An operator-theoretic approach
to the mixed-sensitivity minimization problem (I)

by

Fabio Fagnani

and

Sanjoy K. Mitter

Laboratory for Information and Decision Systems

Massachusetts Institute of Technology

Cambridge, Mass. 02139, U.S.A.

August , 1988

Abstract: in this paper we consider the mixed-sensitivity minimization problem (scalar case).It gives rise to the so-called two-block problem on the algebra H^∞; we analyze this problem from an operator point of view, using Krein space theory. We obtain a necessary and sufficient condition for the uniqueness of the solution and a parameterization of all solutions in the non-uniqueness case. Moreover, an interpolation interpretation is given for the finite-dimensional case.

INTRODUCTION

§1 The mixed-sensitivity minimization problem.

The problem we want to study in this paper is a classical one in Control Theory and it is usually known as the mixed-sensitivity minimization problem; it will be precisely stated later in the paragraph.

Throughout this paragraph all linear systems considered will be causal, time invariant, continuos-time, single-input / single-output; moreover no formal distinction will be done between a system and its transfer function.

Let us consider now the following feedback system:

(I)
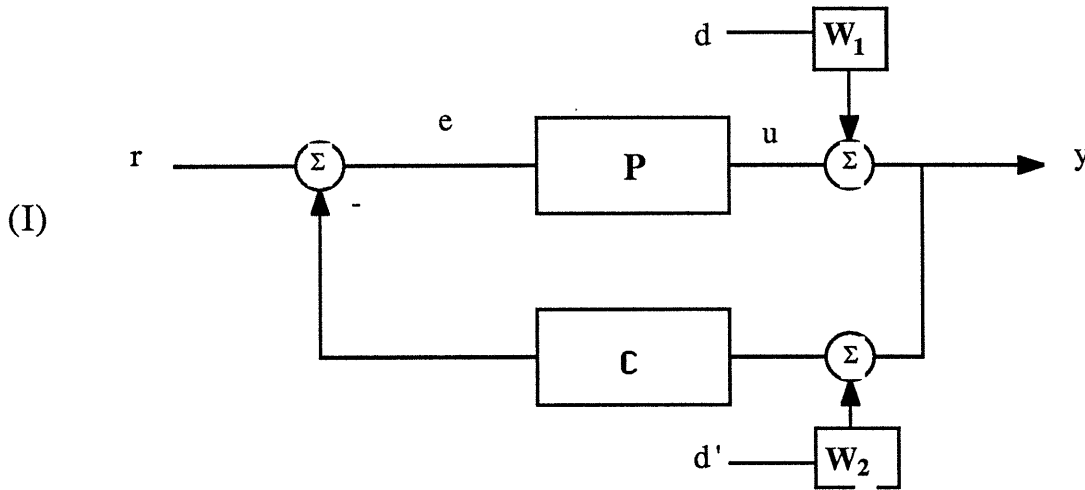
P is the plant and C is the control system; W_1, W_2 are two weighting functions.

Our goal is to minimize (in some sense) the effect of the disturbances d and d' on the plant P. It is easy to verify that the transfer functions from d to y and from d' to y are, respectively:

$$W_1(1+PC)^{-1} \qquad\qquad W_2PC(1+PC)^{-1}$$

Let $H\infty(\Pi^r)$ be the Banach algebra of holomorphic, uniformly bounded, complex functions on P^r (the open right half-plane of $\mathbb{C}$) with the infinity-norm. $H^\infty(\Pi^r)$ may be seen, in a natural way, as a closed subspace of $L^\infty(i\mathbb{R})$, the space of essentially bounded, measurable functions on the imaginary axis (the identification is obtained considering the extension of the holomorphic function to the boundary $i\mathbb{R}$); we will think of a $H^\infty(\Pi^r)$-function in these different ways depending on the context. From a systems point of view $H^\infty(\Pi^r)$ is just the algebra of transfer functions of systems which are linear, causal, time-invariant, continous-time and L^2-stable. Every $H^\infty(\Pi^r)$-function f may be factorized in the following way: $f = f_i f_o$ where $f_i \in H^\infty$ is such that $|f_i| = 1$ almost everywhere on the

1

imaginary axis (it is said the inner factor of f) and f_o is the outer factor of f. f_i and f_o are uniquely determined up to multiplicative complex units. Throughout this paragraph we set $H\infty = H^\infty(P^r)$.

Assume that W_1, W_2 belong to H^∞. C is an admissible feedback control if C is causal and if $(1+PC)^{-1}$ and $P(1+PC)^{-1}$ belong to H^∞. So now it is meaningful to state the following H^∞ optimal problem:

$$(1) \qquad \min_{C \text{ adm.}} \left\| \begin{bmatrix} W_1 (1+PC)^{-1} \\ W_2 PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty$$

The function $S := (1+PC)^{-1}$ is the sensitivity function; for $W_2 = 0$, (1) reduces to the classical optimal sensitivity problem which naturally leads to a Nehari problem. The function $PC(1+PC)^{-1}$ is equal to 1-S and is called complementary sensitivity; for this reason (1) is termed the mixed-sensitivity minimization problem.

The problem (1) looks like as a very hard one because it is non-linear in the control and, moreover, the space on which it is defined, is not well characterized. As in the case of the sensitivity problem, (1) may be transformed into a much simpler minimization problem; we are going to show this fact in the next paragraph.

§2 The canonical form of the minimization problem.

In order to transform (1) in the canonical form, it is necessary to place additional hypotheses:

H1) there exists a coprime factorization for P, that is $\exists$ N, D, a, b $\in H^\infty$ s. t.
$$P = ND^{-1} \qquad aN + bD \equiv 1;$$
moreover N and D have no zeros on the imaginary axis.

H2) $\qquad\qquad\qquad P(a+ib) \to 0 \quad$ if $a \to +\infty \qquad \forall b$

H3) W_1, W_2 are outer invertible in H^∞.

H1) and H2) allow us to use Youla parametrization of admissible controls; we have:

$$(2) \qquad C \text{ is admissible} \Leftrightarrow \exists \, Z \in H^\infty \quad Z \neq N^{-1}b \text{ s. t. } C = (a + DZ)(b - NZ)^{-1}$$

The proof of (2) may be found, for example in [F1].

