$\max\{\|h\|_p, \|\pi_t(x)\|_{C(t_0,t;E)}\} \leqslant \max_{s\in[t_0,t]} \|(h \circ x)_s\|_{M^p}. \tag{4.2}$$

(ii) *The map*

$$(h, x) \mapsto h \circ x: M^p \circ C_{\text{loc}} \to C_{\text{loc}}(t_0, t_1; M^p) \tag{4.3}$$

is an isomorphism (*cf. Definition* 3.2).

Proof. (i) By definition of the memory map.

(ii) If we can prove that the map $t \mapsto (h \circ x)_t$ is continuous, then the map (4.3) is an isomorphism by inequalities (4.1) and (4.2). Firstly it is easy to show that for (h, x) in $C_c(-b, 0; E) \circ C_{\text{loc}}$ the map $t \mapsto (h \circ x)_t$ is continuous. Then we use the density of $C_c(-b, 0; E)$ in $M^p(-b, 0; E)$ [cf. Proposition 2.1(ii)].

Pick any (h, x) in $M^p(-b, 0; E) \circ C_{\text{loc}}(t_0, t_1; E)$. There exists a sequence $\{h_n\}$ in $C_c(-b, 0; E)$ for which $h_n(0) = h(0)$ and $h_n \to h$ in $L^p(-b, 0; E)$ [Proposition 2.1(ii)]. Notice that $h_n \circ x$ is in $C_{\text{loc}}(t_0, t_1; C(-b, 0; E))$ for all n. By definition

$$\begin{aligned}\|(h \circ x)_t - (h_n \circ x)_t\|_{M^p} &= [|h(0) - h_n(0)|^p + \|h - h_n\|_p^p]^{1/p}\\ &= \|h - h_n\|_p.\end{aligned}$$

The continuity of the map $t \mapsto (h \circ x)_t$ is now a consequence of the following sequence of inequalities: for any t and t' in $[t_0, t_1[$

$$\begin{aligned}
&\|(h \circ x)_t - (h \circ x)_{t'}\|_{M^p} \\
&\quad \leqslant \|(h \circ x)_t - (h_n \circ x)_t\|_{M^p} + \|(h_n \circ x)_t - (h_n \circ x)_{t'}\|_{M^p} \\
&\qquad + \|(h \circ x)_{t'} - (h_n \circ x)_{t'}\|_{M^p} \\
&\quad \leqslant 2\| h - h_n \|_p \\
&\qquad + [| x(t) - x(t')|^p + \|(h_n \circ x)_t - (h_n \circ x)_{t'}\|_p^p]^{1/p} \\
&\quad \leqslant 2\| h - h_n \|_p + \max\{1, | t - t' |^{1/p}\} \|(h_n \circ x)_t - (h_n \circ x)_{t'}\|_C .
\end{aligned}$$

This and inequality (4.1) establish that $h \circ x \in C_{\mathrm{loc}}(t_0, t_1; M^p)$. The map (4.3) is clearly linear and it is easy to verify that it is bijective. ∎

PROPOSITION 4.2. $1 \leqslant p \leqslant \infty$, *and* $\theta \in I(-b, 0)$.

(i) *Given* $(h, x) \in M^p \circ C_{\mathrm{loc}}$, *the map* $t \mapsto (h \circ x)_t(\theta)\colon [t_0, t_1[\to E$ *is an element of* $L^p_{\mathrm{loc}}(t_0, t_1; E)$ *that we shall denote by* $(h \circ x)(\theta)$.

(ii) *For all* $t \in]t_0, t_1[$

$$\left[\int_{t_0}^{t} |(h \circ x)_s(\theta)|^p \, ds\right]^{1/p} \leqslant \| h \|_p + (t - t_0)^{1/p} \max_{s \in [t_0, t]} | x(s)|. \tag{4.4}$$

(iii) *The map*

$$(h, x) \mapsto (h \circ x)(\theta) : M^p \circ C_{\mathrm{loc}} \to L^p_{\mathrm{loc}}(t_0, t_1; E) \tag{4.5}$$

is linear, injective and continuous.

