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Generic Solvability of Morgan's Problem

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***Abstract*—For m-input, m-output, linear time-invariant systems of the form**

$$\dot{x}(t)=Ax(t)+Bu(t);\quad y(t)=Cx(t)$$

it is shown that Morgan's problem, (decoupling into an m-single input-single output subsystem) is solvable for almost all real matrix triples (A,B,C) consistently dimensioned.

I. Introduction

We consider linear, time invariant, multivariable systems of the form

$$\dot{x}(t)=Ax(t)+Bu(t),\quad y(t)=Cx(t) \tag{1}$$

where $x(t)\in R^n$, $u(t)$, and $y(t)\in R^m$, $n\geqslant m$, with A, B, and C appropriately dimensioned real matrices. The problem of decoupling an m-input, m-output system (1) into m scalar input–scalar output subsystems by the use of a feedback control law

$$u(t)=Fx(t)+Gv(t) \tag{2}$$

with $v(t)\in R^m$, was first considered by Morgan [1] who constructed a sufficient condition for decoupling. In 1967, Falb and Wolovich [2] completely solved this question showing that decoupling was possible if and only if a certain easily determined matrix, dependent upon the parameters of A, B, and C, was nonsingular. For ease of notation, we shall refer to this issue as Morgan's problem.

Wonham and Morse [3],[4] developed a more general theory of decoupling and found an equivalent geometric condition for the solution of Morgan's problem. Fabian and Wonham [5] extended these geometric results to show, in particular, that (1) is generically (i.e., for almost all parameter sets A, B, and C) decoupleable if dynamic compensation is allowed. Building upon the machinery of [5], we will show that Morgan's problem itself is generically solvable.

II. Problem Formulation

For any positive integer k, let $\boldsymbol{k}$ denote the set $\{1,2,\cdots,k\}$. We designate the ith row of C from (1) by C_i for $i\in\boldsymbol{m}$. For each C_i define the nonnegative integer d_i and the row vector D_i:

$$\begin{aligned} d_i&=\min\{j \mid C_iA^jB\neq 0,\quad j=0,1,\cdots,n-1\}\\ d_i&=n-1,\quad \text{if } C_iA^jB=0 \text{ for all } j\geqslant 0 \\ D_i&=C_iA^{d_i}B.\end{aligned} \tag{3}$$

(For the discrete time analog of (1), d_i+1 represents the minimum time delay for the effect of any input to be visible at output i, and D_i represents the first nontrivial pointwise mapping from inputs to output i.) Then construct the $m\times m$ matrix D whose ith row is given by D_i, i.e.,

$$D=[D_i';\cdots;D_m']'.$$

Falb and Wolovich have shown that (1) is decoupleable if and only if D is nonsingular. If D^{-1} exists, then the control law (2) with $(F,G)=(-D^{-1}A^*,D^{-1})$ where A^* is an $m\times n$ matrix whose ith row is given by $C_iA^{d_i+1}$ for $i\in\boldsymbol{m}$, will decouple. We note that the elements of D are not in general continuous functions of the components of the matrices A, B, and C; hence, the determinant of D is not a continuous function of A, B, and C.

A subspace $\mathfrak{R}$ is said to be a controllability subspace (cs) [3] if for some feedback map F,

$$\mathfrak{R}=\sum_{i\in\boldsymbol{n}}(A+BF)^{i-1}(\mathfrak{B}\cap\mathfrak{R})$$

where $\mathfrak{B}$ denotes the image of B. A cs is then seen to be invariant under the action of $A+BF$ and also completely reachable. If we let $\mathfrak{R}_i$ represent the largest cs contained in the common kernel of $C_1,\cdots,C_{i-1},C_{i+1},\cdots,C_m$,

$$\mathfrak{R}_i\subset\bigcap_{\substack{j\neq i\\ j\in\boldsymbol{m}}}\mathfrak{N}_j,\qquad i\in\boldsymbol{m} \tag{4}$$

where $\mathfrak{N}_j=\operatorname{Ker}C_j$, then Morse and Wonham [4] have shown that Morgan's problem has a solution if and only if

$$\mathfrak{B}=\sum_{i\in\boldsymbol{m}}\mathfrak{B}\cap\mathfrak{R}_i. \tag{5}$$

Fabian and Wonham have used tools of algebraic geometry to show that for almost every time invariant linear system of a particular class, including those of the form (1),

$$\mathfrak{R}_i+\mathfrak{N}_i=R^n$$

which is sufficient to guarantee the existence of a feedback compensator which will effect decoupling. A key element in their work consists of showing that for almost all parameter sets (A,B,C) satisfying several loose dimensional constraints, the subspaces

$$\mathfrak{K}_i=\bigcap_{\substack{j\neq i\\ j\in\boldsymbol{m}}}\mathfrak{N}_j,\qquad i\in\boldsymbol{m}$$

will be controllability subspaces.