From (2) we have:
$$S = D(b-NZ) \,, \ 1\text{-}S = N(a+DZ)$$

so (1) is equivalent to:

$$(3) \qquad \underset{Z \in H^\infty}{\text{Min}} \left\| \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Z \right\|_\infty$$

where:
$$G_1 := W_1 Db \qquad H_1 := W_1 DN$$
$$G_2 := -W_2 Na \qquad H_2 := W_2 DN$$

We are going to transform (3) now, following [JV]. First, we need to remind the concept of spectral factorization. If $A \in H^\infty$ define A^* by $A^*(s) := \overline{A}(-\overline{s}) \; \forall s \in \Pi^1$ (the left open half-plane); clearly A^* admits an L^∞-extension to the immaginary axis and we have $A^*(ix) = \overline{A}(ix) \; \forall x \in \mathbb{R}$ so that $A^* A = |A|^2$ on $i\mathbb{R}$.

<u>Definition</u> Let $f \in L^\infty$; $f \geq 0$ a.e. We say that there exists a spectral factorization for f
$$:\Leftrightarrow \; \exists \, g \in H^\infty \text{ such that } g^* g = f \text{ a.e. on } i\mathbb{R}.$$

<u>Proposition</u> (see [H]) $f \in L^\infty$; $f \geq 0$ a.e. admits a spectral factorization
$$\Leftrightarrow \log f \in L^1(d\lambda/(1+t^2)) \text{ where } \lambda \text{ is the Lebesgue measure on } i\mathbb{R}.$$

<u>Remark</u> If we assume that the spectral factor g is outer than we have the uniqueness of the spectral factorization upto multiplicative complex units.

Set now:
$$T := \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Z$$

$$T^* T = G_1^* G_1 + G_2^* G_2 - (G_1^* H_1 + G_2^* H_2)Z - (H_1^* G_1 + H_2^* G_2)Z^* + (H_1^* H_1 + H_2^* H_2)Z^* Z$$

We have:

$$H_1^* H_1 + H_2^* H_2 = D^* D N^* N \, (W_1^* W_1 + W_2^* W_2)$$

$$G_1^* H_1 + G_2^* H_2 = DN \, (W_1^* W_1 D^* b^* - W_2^* W_2 N^* a^*)$$

From hypotheses H3) and the preceding proposition it follows that $W_1^* W_1 + W_2^* W_2$ admits a spectral factorization with spectral factor M invertible in H^∞. Let us pose $Z' := D_o N_o Z$ where D_o and N_o are the outer factor of, respectively, D and N. Now let G be the L^∞-function such that:

$$G^* = M^{-1}D_i N_i (W_1^* W_1 D^* b^* - W_2^* W_2 N^* a^*)$$

where D_i and N_i are the inner factor of, respectively, D and N. Then:

$$T^*T = (G-MZ')^*(G-MZ') + (G_1^*G_1 + G_2^*G_2 - G^*G)$$

It is easily shown that:

$$G_1^*G_1 + G_2^*G_2 - G^*G = W_1^*W_1 W_2^*W_2 (W_1^*W_1 + W_2^*W_2)^{-1}$$

so, by hypothesis H3) and the proposition $\exists F$ spectral outer factor for the above function. So we obtain:

$$T^*T = (G-MZ')^*(G-MZ') + F^*F$$

Therefore problem (3) is equivalent to the following:

$$(4) \qquad \underset{Z' \in D_o N_o H^\infty}{\text{Min}} \left\| \begin{bmatrix} G - Z' \\ F \end{bmatrix} \right\|_\infty$$

It is important to observe that the function $F \in H^\infty$ does not depend on the plant P, but only on the two weighting functions W_1, W_2; it is rational if W_1, W_2 are.

In general D_o and N_o are not invertible in H^∞ (they may go to 0 at infinity) so that $D_o N_o H^\infty$ is not usually closed in the H^∞ - topology; consequently (4) may not have any optimal solution. As in the case of the sensitivity minimization problem, what we do at this point is to consider a relaxed form of (4); precisely:

$$(5) \qquad \underset{Z' \in H^\infty}{\text{Min}} \left\| \begin{bmatrix} G - Z' \\ F \end{bmatrix} \right\|_\infty$$

Problem (5) is not equivalent to (4); if $Z' \in H^\infty$ is a solution of (5), it may happen that $Z' \notin D_o N_o H^\infty$ or, in other words, that the corresponding parameter Z is improper. If this happenes the best thing we can do is to look for an approximation $Z_n \in H^\infty$ of Z such that:

$$\left\| \begin{bmatrix} G - Z_n' \\ F \end{bmatrix} \right\|_\infty \longrightarrow \left\| \begin{bmatrix} G - Z' \\ F \end{bmatrix} \right\|_\infty$$

The approximation problem will not discussed any longer in this paper; some results in

this sense may be obtained by generalizing corresponding results for the sensitivity minimization problem; see, for example [V].

§3 Our approach to the problem

Our approach to problem (5) will be, essentially, operator theoretic; it will be analyzed using the Ball-Helton theory developped in [BH1] and [BH2], now extended to the more general 4-block problem in [BC], as modified by Sarason in [S2].

The contribution of this paper is to give finer details as to the structure of the solution; in chapter two we show how the matrix U which parametrizes the suboptimal solution of (5), can be written down in terms of the Hankel-Toeplitz operator $\mathcal{A}$ associated to our problem. This permits us to analyze the optimal case as a limit of the suboptimal case: in this way we obtain, in chapter three, a necessary and sufficient condition for the uniqueness of the optimal solution (still in terms of the operator $\mathcal{A}$) which generalizes the well-known Adamjan-Arov-Krein uniqueness criterion for the Nehari problem. An extension of the sufficient maximal vector uniqueness criterion is also presented. In the same chapter we give a parameterization of all the optimal solution of problem (5) going beyond the result obtained in [BH1] where the authors restricted themselves to the "pseudo-regular" case. Finally, in chapter four, we carry on a detailed analysis of the finite-dimensional case ($G = \overline{\psi}W$ where ψ is a finite Blaschkee product, $W \in H^\infty$ rational); we show that our problem is, in this case, equivalent to solving a suitable Nevanlinna-Pick interpolation problem. This kind of approach to the finite-dimensional case is, conceptually, quite close to the ε-iteration formula in [JV]: the difference consists in working with interpolation problems instead of Hankel operators. Nevertheless, our approach permits to overcome the main problem connected to the ε-iteration formula: in fact, the difficulty in treating the critical case in which the optimal value is equal to the ∞-norm of F, is overcome by the introduction of a natural limit interpolation problem.