Proof. By definition of the memory map $(h \circ x)(\theta)$ is clearly m measurable. Moreover it is easy to show that

$$\left[\int_{t_0}^{t} |(h \circ x)_s(\theta)|^p \, ds\right]^{1/p} \leqslant \| h \|_p + (t - t_0)^{1/p} \| \pi_t(x)\|_{C(t_0, t; E)} .$$

Hence $(h \circ x)(\theta)$ is in $L^p_{\mathrm{loc}}(t_0, t_1; E)$. This establishes the proposition since the linearity and the injectivity of the map (4.5) are obvious. ∎

Counterexample to Proposition 4.1 *for* M^∞

For completeness we construct an example of a pair $(h, x) \in M^\infty \circ C_{\mathrm{loc}}$ for which the map $t \mapsto (h \circ x)_t : [t_0, t_1[\to M^\infty$ is not continuous. Let $t_0 = 0$, $t_1 = 2$, $x = 0$ and

$$h(\theta) = \begin{cases} e, & -2 \leqslant \theta < -1 \\ 0, & -1 \leqslant \theta \leqslant 0, \end{cases}$$

where $e \in E$ has norm equal to 1. Then

$$(h \circ x)_t(\theta) = \begin{cases} e, & -2 \leqslant \theta < -(1+t) \\ 0, & -(1+t) \leqslant \theta \leqslant 0. \end{cases}$$

Hence $(h \circ x)_1 = 0$ and

$$\|(h \circ x)_t - (h \circ x)_1\|_{M^\infty} = \|(h \circ x)_t\|_{M^\infty} = \begin{cases} 1, & 0 \leqslant t < 1 \\ 0, & 1 \leqslant t \leqslant 2. \end{cases} \quad \blacksquare$$

Propositions 4.1 and 4.2 are now combined to obtain the precise results necessary in the proofs of Theorems 3.3 and 3.5.

PROPOSITION 4.3. *Let p be an integer, $1 \leqslant p < \infty$.*

(i) *The map*

$$\sigma : M^p(-b, 0; E) \circ C_{\mathrm{loc}}(t_0, t_1; E) \to L^p_{\mathrm{loc}}(t_0, t_1; B^p), \tag{4.6}$$

defined as

$$t \mapsto \sigma(h, x)(t) = ((h \circ x)_t(\theta_N), \ldots, (h \circ x)_t(\theta_1), (h \circ x)_t): [t_0, t_1[\to B^p, \tag{4.7}$$

is linear and continuous.

(ii) *For all $t \in [t_0, t_1[$*

$$\left[\int_{t_0}^{t} \|\sigma(h, x)(s)\|_{B^p}^p \, ds\right]^{1/p} \leqslant k(p, t - t_0)[\|h\|_p + \|\pi_t(x)\|_{C(t_0, t; E)}], \tag{4.8}$$

where $k(p, t - t_0) = [1 + (t - t_0)^{1/p}][N + 1 + (t - t_0)^{1/p}] > 0$.

(iii) *Let κ: $M^p \to E \times L^p$ be the isomorphism of Proposition 2.1. For $(x, g) \in C_{\mathrm{loc}} \times L^p$ let*

$$g_x = \kappa^{-1}(x(t_0), g). \tag{4.9}$$

Fix g in L^p and t in $[t_0, t_1[$. Then for all x, y in C_{loc}

$$\|\sigma(g_x, x)(t) - \sigma(g_y - y)(t)\|_{B^p} \leqslant c'(p, t - t_0)\,\|\pi_t(x - y)\|_{C(t_0, t; E)}, \tag{4.10}$$

where $c'(p, t - t_0) = (N + 1 + c(p, t - t_0)^p)^{1/p}$.

Proof. (i) $\sigma(h, x)$ is an element of $L^p_{\mathrm{loc}}(t_0, t_1; B^p)$ by Propositions 4.1(ii) and 4.2(iii) and the fact that $C_{\mathrm{loc}}(t_0, t_1; M^p)$ is a subspace of $L^p_{\mathrm{loc}}(t_0, t_1; M^p)$. The map σ is linear by its very construction. It is continuous by continuity of the maps (4.3) and (4.5).

(ii) From Eqs. (4.1) and (4.4) and the definition of $\sigma(h, x)(t)$, we obtain inequality (4.8).