Relying heavily upon the work of Fabian and Wonham, we will show that (5) is generically true for systems of the form (1). That is, if we consider real matrix triples (A,B,C) as points in $R^{n(n+m+m)}$, then those points at which (5) fails to hold lie on a proper algebraic variety in $R^{n(n+m+m)}$ and hence constitute a set of zero measure.

III. Main Result

Let $N=n(n+m+m)$ and consider the ring of polynomials in N indeterminates over the reals, $R[\lambda_1,\cdots,\lambda_N]$. An algebraic variety $V\subset R^N$ is the set of common zeros of a finite number of such polynomials. A variety is called proper if it is not equal to R^N, and nontrivial if it is not empty.

A property Π is a function on R^N to a two element set, {true, false} for example. If V is a proper variety of R^N, we say Π is generic relative to V if Π is true everywhere on R^N except for a subset of V. Π is deemed generic if such a V exists. Since a proper variety is closed in the usual topology, it follows that if Π is generic relative to V, for every $x\in V^c$ (the complement of V) Π is true on some neighborhood of x. As a proper variety V cannot contain any open set in R^N (if this were so, the defining polynomials would all be identically zero). It follows that if Π is false for some $x\in v$, then there exist points arbitrarily close to x such that Π is true at these points.

The key lemma of our development will be applicable to a more general class of linear systems than only m input–m output systems. Indeed consider a matrix triple (A,B,C) representing a system of the form (1) with $A\, n\times n$, $B\, n\times n$, and $C\, q\times n$. We assume an arbitrary partition of C into k submatrices.

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$$C=[C_1',\cdots,C_k']'$$

with each $C_i q_i \times n$. Again, we let $\mathcal{K}_i = \operatorname{Ker} C_i$ with $\mathcal{R}_i$ defined as in (4) for all $i \in \mathbf{k}$.

Lemma: Given a linear system of the form described above, if

$$n \geq m \geq \sum_{i \in \mathbf{k}} q_i \tag{6}$$

then (5) is generically true.

Proof: We shall make use of the results on the generic dimensions of subspaces from [5] without specific reference. Further identities which hold everywhere except possibly on a subset of a proper algebraic variety will be indicated by a postscripted (g).

We note that any $r \times s$ matrix Q generically has rank $t = \min(r,s)$. For otherwise all $t \times t$ minors of Q must vanish identically, in which case the elements of Q constitute a zero for a set of polynomials defined on $R^{r \times s}$. Thus, letting $\mathcal{C}_i$ denote the image of the map C_i, we have $\dim \mathcal{C}_i' = q_i(g)$ and $\dim(\operatorname{Ker} C_i) = n - q_i(g) i \in \mathbf{k}$. Then from (6)

$$\dim\Big(\sum_{\substack{j \neq i \\ i \in \mathbf{k}}} \mathcal{C}_j'\Big) = \min(n, q_i^*) = q_i^*(g), \qquad i \in \mathbf{k}$$

where

$$q_i^* = \sum_{\substack{j \neq i \\ i \in \mathbf{k}}} q_j$$

and by complementation

$$\dim \mathcal{K}_i = n - \min(n, q_i^*) = n - q_i^*(g), \qquad i \in \mathbf{k}. \tag{7}$$

We will first demonstrate that under the hypothesis of the lemma

$$\sum_{i \in \mathbf{k}} \mathcal{B} \cap \mathcal{K}_i = \mathcal{B}\ (g).$$

Since $\Sigma_{i \in \mathbf{k}}(\mathcal{B} \cap \mathcal{K}_i) \subset \mathcal{B}$, we need only prove that (6) implies

$$\dim\Big(\sum_{i \in \mathbf{k}} \mathcal{B} \cap \mathcal{K}_i\Big) = \dim \mathcal{B} = m(g). \tag{8}$$