§4 Some mathematical preliminaries

Def 1.0   A **Krein space** is a pair (H,J) where H is a complex Hilbert space and J is a symmetry on H, that is, a self-adjoint unitary operator on H. To eliminate trivial cases we assume that J is different from $\pm I$.

The symmetry J induces an indefinite inner product on H given by (Jx,y) where $x, y \in H$, denoted by [x,y]. Obviously $\sigma(J) = \{-1, +1\}$; let us denote by H_+ and H_- the corresponding eigenspaces and by P_+ and P_- the orthogonal projections. Thus $J = P_+ - P_-$ and

$$[x,y] = (P_+x, P_+y) - (P_-x, P_-y).$$

<u>Def 2.0</u> A vector $x \in H$ is called positive iff $[x,x] \geq 0$. A subspace of H is called positive iff it consists of positive vectors. A positive subspace is said **maximal positive** iff it is not properly contained in another positive subspace. Negative vectors and subspaces are analogously defined.

<u>Prop 3.0</u> $K \leq H$ is a positive subspace $\Leftrightarrow \exists\ T : D \leq H_+ \to H_-$ contraction such that $G(T)=K$ (where $G(T)$ is the graph of T). Moreover K is maximal positive $\Leftrightarrow D= H_+$. The operator T is said to be the angular operator of K.

<u>Def 4.0</u> A positive subspace is said to be **uniformly positive** iff the norm of its angular operator is less than one.

Using the indefinite inner product $[\ ,\ ]$ one can define the J-orthogonality between vectors and subspaces (indicated by $[\perp]$), the J-adjoint of an operator (indicated by a $^{[*]}$) and so on.

<u>Def 5.0</u> $K \leq H$ is called **regular** iff $\exists\ M_+ \leq H$ uniformly positive and $M_- \leq H$ uniformly negative, J- ortogonal, such that $K = M_+ \oplus M_-$.

<u>Prop 6.0</u> $K \leq H$ is regular $\Leftrightarrow H = K \oplus K^{[\perp]}$. In particular K is regular if and only if $K^{[\perp]}$ is regular.

<u>Example</u> The simplest example of a Krein space is the following: let us consider the finite-dimensional Hilbert space $\mathbb{C}^{m+n}$; it is a Krein space with the isometry $J_{m,n}$ given by $J_{m,n}(x,y) := (x,-y)$ where $x \in \mathbb{C}^m$; $y \in \mathbb{C}^n$.

Problem (5) is stated on the Banach algebra $H^\infty(\Pi^r)$; it is simpler to work on the unit disk Δ instead of the right-half plane; in the sequel we will briefly recall the definition of Hardy spaces on the unit disk.

Set the following notation: Δ is the unit open disk in the complex plane; $\mathbb{T} := \partial\Delta$. If $f \in \text{Hol}(\Delta,\mathbb{C})$ (algebra of the holomorphic, complex-valued functions on Δ) then f_r indicates the function $\theta \to f(re^{i\theta})$; $\|\ \|_p$ is the canonical norm on $L^p(\mathbb{T},\mathbb{C})$ where $p \geq 1$.

<u>Def 7.0</u>

$$H^p(\Delta) := \{\ f \in \text{Hol}(\Delta,\mathbb{C})\ \text{s.t.}\ \sup\{\| f_r \|_p\ |\ r \in (0,1)\} < +\infty\}$$

It is a Banach space with the norm: $\| f \|_p := \sup\{\| f_r \|_p\ |\ r \in (0,1)\}$

<u>Prop 8.0</u>   Let $f \in H^p(\Delta)$ then $\exists\ f' \in L^p(\mathbb{T},\mathbb{C})$ such that $f_r \to f'$ a.e.
Moreover the map $f \to f'$ is an isometry between $H^p(\Delta)$ and $L^p(\mathbb{T},\mathbb{C})$

From now on we will simply indicate by H^p and L^p the spaces, respectively, $H^p(\Delta)$ and $L^p(\mathbb{T},\mathbb{C})$. In particular H^∞ will be the algebra $H^\infty(D)$.

Remark   From the preceding proposition we deduce that H^p may be identified with a closed subspace of L^p. This identification will be used throughout this paper; depending on the case, a H^p-function will be thought as an analytic function on Δ or as an L^p-function on $\mathbb{T}$.

The results about Hardy spaces theory used in the paper can be found in [H].

We finish this introduction with the following:

Def 9.0   Let $W \in L^\infty$. We define:

$$M_W: L^2 \to L^2 \qquad\qquad M_W(f) := Wf \qquad \textbf{Laurent operator}$$
$$\mathcal{H}_W: H^2 \to (H^2)^\perp \qquad \mathcal{H}_W(f) := P_-(Wf) \qquad \textbf{Hankel operator}$$
$$T_W: H^2 \to H^2 \qquad\qquad T_W(f) := P_+(Wf) \qquad \textbf{Toeplitz operator}$$

W is said to be the symbol of the corresponding operator.

CHAPTER ONE
Statement of the problem in the operator theory context.

Let us begin by writing down again the H^∞ optimal problem:

$$(1) \qquad\qquad \underset{Z \in H^\infty}{\text{Min}} \left\| \begin{bmatrix} G - Z \\ F \end{bmatrix} \right\|_\infty = \eta$$

Let us consider now the following operator:

$$(2) \qquad\qquad \mathcal{A} : H^2 \to (H^2)^\perp \oplus H^2$$

given by:

$$(3) \qquad\qquad \mathcal{A}\phi := (\mathcal{H}_G\phi , T_F\phi)$$

The operator $\mathcal{A}$ has been already introduced in [CCL]. $\mathcal{A}$ does not change if we modify G by adding an H^∞ function.

The pairs:

$$\begin{bmatrix} G - Z \\ F \end{bmatrix}$$

where $Z \in H^\infty$, are said to be symbols of $\mathcal{A}$. Obviously:

$$\| \mathcal{A} \| \leq \eta$$

The operator $\mathcal{A}$ behaves as a Hankel operator; the preceding inequality is, in fact, an equality so that the solutions of (1) are in one-to-one correspondence with the minimal symbols of $\mathcal{A}$.