(iii) Again by definition

$$|(g_x \circ x)_t(\theta_i) - (g_y \circ y)_t(\theta_i)| = \begin{cases} 0, & t_0 \leqslant t < t_0 - \theta_i \\ x(t) - y(t), & t_0 - \theta_i \leqslant t < t_1 \end{cases} \quad (4.11)$$

and

$$\begin{aligned} \|(g_x \circ x)_t - (g_y \circ y)_t\|_{M^p} &= \|((g_x - g_y) \circ (x - y))_t\|_{M^p} \\ &\leqslant \|g_x - g_y\|_p^p + c(p, t - t_0)\|\pi_t(x - y)\|_{C(t_0,t;E)} \\ &\leqslant c(p, t - t_0)\|\pi_t(x - y)\|_{C(t_0,t;E)} \end{aligned} \quad (4.12)$$

[by linearity of $(h \circ x)_t$ in (h, x) and the inequality (4.1)]. Inequalities (4.11) and (4.12) establish inequality (4.10). ∎

To summarize the situation we have constructed a map

$$\sigma : M^p \circ C_{\mathrm{loc}} \to L^p_{\mathrm{loc}}(t_0, t_1; B^p) \quad (4.13)$$

with the following two properties for all $t \in]t_0, t_1[$:

$$\left[\int_{t_0}^{t} \|\sigma(h, x)(s)\|_{B^p}^p \, ds\right]^{1/p} \leqslant k(p, t - t_0)[\|h\|_p + \|\pi_t(x)\|_C], \quad (4.14)$$

$(h, x) \in M^p \circ C_{\mathrm{loc}}$, and

$$\|\sigma(g_x, x)(t) - \sigma(g_y, y)(t)\|_{B^p} \leqslant c'(p, t - t_0)\|\pi_t(x - y)\|_C \quad (4.15)$$

for all $x, y \in C_{\mathrm{loc}}$ and $g \in L^p$. All structural properties of the system are contained in the couple (B^p, σ). The map σ bears a certain similarity to the *lag function* introduced by G. S. Jones [16], but there is a fundamental difference. Jone's lag function α is defined on the time variable to generate a hereditary time set:

$$\alpha : R \to \Omega, \quad (4.16)$$

where Ω denotes the set of all closed subsets of R which are bounded above. Here α would be defined as

$$\alpha(t) = \{t + \theta_N, \ldots, t + \theta_1, t, I(t - b, t)\}. \quad (4.17)$$

The map σ acts on the "information $(h \circ x)_t$ stored in a memory at time t" and samples what the system needs at time t, precisely $\sigma(h, x)(t)$. It is felt that it is technically advantageous to work with the map σ as defined here rather than working with set-valued maps in the lag-function approach. In fact the σ-map approach allows us to handle existence, uniqueness and continuity questions for variable time delays in a natural manner, once certain measurability questions are taken care of.

LEMMA 4.4. (Carathéodory). *Assume the map* $f\colon [t_0, t_1[\times B^p \to E$ *satisfies the first two Carathéodory hypotheses*:

(CAR-1) $t \mapsto f(t, z)$ *is m-measurable for fixed* z;

(CAR-2) $z \mapsto f(t, z)$ *is continuous on* B^p *for almost all* t *in* $[t_0, t_1[$.

Then for any m-measurable map $y\colon [t_0, t_1[\to B^p$ *the map*

$$t \mapsto f_y(t) = f(t, y(t))\colon [t_0, t_1[\to E \tag{4.18}$$

is m measurable.

Proof. There exists a sequence of step maps s_n converging almost everywhere to y. It is sufficient to show that

(1) $\{f_{s_n}\}$ is a sequence of m-measurable maps, and

(2) $\{f_{s_n}\}$ converges almost everywhere to f_y.

By the hypothesis (CAR-2)

$$f_{s_n}(t) = f(t, s_n(t)) \to f(t, y(t)) = f_y(t)$$

for almost all t since $s_n(t) \to y(t)$ for almost all t. Let $s(t)$ be an arbitrary step map defined on $[t_0, t_1[$. Its most general form is

$$s(t) = \sum_{i=1}^{r} a_i \chi_{A_i}(t),$$

where $r \geqslant 1$ is a positive integer and the A_i's are Lebesgue measurable disjoint subsets of $[t_0, t_1[$ the union $A = \bigcup_{i=1}^{r} A_i$ of which has finite measure. Then

$$f_s(t) = f(t, s(t)) = f(t, 0)[1 - \chi_A(t)] + \sum_{i=1}^{N} f(t, a_i)\, \chi_{A_i}(t)$$

is clearly the sum of $N + 1$ m-measurable maps by the hypothesis (CAR-1). ∎

Remark. The proof of the above lemma is essentially Carathéodory's original proof [3, p. 665].