Using the geometric identity

$$\dim(\mathcal{S} \cap \mathcal{J}) = \dim(\mathcal{S}) + \dim(\mathcal{J}) - \dim(\mathcal{S} + \mathcal{J})$$

to expand the left side of (8) results in

$$\dim\Big(\sum_{i \in \mathbf{k}} \mathcal{B} \cap \mathcal{K}_i\Big) = \dim(\mathcal{B} \cap \mathcal{K}_k) + \dim\Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big) - \dim\Big((\mathcal{B} \cap \mathcal{K}_k) \cap \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big).$$

But

$$\dim\Big((\mathcal{B} \cap \mathcal{K}_k) \cap \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big) = \dim\Big(\mathcal{K}_k \cap \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big)$$
$$= \dim \mathcal{K}_k + \dim\Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big) - \dim\Big(\mathcal{K}_k + \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big)$$

which yields

$$\dim\Big(\sum_{i \in \mathbf{k}} \mathcal{B} \cap \mathcal{K}_i\Big) = \dim(\mathcal{B} \cap \mathcal{K}_k) - \dim \mathcal{K}_k + \dim\Big(\mathcal{K}_k + \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big). \tag{9}$$

Now

$$\dim(\mathcal{B} \cap \mathcal{K}_k) = \dim \mathcal{B} + \dim \mathcal{K}_k - \dim(\mathcal{B} + \mathcal{K}_k)$$
$$= m + (n - q_k^*) - \min(n, m + n - q_k^*)(g)$$
$$= m - q_k^*(g) \tag{10}$$

as $m > q_k^*$ by (6). Since

$$\dim(\mathcal{B} + \mathcal{K}_k) = \min(n, \dim \mathcal{B} + \dim \mathcal{K}_k)(g)$$
$$= \min(n, m + n - q_k^*) = n(g)$$

it follows from (10) that

$$\dim\Big(\mathcal{K}_k + \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big) = \min\Big(n, n - q_k^* + \sum_{i \in \mathbf{k}-1} (m - q_i^*)\Big)(g). \tag{11}$$

But

$$q_k^* + \sum_{i \in \mathbf{k}-1} q_i^* = (k-1) \sum_{i \in \mathbf{k}} q_i$$

where (11) becomes

$$\dim\Big(\mathcal{K}_k + \Big(\sum_{i \in \mathbf{k}-1} \mathcal{B} \cap \mathcal{K}_i\Big)\Big) = \min\Big(n, n + (k-1)\Big(m - \sum_{i \in \mathbf{k}} q_i\Big)\Big) = n(g). \tag{12}$$

Combining (7), (10), and (12) together with (9) gives

$$\dim\Big(\sum_{i \in \mathbf{k}} \mathcal{B} \cap \mathcal{K}_i\Big) = m - q_k^* - (n - q_k^*) + n = m(g).$$

Fabian and Wonham have demonstrated that under condition (6), the subspaces $\mathcal{K}_i$, $i \in \mathbf{m}$ are generically controllability subspaces themselves. Thus,

$$\mathcal{R}_i = \mathcal{K}_i\ (g)$$

and (8) implies (5) proving the lemma.

Theorem: Morgan's problem is generically solvable.

Proof: For this problem $m = k$, $q_i = 1$ for all $i \in \mathbf{k}$ and the hypothesis of the lemma is satisfied. Hence, (5) follows for almost all parameter sets (A,B,C).

IV. Discussion

It has been shown that the decoupling problem originally considered by Morgan and solved by Falb and Wolovich is generically solvable. The development made use of the geometric problem formulation of Wonham and Morse and the results on generic dimension by Fabian and Wonham.

We note that if the elements of D were polynomial functions of the components of the matrices A, B, and C, then by elementary methods we could deduce the desired theorem directly from the result of Falb and Wolovich. However the dependence of the rows of D on the discrete valued functions d_i precludes this approach.

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On The Decomposition of State Space

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***Abstract*—The response of a linear control system is often viewed as a superposition of independent, modal responses. For complex systems, traditional techniques for resolving modal responses may be either inapplicable, quite expensive, or numerically unstable.**

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