We introduce now the following Krein space:

$$(4) \qquad \mathcal{H} := L^2 \oplus H^2 \oplus H^2$$

with the indefinite scalar product given by:

$$(5) \qquad [\,(f_1, f_2, f_3)\,,\,(g_1, g_2, g_3)\,] := \,<f_1, g_1>_{L^2} + \,<f_2, g_2>_{H^2} - \,<f_3, g_3>_{H^2}$$

Let J be the corresponding symmetry. On $\mathcal{H}$ we may consider the right shift S given by:

$$S\,(f_1, f_2, f_3) = (\,S_{L^2}f_1\,, S_{H^2}f_2\,, S_{H^2}f_3\,)$$

where S_{L^2} is the bilateral right shift on L^2 and S_{H^2} the unilateral right shift on H^2. It is easy to realize that:

$$\mathcal{A}^*(\,S_{L^2}{}^*|_{(H^2)^\perp}\phi_1\,,\,S_{H^2}{}^*\phi_2) = \,S_{H^2}{}^*\mathcal{A}^*(\,\phi_1, \phi_2)$$
$$\forall\phi_1 \in (H^2)^\perp,\,\forall\phi_2 \in H^2$$

from which we easily derive that $G(\mathcal{A}^*)$, the graph of $\mathcal{A}^*$, seen as a subspace of $\mathcal{H}$, is S^*-invariant.

Let us state now the following fundamental:

Theorem 1.1 $\qquad \|\mathcal{A}\| < 1 \qquad \Rightarrow \qquad \exists\, Z \in H^\infty \qquad$ s. t.

$$\left\| \begin{bmatrix} G-Z \\ F \end{bmatrix} \right\|_\infty \leq 1$$

that is $\mathcal{A}$ has a symbol whose norm is not greater than one.

Proof
It is contained in [BH1]; I need to briefly recall it.

$G(\mathcal{A}^*) \leq H$ is S^*-invariant and is also (because of $\|\mathcal{A}\| < 1$) uniformly positive. To find symbols of $\mathcal{A}$ whose norm is not greater than one is equivalent to finding extensions of the operator $\mathcal{A}^*$ to the domain $L^2 \oplus H^2$ whose operator norm is not greater than one and whose graph is S^*-invariant. So, it is equivalent to find maximal positive, S^*-invariant subspaces of $\mathcal{H}$, containing $G(\mathcal{A}^*)$.

8

Let us set $N := G(\mathcal{A}^*)^{[\perp]}$. Using the regularity of N (Prop 6.0) we obtain the following representation:

$$N = \text{span}\{S^k L \mid k \geq 0\}.$$

where L is a regular subspace which is neither positive, nor negative; in particular $\exists x_1 \in L$: $[x_1, x_1] = 1$. Let us set $N_+ := \text{span}\{S^k x_1 \mid k \in \mathbb{N}\}$ and let us consider the S^*-invariant subspace $G(\mathcal{A}^*) + N_+$. We easily check that is maximal positive so that the proof is complete. Q.E.D.

<u>Cor 2.1</u> $\| \mathcal{A} \| = 1 \quad \Rightarrow \quad \exists Z \in H^\infty$ s. t.

$$\left\| \begin{bmatrix} G-Z \\ F \end{bmatrix} \right\|_\infty = 1$$

<u>Proof</u>

It easily follows by a standard compactness argument in the weak-* topology of $L^\infty \oplus H^\infty$.

Q.E.D

CHAPTER TWO
The parameterization of the sub-minimal symbols.

Let us assume that $\| \mathcal{A} \| < 1$. The problem of describing all the sub-minimal symbols of $\mathcal{A}$ (that is symbols whose infinity-norm is not greater than one) is solved in [BH1]. I will briefly recall the parameterization showing how, in this case, we can give finer details on the structure of the parameterizer U.

In the terms of the last chapter we want to classify all the S^*-invariant, maximal-positive subspaces of $\mathcal{H}$, containing $G(\mathcal{A}^*)$; but this is equivalent to classifying all the S-invariant, maximal-negative subspaces of $\mathcal{H}$, contained in the space N introduced in the proof of proposition 1.1. Such subspaces are, exactly, the graphs of the multiplicative operators whose inducing functions are the symbols for $\mathcal{A}$ having L^∞-norm of at most 1.

Let us consider again the subspace L introduced before; being regular, it may be written as the J-orthogonal sum of a uniformly positive subspace L^+ and a uniformly negative one L^-. It is easy to see that $\dim L^+ = \dim L^- = 1$; let us choose $x_1, x_2 \in L$ s. t.

$$[x_1, x_1] = -[x_2, x_2] = 1 \qquad [x_1, x_2] = 0$$

We may write:

$$x_1 = p_1 \oplus q_1 \oplus r_1 \qquad p_i \in L^2 \quad q_i, r_i \in H^2$$
$$x_2 = p_2 \oplus q_2 \oplus r_2$$

<u>Obs 1.2</u>   It follows from [S2] that the above equalities also hold pointwise on the boundary $\partial\Delta$. In other words:

$$(1) \qquad [\, x_1(e^{i\theta}),\, x_1(e^{i\theta})\,]_{2,1} = 1 = -[\, x_2(e^{i\theta}),\, x_2(e^{i\theta})\,]_{2,1} \quad \text{a.e. on } \partial\Delta$$
$$[\, x_1(e^{i\theta}),\, x_2(e^{i\theta})\,]_{2,1} = 0 \ \text{ a.e. on } \partial\Delta$$

where $[\ ,\ ]_{2,1}$ is the indefinite product on the Krein space $(\mathbb{C}^3, J_{2,1})$ as defined in the introduction.

Let us consider now the matrix:

$$(2) \qquad U := \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix}$$

The matrix U may be thought as a linear map between $H^\infty \oplus H^\infty$ and N; moreover the Hilbert space $H^2 \oplus H^2$ may be seen as a Krein space with the indefinite product induced by that of $(\mathbb{C}^2, J_{1,1})$. The following result has been stated and proved in [BH1]:

<u>Theo.2.2</u> $N' \leq \mathcal{H}$ is a S-invariant maximal negative subspace contained in N $\Leftrightarrow$
$\exists\, N'' \leq H^2 \oplus H^2$ be S-invariant, maximal-negative s.t.

$$(3) \qquad N' := \text{clos } U(N'' \cap H^\infty \oplus H^\infty) \leq \mathcal{H}$$
where clos is the closure operator in $\mathcal{H}$.