We now know that under hypotheses (CAR-1) and (CAR-2) the right-hand side of Eq. (3.8) as a function of t,

$$t \mapsto f(t, \sigma(h, x)(t))\colon [t_0, t_1[\to E, \tag{4.19}$$

is m-measurable. Under the hypotheses of Theorems 3.3 or 3.5 it will be established that the map (4.19) is an element of $L^1_{loc}(t_0, t_1; E)$. It would be good to have this formulation of the problem right from the beginning. However this last fact is obtained via different sets of hypotheses (Theorems 3.3 and 3.5).

5. Proofs of Theorems 3.3, 3.5 and 3.7

5.1. *Auxiliary Results*

The proof of Theorem 3.3 will make use of the Banach fixed-point theorem [12, p. 305] and some techniques borrowed from A. Bielecki [2] and C. Corduneanu [7].

Lemma 5.1. (Bielecki, Corduneanu). *Let $n \in \mathscr{L}^1_{loc}(t_0, t_1; R)$ be a nonnegative function, and α, $0 < \alpha < 1$, be given. The inequality*

$$\int_{t_0}^{t} n(s)\, g(s)\, ds \leqslant \alpha g(t), \qquad t \in [t_0, t_1[\tag{5.1}$$

has a solution in $C_{loc}(t_0, t_1; R)$ which is strictly positive and nondecreasing. In particular the following function is a solution of (5.1):

$$g_\alpha(t) = \exp\left\{\alpha^{-1} \int_{t_0}^{t} n(s)\, ds\right\}, \qquad t \in [t_0, t_1[. \tag{5.2}$$

Remark. The introduction of the function g_α in the context of "global differential equations" is due to A. Bielecki [2]. Thereafter this idea was successfully used by C. Corduneanu [7, 6] in the global case.

Definition 5.2. Let α, $0 < \alpha < 1$, g_α be given by (5.2) and $t \in]t_0, t_1[$. $C_\alpha(t_0, t; E)$ will denote the space of all continuous maps defined on $[t_0, t]$ with values in E endowed with the norm

$$\| x \|_\alpha = \max_{s \in [t_0, t]} \{| x(s)| / g_\alpha(s)]. \quad \blacksquare$$

Remark. $C_\alpha(t_0, t; E)$ and $C(t_0, t; E)$ are equal as sets and equivalent as Banach spaces.

Lemma 5.3 (Corduneanu [7]). *For all α, $0 < \alpha < 1$, $x \in C_{loc}(t_0, t_1; E)$ and $t \in]t_0, t_1[$,*

$$\max_{[t_0, t]}\{| x(s)| / g_\alpha(s)\} = \max_{[t_0, t]}\{\| \pi_s(x)\|_{C(t_0, s; E)} / g_\alpha(s)\}. \tag{5.3}$$

5.2. *Proof of Theorem* 3.3

(1) For arbitrary $(h, x) \in M^p \circ C_{loc}$ consider the map

$$t \mapsto \tilde{f}(h, x)(t) = f(t, \sigma(h, x)(t)) : [t_0, t_1[\to E. \tag{5.4}$$

We claim $\tilde{f}(h, x) \in L^1_{loc}(t_0, t_1; E)$. By the hypotheses (LIP) the map f is continuous in z for almost all $t \in [t_0, t_1[$; hence hypotheses (CAR-1) and (CAR-2) are satisfied. Also $\sigma(h, x) \in L^p_{loc}(t_0, t_1; B^p)$ (Proposition 4.3). By Lemma 4.4 $\tilde{f}(h, x)$ is m-measurable. Also for all $t \in]t_0, t_1[$

$$\int_{t_0}^{t} |\tilde{f}(h, x)(s)| \, ds \leqslant \int_{t_0}^{t} |\tilde{f}(h, x)(s) - f(s, 0)| \, ds + \int_{t_0}^{t} |f(s, 0)| \, ds. \tag{5.5}$$

By the hypothesis (BC), the last term on the right is finite. By hypothesis (LIP) and by Proposition 4.3

$$\begin{aligned} \int_{t_0}^{t} |\tilde{f}(h, x)(s) - f(s, 0)| \, ds &\leqslant \| \pi_t(n) \|_q \, \| \pi_t(\sigma(h, x)) \|_p \\ &\leqslant \| \pi_t(n) \|_q \, k(p, t - t_0) [\| h \|_p + \| \pi_t(x) \|_C]. \end{aligned}$$

This shows that $\tilde{f}(h, x) \in L^1_{loc}(t_0, t_1; E)$.