<u>Obs.3.2</u>   By Prop.3.0 S-invariant maximal-negative subspaces of $H^2 \oplus H^2$ are exactly the graphs of the multiplicative operators whose corresponding symbols are in $B(H^\infty)$. If $N'' = \{\psi h \oplus h \mid h \in H^2\}$ where $\psi \in B(H^\infty)$, then the angular operator associated to the subspace N' in (3) is the multiplicative operator induced by the pair:

$$(4) \qquad (\ (p_1\psi + p_2)(r_1\psi + r_2)^{-1},\ (q_1\psi + q_2)(r_1\psi + r_2)^{-1}\ )$$

As ψ varies in $B(H^\infty)$ these are, exactly, the symbols (of $\mathcal{A}$) whose norm is not greater than one. Necessarily:

$$(q_1\psi + q_2)(r_1\psi + r_2)^{-1} = F \quad \forall \psi \in B(H^\infty) \quad \Rightarrow$$

$$(5) \qquad q_1 = Fr_1 \quad q_2 = Fr_2$$

Let us set:

$$U' := \begin{bmatrix} p_1 & p_2 \\ r_1 & r_2 \end{bmatrix}$$

Let us set the following notation:

$$(6) \qquad U'^t(\psi, 1) = (p_1\psi + p_2)(r_1\psi + r_2)^{-1}$$

Therefore all the subminimal symbols of $\mathcal{A}$ are given by:

$$(U'^t(\psi, 1), F) \qquad \psi \in B(H^\infty)$$

Let us observe that $U'^t(1,1)$, $U'^t(0,1)$ are symbols for $\mathcal{H}_G$ so that: $U'^t(1,1) - U'^t(0,1) \in H^\infty$. Furthemore:

$$r_2^{-1}(r_1 + r_2)^{-1}\det U' = U'^t(1,1) - U'^t(0,1) \implies \det U' \in H^1.$$

Prop.4.2 $\det U' \in H^\infty$ is outer, determined (up to a complex unit factor) by the relation:

$$(6) \qquad |\det U'|^2 = (1 - |F|^2)^{-1} \text{ a.e. on } \partial\Delta$$

Proof

From relations (1) and (5) we easily obtain:

$$(7) \qquad U'^* \begin{bmatrix} 1 & 0 \\ 0 & -(1 - |F|^2) \end{bmatrix} U' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ a.e. on } \partial\Delta \implies$$

$$|\det U'|^2 = (1 - |F|^2)^{-1} \text{ a.e. on } \partial\Delta$$

This implies that $\det U'$ belongs to H^∞ (we use the fact $\| F \|_\infty \le \| \mathcal{A} \| < 1$) and its outer factor is determined (up to a complex unit factor) by the spectral factorization of $(1 - |F|^2)^{-1}$. By a similar argument to the one in [S2] we, finally, prove that $((\det U')H^2)^\perp = (0)$ so that $\det U'$ is outer.   Q.E.D.

Obs.5.2   By manipulating relation (7) we have:

$$(8) \qquad p_1 = \overline{r_2}\, d^{-1} \qquad p_2 = \overline{r_1}\, d^{-1}$$

where $d = \det U'$.

We want now to give an explicit expression of the elements of the matrix U in terms of the operator $\mathcal{A}$.

<u>Prop.6.2</u> A possible choice of the elements of U is the following:

$$p_2 = \| \, (\, I - \mathcal{A}^*\mathcal{A})^{-1/2}1\|_2^{-1} \, \mathcal{H}_G \, (\, I - \mathcal{A}^*\mathcal{A})^{-1}1$$

$$r_2 = \| \, (\, I - \mathcal{A}^*\mathcal{A})^{-1/2}1\|_2^{-1}(\, I - \mathcal{A}^*\mathcal{A})^{-1}1$$

the other elements of U are linked to p_2 and r_2 by relations (5) and (8).

<u>Proof.</u>

It is easy to see that: $N = H^2 \oplus \{0\} \oplus \{0\} \; \bigoplus \; G(\mathcal{A})$. On the other hand $L \leq N$ and $S^*L \leq G(\mathcal{A}^*)$. Therefore, given $x \in L \; \exists \; \xi \in H^2, \exists \; \alpha \in \mathbb{C}$ s.t.

$$(9) \qquad\qquad x = (\, \mathcal{H}_G \xi + \alpha \,, T_F \, \xi \,, \xi \,)$$

It is not difficult to show that necessary and sufficient condition on the pair (ξ, α) s.t. x, as defined in (9), belongs to L, is: $\exists \beta \in \mathbb{C}$ s.t.

$$\exists \beta \in \mathbb{C} \;\; \text{s.t} \qquad \xi = \alpha(\, I - \mathcal{A}^*\mathcal{A})^{-1} S_{H^2} \mathcal{H}^*_G S_{L^2}^*1 + \beta(\, I - \mathcal{A}^*\mathcal{A})^{-1}1$$

Taking $\alpha = 0$ we obtain:

$$x = (\, \beta\mathcal{H}^*_G(\, I - \mathcal{A}^*\mathcal{A})^{-1}1 \,, \beta T_F(\, I - \mathcal{A}^*\mathcal{A})^{-1}1, \beta(\, I - \mathcal{A}^*\mathcal{A})^{-1} \,)$$

which is strictly negative $\forall \beta \neq 0$ and so, after the scalar b has been properly choosen, becomes a possible choice for x_2. This completes the proof.   Q.E.D.

CHAPTER THREE
Uniqueness criteria. A parametrization of the minimal symbols in non-uniqueness case.

§1 A generalization of Krein's uniqueness condition.