(2) To establish the continuity of the map

$$(h, x) \mapsto \tilde{f}(h, x) : M^p \circ C_{loc} \to L^1_{loc}(t_0, t_1; E)$$

we use the following inequalities [Proposition 4.3, Eq. (4.8)]:

$$\begin{aligned} \| \pi_t(\tilde{f}(h, x) - \tilde{f}(k, y)) \|_p &\leqslant \int_{t_0}^{t} n(s) \, \| \sigma(h, x)(s) - \sigma(k, y)(s) \|_{B^p} \, ds \\ &\leqslant \int_{t_0}^{t} n(s) \, \| \sigma(h - k, x - y)(s) \|_{B^p} \, ds \\ &\leqslant \| \pi_t(n) \|_q \, k(p, t - t_0) [\| h - k \|_p + \| \pi_t(x - y) \|_C]. \end{aligned} \tag{5.6}$$

(3) Fix $\tilde{h} \in M^p$ and let $(\tilde{h}^0, \tilde{h}^1) = \kappa(\tilde{h})$ (Definition 2.1). Consider the map

$$x \mapsto \tilde{h}_x^1 = \kappa^{-1}(x(t_0), \tilde{h}^1) : C_{loc}(t_0, t_1; E) \to M^p.$$

By construction $(\tilde{h}_x^1, x) \in M^p \circ C_{loc}$ for all $x \in C_{loc}$ [Proposition 4.3(iii)]. We define the map

$$t \mapsto \tilde{U}_f(x)(t) = \tilde{h}^0 + \int_{t_0}^{t} \tilde{f}(\tilde{h}_x^1, x)(s) \, ds : [t_0, t_1[\to E.$$

From part (1) we conclude that $\tilde{U}_f \in AC^1_{\text{loc}}(t_0, t_1; E)$ which is a subspace of $C_{\text{loc}}(t_0, t_1; E)$. So for each $t \in]t_0, t_1[$ and an arbitrary α, $0 < \alpha < 1$, to be chosen later, we have a map

$$\pi_t(\tilde{U}_f): C_\alpha(t_0, t; E) \to C_\alpha(t_0, t; E) \tag{5.7}$$

since $C_\alpha(t_0, t; E)$ and $C(t_0, t; E)$ are equal as sets and have equivalent norms. We show that for some $\alpha = \tilde{\alpha}$ such a map is a contraction mapping. For arbitrary x and y in $C_\alpha(t_0, t; E)$ and $t' \in [t_0, t]$,

$$| \pi_t(\tilde{U}_f(x) - \tilde{U}_f(y))(t')| \leqslant \int_{t_0}^{t'} n(s)c'(p, t-t_0)[\max_{r\in[t_0,s]} | x(r) - y(r)|] \, ds \tag{5.8}$$

[Proposition 4.3(iii)]. But

$$\begin{aligned}\int_{t_0}^{t'} & n(s)[\max_{[t_0,s]} | x(r) - y(r)|] \, ds \\ & \leqslant \int_{t_0}^{t'} n(s)\, g_\alpha(s)\, ds \max_{s\in[t_0,t']} \{\max_{r\in[t_0,s]} | x(r) - y(r)|/g_\alpha(s)\} \\ & \leqslant \alpha g_\alpha(t') \| \pi_{t'}(x-y)\|_{C_\alpha}\end{aligned}$$

by Lemmas 5.1 and 5.3 and

$$\| \pi_t(\tilde{U}_f(x) - \tilde{U}_f(y))\|_{C_\alpha} \leqslant c'(p, t-t_0)\, \alpha \, \| \pi_t(x-y)\|_{C_\alpha}. \tag{5.9}$$

Since $c'(p, t-t_0) > 1$ (Proposition 4.3) there exists $\tilde{\alpha}$, $0 < \tilde{\alpha} < 1$, such that $\tilde{\alpha}c'(p, t-t_0) = 1/2$. By the Banach fixed-point theorem [12, p. 305], $\pi_t(\tilde{U}_f)$ has a unique fixed point. By definition this fixed point is in $AC^1(t_0, t; E)$. So it is the unique solution in $AC^1(t_0, t; E)$ of the differential equation (3.4). Finally for each t in $]t_0, t_1[$ there exists a unique solution x^t on $[t_0, t]$. For $t_0 < t \leqslant t' \leqslant t_1$ the restriction $\pi_{tt'}(x^{t'})$ to $[t_0, t]$ of the solution $x^{t'}$ on $[t_0, t']$ is also a solution on $[t_0, t]$ which is necessarily equal to x^t by uniqueness. Hence there exists a unique $x \in AC^1_{\text{loc}}(t_0, t_1; E)$ such that $\pi_t(x) = x^t$, which is the sought global solution.

(4) The continuity of the map (3.13) is also obtained via the spaces C_α. Let x and y be solutions corresponding to h and k, respectively. Fix $t \in]t_0, t_1[$. To compute $\| \pi_t(x-y)\|_{AC^1}$ we need $\| \pi_t(x-y)\|_C$. For all $r \in [t_0, t]$

$$| x(r) - y(r)| \leqslant \left| \int_{t_0}^{r} (\tilde{f}(h, x)(s) - \tilde{f}(k, y)(s)) \, ds \right| + | h(0) - k(0)| \tag{5.10}$$

and by Proposition 4.3(iii), hypothesis (LIP) and inequality (4.8)

$$\int_{t_0}^{r} |\tilde{f}(h, x)(s) - \tilde{f}(k, y)(s)| \, ds$$

$$\leqslant \int_{t_0}^{r} |\tilde{f}(h_x^1, x)(s) - \tilde{f}(h_y^1, y)(s)| \, ds + \int_{t_0}^{r} |\tilde{f}(h_y^1, y)(s) - \tilde{f}(k_y^1, y)(s)| \, ds$$

$$\leqslant \int_{t_0}^{r} n(s) \, c'(p, t - t_0) \max_{[t_0, s]} | x(r) - y(r)| \, ds$$
$$+ \| \pi_t(n)\|_q \, k(p, t - t_0) \, \| h^1 - k^1 \|_p \tag{5.11}$$

$$\leqslant ac'(p, t - t_0) \, g_\alpha(r) \, \| \pi_r(x - y)\|_{C_\alpha} + \| \pi_t(n)\|_q \, k(p, t - t_0) \, \| h^1 - k^1 \|_p . \tag{5.12}$$

If we substitute (5.12) in inequality (5.10) and divide both sides by $g_\alpha(r)$ we obtain by taking the maximum over $[t_0, t]$,

$$\| \pi_t(x - y)\|_{C_\alpha} \leqslant \alpha c'(p, t - t_0) \, \| \pi_t(x - y) \|_{C_\alpha} + d_1(p, t - t_0) \, \| h - k \|_{M^p} ,$$

where $d_1(p, t - t_0) = 1 + \| \pi_t(n)\|_q \, k(p, t - t_0)$, and

$$\| \pi_t(x - y)\|_C \leqslant \| \pi_t(x - y)\|_{C_{\tilde{\alpha}}} \leqslant 2d_1(p, t - t_0) \, \| h^1 - k^1 \|_p \tag{5.13}$$

by choice of $\tilde{\alpha}$. (In the above we made use of the fact that $(g_\alpha(r))^{-1} \leqslant 1$, $r \in [t_0, t]$ [see Eq. (5.2)].) Finally from inequalities (5.11) and (5.13) we obtain $\| \pi_t(x - y)\|_{AC^1} \leqslant d(p, t - t_0) \, \| h - k \|_{M^p}$, where

$$d(p, t - t_0) = 1 + \| \pi_t(n)\|_q \, [2c'(p, t - t_0)(t - t_0)^{1/p} \, d_1(p, t - t_0) + k(p, t - t_0)]. \quad \blacksquare \tag{5.14}$$