From now on we will suppose $\| \, \mathcal{A} \, \| = 1$. Let us consider $\mathcal{A}_\varepsilon := (1-\varepsilon)\mathcal{A}$ $\varepsilon \in (0,1)$; $\|\mathcal{A}_\varepsilon\| < 1$. By the result obtained in the last chapter we have that the symbols of $\mathcal{A}_\varepsilon$ in $B(H^\infty)$ are parameterized in terms of a given matrix U'_ε:

$$U'_\varepsilon := \begin{bmatrix} p_{1\varepsilon} \, p_{2\varepsilon} \\ r_{1\varepsilon} \, r_{2\varepsilon} \end{bmatrix}$$

where:

(1)
$$\begin{bmatrix} r_{2\varepsilon} & = & \| (I - \mathcal{A}_\varepsilon^* \mathcal{A}_\varepsilon)^{-1/2}1 \|_2^{-1}(I - \mathcal{A}_\varepsilon^* \mathcal{A}_\varepsilon)^{-1}1 \\ p_{2\varepsilon} & = & \mathcal{H}_{f_\varepsilon} r_{2\varepsilon} \quad f_\varepsilon = (1-\varepsilon)G \\ r_{1\varepsilon} & = & d_\varepsilon \, \overline{p_{2\varepsilon}} \\ p_{1\varepsilon} & = & \overline{r_{2\varepsilon}} \, \overline{d_\varepsilon}^{-1} \end{bmatrix}$$

$d_\varepsilon = \det U'_\varepsilon$ is outer invertible in H^∞ $\forall \varepsilon \in (0,1)$; it may be computed by the relation $|d_\varepsilon|^2 = (1-|F|^2)^{-1}$ where $F_\varepsilon = (1-\varepsilon)F$. We have that:

(2)
$$r_{2\varepsilon}(0) = <r_{2\varepsilon}, 1>_{H^2} = \| (I - \mathcal{A}_\varepsilon^* \mathcal{A}_\varepsilon)^{-1/2}1 \|_2$$

Let us state the fundamental result:

<u>Theo.1.3</u>   The operator $\mathcal{A}$ admits a unique minimal symbol $\Leftrightarrow$

(3)
$$\lim_{\varepsilon \to 0} \| (I - \mathcal{A}_\varepsilon^* \mathcal{A}_\varepsilon)^{-1/2}1 \|_2 = +\infty$$

<u>Proof.</u>

It is similar to the proof given in [S2] for Hankel operators, with some slight technical modifications.

Let $\psi \in B(H^\infty)$; $U'_\varepsilon \psi$ is the corresponding symbol of $\mathcal{A}_\varepsilon$. Then:

$$U'_\varepsilon \psi - U'_\varepsilon 0 = [r_{2\varepsilon}(r_{1\varepsilon}\psi + r_{2\varepsilon})]^{-1}\psi d_\varepsilon \in H^\infty$$

The set of values taken by the preceding function in a point z of the unit open disk when ψ varies in $B(H^\infty)$, is a closed disk whose ray is given by:

$$\rho_\varepsilon(z) = |d_\varepsilon(z)| [\, | r_{2\varepsilon}(z) |^2 - | r_{1\varepsilon}(z) |^2 \,]$$

Necessary and sufficient condition for the uniqueness of the minimal symbol of $\mathcal{A}$ is that

(4)
$$\lim_{\varepsilon \to 0} \rho_\varepsilon(z) = 0 \quad \forall z \in \Delta$$

Let us observe now that $\{d_\varepsilon^{-1}\}$ is a normal family and $d_\varepsilon^{-1}(z) \neq 0$ $\forall z$ $\forall \varepsilon$; therefore, there are only two possibilities: either inf $|d_\varepsilon(z)|^{-1} > 0$ $\forall z$ or every limit point of $\{d_\varepsilon^{-1}\}$ when $\varepsilon \to 0$ in

13

the open-compact topology is the null function. In the first case, the analysis goes on as in [S2] (pg.308-309) showing that (4) is equivalent to:

$$(5) \qquad \lim_{\varepsilon \to 0} |r_{2\varepsilon}(0)| = +\infty$$

which is exactly (4). In the second case it is easy to check, explicity, that $\mathcal{A}$ has a unique minimal symbol and that, on the other hand, limit in (3) is finite, so that the theorem is still true. Q.E.D.

Obs.2.3 It is easy to see that (3) is equivalent to the two following conditions:

$$(6) \qquad 1 \notin \mathcal{R}(I - \mathcal{A}^*\mathcal{A})^{1/2}$$

$$(7) \qquad \lim_{\varepsilon \to 0} <(I - \mathcal{A}_\varepsilon^*\mathcal{A}_\varepsilon)^{-1}1, 1> = +\infty$$

§2 The maximal vector uniqueness criterion

The criterion we now expose is the generalization of a well-known uniqueness criterion for Hankel operators (see, for example, [AAK] and [S1]).

Def.3.3 Let T: H → K be a bounded operator acting on Hilbert spaces. A vector $g \in H$, $\|g\| = 1$ is said a **maximal vector** for T $:\Leftrightarrow \|Tg\| = \|T\|$.

Theo.4.3 Let us suppose that $\mathcal{A}$ has a maximal vector g. Then $\mathcal{A}$ has a unique minimal symbol given by:

$$(8) \qquad \begin{bmatrix} g^{-1}\mathcal{H}_G g \\ F \end{bmatrix}$$

Moreover:

$$(9) \qquad |g^{-1}\mathcal{H}_G g|^2 + |F|^2 = \|\mathcal{A}\|^2 \qquad \text{a.e. on } \partial\Delta.$$

Proof.

It is a straightforward generalization of the corresponding one for Hankel operators.

Obs.5.3 The relation (9) generalizes the result that the minimal symbol of a Hankel operator having a maximal vector is unimodular.

Obs.6.3 The operator $\mathcal{A}$ admits a maximal vector $\Leftrightarrow \mathcal{A}^*\mathcal{A} = \mathcal{H}_G^*\mathcal{H}_G + T_F^*T_F$ admits a maximal eigenvalue λ. In this case every eigenvector of $\mathcal{A}^*\mathcal{A}$ relative to λ is a

maximal vector of $\mathcal{A}$ and vice versa.

Obs.7.3   It follows from the obs.6.3 that a sufficient condition for the existence of a maximal vector for $\mathcal{A}$ and consequently for the uniqueness of the minimal symbol is that:

$$(10) \qquad \rho_{ess}(\mathcal{A}^*\mathcal{A}) < \| \mathcal{A}^*\mathcal{A} \|$$

where ρ_{ess} is the essential ray of the operator.

So it may be fruitful to analyze the spectrum and the essential spectrum of $\mathcal{A}^*\mathcal{A}$ to verify (10); this has been done in some special cases: in [JV] and, in more generality, in [ZM].

§3   A parameterization of the minimal symbols.

Let us now suppose that we are in the case of non-uniqueness of the minimal symbols of the operator $\mathcal{A}$ whose norm is supposed to be equal to one. From relation (5) and a standard normality argument, it follows that $r_{2\varepsilon}$ is uniformly bounded on the compact sets of Δ. Furthemore, from relations (1) of chapter two, it easily follows that:

$$(11) \qquad r_{2\varepsilon}^{-1}, r_{1\varepsilon}r_{2\varepsilon}^{-1} \in B(H^\infty)$$

In particular, $r_{1\varepsilon}$ is uniformly bounded on the compact sets of Δ. Therefore: $\exists\ \varepsilon_n \downarrow 0$, $\exists r_1, r_2 \in \text{Hol}\ (\Delta)$:

$$r_{1n} := r_{1\varepsilon_n} \to r_1$$
$$r_{2n} := r_{2\varepsilon_n} \to r_2$$

uniformly on the compact sets of Δ. From (11) it follows that r_1, r_2 are two holomorphic functions with bounded characteristic and so they have well-defined values on the boundary. From the non-uniqueness assumption it follows that $\{d_\varepsilon\}$ is a normal family so that it is not restrictive to assume that $\exists d \in \text{Hol}(\Delta)$ such that $d_{\varepsilon_n} \to d$ uniformly on the compact sets of Δ.

We, now, need to make a technical assumption on the function F, precisely that there exists spectral factorization for $1-|F|^2$; with this hyphothesis it is not difficult to show that d is a function with bounded characteristic such that $|d|^2 = (1-|F|^2)^{-1}$ a.e. on $\partial\Delta$.

We are now ready to state the parameterization result. Let us set:

$$U' := \begin{bmatrix} \bar{d}^{-1}\,\bar{r}_1 & \bar{d}^{-1}\,\bar{r}_1 \\ r_1 & r_2 \end{bmatrix}$$

<u>Theo.8.3</u> $\{(U^t\psi, F) \mid \psi \in B(H^\infty)\}$ is the set of all the minimal symbols of $\mathcal{A}$.

<u>Proof.</u>

It is very technically involved but it does not presesent any conceptual difference respect to the corresponding one in [S2] (pg. 309-313); therefore, it will be omitted. Q.E.D.

CHAPTER FOUR
The finite-dimensional case; an interpolation approach

We want to analyze deeply the optimal problem in the finite-dimensional case that is in the case when $G = \overline{B}W$ where B is a pure finite Blaschke product with simple zeros $\{z_1, \ldots z_n\}$ in Δ and $W \in H^\infty$; F is supposed to be rational too. In this case we can write the optimal problem in the following form:

$$(1) \qquad \underset{Z \in H^\infty}{\text{Min}} \left\| \begin{bmatrix} W - BZ \\ F \end{bmatrix} \right\|_\infty$$

It is well-known that if we set $w_i := W(z_i)$, then $\{W-Bh \mid h \in H^\infty\}$ is exactly the set of the bounded holomorphic functions interpolating the points (z_i, w_i). So we have that :

$$(2) \quad \varepsilon_0 = \underset{Z \in H^\infty}{\text{Min}} \left\| \begin{bmatrix} W - BZ \\ F \end{bmatrix} \right\|_\infty = \min \{ \|t(f,F)\| \; f \in H^\infty \; f(z_i) = w_i\}$$

so, as in the case of the finite-dimensional Nehari problem, there is an interpolation problem linked to the original H^∞-optimal problem. A function $f \in H^\infty$ solving problem (2) in the interpolation form is called a minimal interpolating function of (2). We have the following:

<u>Prop.1.4</u>   Let us assume that $\varepsilon_0 > \|F\|_\infty$. Then:

(i)   there exists a unique minimal interpolation function f which is rational;

(ii)   the outer factor g of f is determined by the condition $|g|^2 + |F|^2 = \varepsilon_0^2$ a.e.

(iii) the inner factor of f is a Blaschke product B' of degree at most n-1 which is the minimal solution of the Nevanlinna-Pick interpolation problem relative to the pairs $(z_i, w_i g(z_i)^{-1})$

<u>Proof.</u>

Let us note that the operator $\mathcal{H}_{\overline{B}W}$ is compact because $\overline{B}W \in H^\infty + C(i\mathbb{R})$. So we

16

have :

$$\rho_{ess}(\mathcal{H}_{\overline{BW}}^*\mathcal{H}_{\overline{BW}} + T_F^*T_F) = \rho_{ess}(T_F^*T_F) = \rho_{ess}(T_{|F|^2}) = \|F\|_\infty^2$$

Therefore in the case $\varepsilon_o > \|F\|_\infty$ there exists a maximal vector for $\mathcal{A}$; by applying theorem 4.3, we prove the uniqueness of the solution.

Now, consider the inner-outer factorization of the minimal solution f: f = B'g. From theorem 4.3 it follows that the outer factor g is determined by the condition $|g|^2 + |F|^2 = \varepsilon_o^2$ a.e.. On the other hand B' is, obviously, a function interpolating the pairs $(z_i, w_i g(z_i)^{-1})$; it has to be the interpolating function of minimal norm because, otherwise, f could not be the minimal solution of the original problem; in particular, this shows that B' is a Blashke product of degree at most n-1. Finally, f is rational because g and B' are.   Q.E.D.

In the case $\varepsilon_o = \|F\|_\infty$ the existence of a maximal vector is not assured anymore and, therefore, we can not carry out the same analysis as before.

Consider the outer function g_ε determined by the condition $|g_\varepsilon|^2 + |F|^2 = \varepsilon^2$ a.e., where $\varepsilon \geq \|F\|_\infty$. It turns out that g_ε is invertible in H^∞ if and only if $\varepsilon > \|F\|_\infty$. Now, consider the Nevanlinna-Pick interpolation problem (NP_ε) relative to the pairs $(z_i, w_i g_\varepsilon(z_i)^{-1})$; the Hankel operator canonically associated to this problem, when $\varepsilon > \|F\|_\infty$, is $\mathcal{H}_{\overline{B}W g_\varepsilon - 1}$. It is easy to see that:

(1) $\| \mathcal{H}_{\overline{B}W g_\varepsilon - 1} \| \leq 1 \quad \Leftrightarrow \quad \varepsilon \geq \varepsilon_o$

and, if $\varepsilon_o > \|F\|_\infty$, then

$$\| \mathcal{H}_{\overline{B}W g_\varepsilon - 1} \| = 1 \quad \Leftrightarrow \quad \varepsilon = \varepsilon_o$$

This observation leads to an algorithm to find the optimal value called the ε-algorithm (see [JV] and [CDL]; the main problem connected to the ε-algorithm is that $g_{\|F\|_\infty}$ is not invertible in H^∞ so that, in the case $\varepsilon_o = \|F\|_\infty$, we can not get the optimal value. In the sequel to the paragraph we shall analyze the case $\varepsilon_o = \|F\|_\infty$, showing, in particular, how it is possible to overcome the above difficulty.