Remark. With $C(-a, 0; E)$ in place of $B^p(-a, 0; E)$ and $M^p(-a, 0; E)$ the map σ becomes $\sigma(h, x)(t) = (h \circ x)_t$, $t \in [t_0, t_1[$, and the map $\tilde{f}$ is defined as in Eq. (5.4). Fix $\tilde{h} \in C(-a, 0; E)$ and define the map $\tilde{U}_f$ on the subspace $S = \{x \in C_{\text{loc}}(t_0, t_1; E) \mid x(t_0) = \tilde{h}(0)\}$ of $C_{\text{loc}}(t_0, t_1; E)$ as follows:

$$t \mapsto \tilde{U}_f(x)(t) = \tilde{h}(0) + \int_{t_0}^{t} \tilde{f}(\tilde{h}, x)(s) \, ds : [t_0, t_1[\to E.$$

With the above definitions the proof of Theorem 3.3 remains correct.

5.3. *Proof of Theorem* 3.5

(1) V_0 is clearly closed and convex by definition. We show that for all $x \in V_0$, the map

$$t \mapsto f(t, \sigma(h, x)(t)) \colon [t_0, t_1[\to E \tag{5.15}$$

is in $L^1_{\text{loc}}(t_0, t_1; E)$. For each $x \in V_0$, the map

$$t \mapsto \sigma(h, x)(t): [t_0, t_1[\to B^p$$

is in $L^p_{\text{loc}}(t_0, t_1; B^p)$ by Proposition 4.3 since $(h, x) \in M^p \circ C_{\text{loc}}$. Hence hypotheses (CAR-1), (CAR-2) and Lemma 4.4 establish the m measurability of the map (5.15). The assertion is now true by part (b) of the hypothesis (CAR-3) and the Lebesgue dominated convergence theorem.

(2) The map $x \mapsto U_f(x): V_0 \to C_{\text{loc}}(t_0, t_1; E)$ which is defined by

$$(U_f(x))(t) = h(0) + \int_{t_0}^{t} f(s, \sigma(h, x)(s))\, ds, \qquad \forall t \in [t_0, t_1[, \tag{5.16}$$

now makes sense. Moreover for all $t \in [t_0, t_1[$,

$$| U_f(x)(t) - h(0)| = \left| \int_{t_0}^{t} f(s, \sigma(h, x)(s))\, ds \right| \leqslant \int_{t_0}^{t} m(s)\, ds,$$

and $U_f(x)(t_0) = h(0)$. Thus $U_f(x) \in V_m \subset V$. But

$$V_m = V_m \cap V_0 \subset V \cap V_0 = V_0 .$$

So the image of the map U_f is contained in V_0.

(3) Let

$$M(s) = \int_{t_0}^{s} m(v)\, dv, \qquad s \in [t_0, t_1[. \tag{5.17}$$

M is uniformly continuous on compact subsets of $[t_0, t_1[$ and monotonically increasing in $[t_0, t_1[$. For arbitrary s and t in $[t_0, t_1[$ and any $x \in V_0$

$$\begin{aligned} | U_f(x)(t) - U_f(x)(s)| &\leqslant \left| \int_{t_0}^{t} f(v, \sigma(h, x)(v))\, dv - \int_{t_0}^{s} f(v, \sigma(h, x)(v))\, dv \right| \\ &\leqslant | M(t) - M(s)|. \end{aligned} \tag{5.18}$$

For each $t \in]t_0, t_1[$ the family $\pi_t(U_f(V_0))$ is equicontinuous. Also for $s \in [t_0, t]$

$$\begin{aligned} | U_f(x)(s)| &= \left| h(0) + \int_{t_0}^{s} f(v, \sigma(h, x)(v))\, dv \right| \\ &\leqslant | h(0)| + \int_{t_0}^{s} m(v)\, dv \leqslant | h(0)| + M(t) \end{aligned}$$

and $\pi_t(U_f(V_0))$ is an equicontinuous and uniformly bounded family, hence a relatively compact subset of $C(t_0, t; E)$ by Ascoli's lemma [18, p. 211]. Finally $U_f(V_0)$ is relatively compact in $C_{loc}(t_0, t_1; E)$, since for all t in $]t_0, t_1[$ the set $\pi_t(C)$ is relatively compact in $C(t_0, t_1; E)$.

(4) Finally to show U_f is continuous on V_0 pick any point $x \in V_0$ and consider an arbitrary Cauchy sequence $\{x_n\}$ of points in V_0 converging to x. Let $g_n(t) = f(t, \sigma(h, x_n)(t))$, $g(t) = f(t, \sigma(h, x)(t))$. By definition of σ

$$\sigma(h, x_n)(t) \to \sigma(h, x)(t) \qquad \text{a.e. in} \quad [t_0, t_1[$$

when $x_n \to x$; and by hypothesis (CAR-2)

$$g_n(t) \to g(t), \qquad \text{a.e. in} \quad [t_0, t_1[.$$

By part (b) of the hypothesis (CAR-3), the Lebesgue dominated convergence theorem can be applied and for all $t \in]t_0, t_1[$

$$\pi_t(g_n) \to \pi_t(g) \qquad \text{in} \quad L^1(t_0, t; E),$$

that is, $g_n \to g$ in $L^1_{loc}(t_0, t_1; E)$. By continuity of the integral

$$\int_{t_0}^{t} g_n(s)\, ds \to \int_{t_0}^{t} g(s)\, ds, \qquad t \in]t_0, t_1[,$$

and hence $U_f(x_n) \to U_f(x)$.

(5) The map $U_f : V_0 \to V_0$ is continuous and $U_f(V_0)$ is relatively compact. The theorem is now true by the Schauder–Tychonoff theorem [13, Corollary 3.6.2, p. 163 and Remarks, p. 164]. ∎

Remark. The proof also holds with $C(-a, 0; E)$ in place of $M^p(-a, 0; E)$ and $B^p(-a, 0; E)$.

5.4. *Proof of Theorem* 3.7

The existence of a solution $\phi(k) \in AC^1_{loc}(t_0, t_1; E)$ is guaranteed for all $k \in N(h)$. Pick a sequence $\{k_n\}$ in $N(h)$ converging to h. There exists a subsequence of $\{k_n\}$ (also denoted $\{k_n\}$) which converges almost everywhere to h. For each k_n choose a solution x_n. For $t \in]t_0, t_1[$ the set $\{\pi_t(x_n)\}_{n\in\mathbb{N}}$ is an equicontinuous family of maps on $[t_0, t]$ since

$$| x_n(s) - x_n(s')| \leqslant | M(s) - M(s')|$$

[Eq. (5.18)] and $s \mapsto \pi_t(M)(s): [t_0, t] \to R$ is uniformly continuous on $[t_0, t]$; it is also uniformly bounded since

$$| x_n(s) - h(0)| \leqslant | k_n(0) - h(0)| + M(t)$$

and a converging sequence is bounded. By Ascoli's lemma [18, p. 211] the set $\{\pi_t(x_n) \mid n \in \mathbb{N}\}$ is relatively compact. Hence the set $\{x_n \mid n \in \mathbb{N}\}$ is relatively compact in $C_{\text{loc}}(t_0, t_1; E)$ (as in the proof of Theorem 3.5). Therefore there exists a subsequence, also denoted $\{x_n\}$, converging to some $y \in C_{\text{loc}}(t_0, t_1; E)$ as $k \to \infty$. By definition of σ

$$k_n \to h \qquad \text{a.e. in} \quad I(-b, 0)$$

and

$$x_n \to y \Rightarrow \sigma(k_n, x_n)(t) \to \sigma(h, y)(t) \qquad \text{a.e. in} \quad [t_0, t_1[.$$

Thus for $t \in]t_0, t_1[$

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, \sigma(k_n, x_n)(s))\, ds = \int_{t_0}^{t} f(s, \sigma(h, y)(s))\, ds$$

by the continuity of the map $z \mapsto f(s, z)\colon M^p \to E$ for fixed s, hypothesis (CAR-3) on $N(h)$ and the Lebesgue dominated convergence theorem. Finally

$$y(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \left[k_n(0) + \int_{t_0}^{t} f(s, \sigma(k_n, x_n)(s))\, ds\right]$$

$$= h(0) + \int_{t_0}^{t} f(s, \sigma(h, y)(s))\, ds, \qquad t \in [t_0, t_1[.$$

This means that y is a solution with initial datum h. By uniqueness y is necessarily equal to x and $x_n \to x$ as $k_n \to h$. ∎

Remark. The proof also applies with $C(-a, 0; E)$ in place of $B^p(-a, 0; E)$ and $M^p(-a, 0; E)$.

Remark. The proof of Theorem 3.7 is essentially Carathéodory's proof [3] with obvious technical changes.

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