If $F(z) = \|F\|_\infty \ \forall z \in \Delta$, then the optimal problem (2) is trivial with unique solution given by f = 0. Therefore, by the maximum principle, we may assume that F does not assume its maximum value on the open disk Δ. Set $g := g_{\|F\|_\infty}$; we have: $g(z_i) \neq 0 \ \forall i$. Therefore it is meaningful to consider the Nevanlinna-Pick interpolation problem (NP) relative to the pairs $(z_i, w_i g(z_i)^{-1})$. Let y a some interpolating function of (NP); the Hankel operator associated to (NP) is, thus, given by $\mathcal{H}_{\overline{B}y}$. Moreover let y_ε the minimal interpolating function of the Nevanlinna-Pick interpolation problems relative to the pairs $(z_i, w_i g(z_i)^{-1} - w_i g_\varepsilon(z_i)^{-1})$; it is easy to see that $\|y_\varepsilon\|_\infty \to 0$ (eventually passing to a sequence). If we consider now, the relative

Hankel operators, we have:

$$\mathcal{H}_{\bar{B}Wg_\varepsilon-1} - \mathcal{H}_{\bar{B}y} = \mathcal{H}_{\bar{B}(Wg_\varepsilon-1-y)} = \mathcal{H}_{\bar{B}y_\varepsilon} \to 0$$

in the operator norm. So $\mathcal{H}_{\bar{B}Wg_\varepsilon-1} \to \mathcal{H}_{\bar{B}y}$ in the operator norm. From (1) and the assumption that $\varepsilon_0 = \|F\|_\infty$ we obtain that $\|\mathcal{H}_{\bar{B}Wg_\varepsilon-1}\| \le 1 \ \forall \varepsilon > \|F\|_\infty$. Therefore, we have $\|\mathcal{H}_{\bar{B}y}\| \le 1$.

<u>Theo.2.4</u>   Suppose $\varepsilon_0 = \|F\|_\infty$; then:

(i) $\|\mathcal{H}_{\bar{B}y}\| = 1 \Rightarrow$ there is a unique minimal interpolating function f_0 of (2) given by $f_0 = B'g$ where B' is the interpolating function of minimal norm relative to (NP); B' is a pure Blaschkee product, so that B'g is the inner-outer factorization of f_0.

(ii) $\|\mathcal{H}_{\bar{B}y}\| < 1 \Rightarrow$ there are infinitely many minimal interpolating function of (2) given by $f_\phi = \phi g$ where ϕ is any interpolating function of (NP) whose norm is not greater than 1.

<u>Proof.</u>
Let us note that every function f of the form $f_\phi = \phi g$, where ϕ is an interpolating function of (NP) whose norm is not greater than 1, is a minimal interpolating function of our original problem. Therefore, the proof is complete if we show that every minimal interpolating function is necessarily of this form.

Let $f_0 \in H^\infty$ a minimal interpolating function of problem (2) ; clearly $\|g_\varepsilon^{-1}f_0\|_\infty \le 1$ $\forall \varepsilon > \|F\|_\infty$. Therefore $\exists \varepsilon_k \to \|F\|_\infty$ and $\exists \ \phi \in B(H^\infty)$ s.t. $g_{\varepsilon k}^{-1}f_0 \to \phi$ in the compact-open topology; on the other hand $g_{\varepsilon k} \to g$ in the compact-open topology; we conclude that $f_0 = \phi g$. We show now that ϕ is an interpolating function of (NP). Let y some interpolating function of problem (NP). $\bar{B}g_{\varepsilon k}^{-1}f_0$ are symbols for the Hankel operators $\mathcal{H}_{\bar{B}Wg_{\varepsilon k}-1}$; we know that there exist $y_\varepsilon \in H^\infty$ such that $\bar{B}y_{\varepsilon k}$ are symbols for $\mathcal{H}_{(\bar{B}Wg_{\varepsilon k}-1- \bar{B}y)}$: $\|y_{\varepsilon k}\|_\infty \to 0$. Then $\forall k$ $\bar{B}g_{\varepsilon k}^{-1}f_0 - \bar{B}y_{\varepsilon k}$ are symbols of $\mathcal{H}_{\bar{B}y}$ converging to $\bar{B}\phi$ from which we derive that $\bar{B}\phi$ is a symbol of $\mathcal{H}_{\bar{B}y}$ and, consequently, ϕ is an interpolating function of (NP).   Q.E.D.

<u>Obs.3.4</u> From the preceding proposition we see that the solution of our initial problem may be unique even if the operator $\mathcal{A}$ does not have a maximal vector; in fact it is quite easy to build an example where this happens.

<u>Obs.4.4</u> The result contained in theorem 2.4 permits to overcome the difficulty connected to the ε-algorithm; in fact, instead of starting the algorithm from an arbitrary value of ε, now, we can start from $\varepsilon = \|F\|_\infty$ calculating $\|\mathcal{H}_{\bar{B}y}\|$. If $\|\mathcal{H}_{\bar{B}y}\| \le 1$ then, $\varepsilon_0 = \|F\|_\infty$; if $\|\mathcal{H}_{\bar{B}y}\| > 1$ then $\varepsilon_0 > \|F\|_\infty$; in the latter case we increase the value of ε and we continue the algorithm.

REFERENCES

Adamyan V.M.,Arov D.Z.,Krein M.G.

[AAK]   Infinite Hankel matices and generalized problems of Caratheodory,Fejér and I.Schur; *Functional Anal. Appl.* **2** ,1968, 269-281.

Ball J.A. , Cohen N.

[BC]   Sensitivity minimization in an H$^\infty$ norm: parametrization of all the suboptimal solution; *Int. J. Control* **46**, 1987, 785-816.

Ball J.A., Helton J.W.

[BH1]   A Beurling-Lax theorem for the Lie group U(m,n) which contains most classical interpolation theory. *Journal Operator Theory* **9** , 1983, 107-142.

[BH2]   Interpolation problems of Pick-Nevanlinna and Loewner types for meromorphic matrix functions: parametrization of the set of all solutions. *Integral Equations and Operator Theory* **9**, 1986, 155-203.

Chu C.,Doyle J.C.,Lee E.B.

[CCL]   The general distance problem in H$^\infty$ optimal control theory; *Int.J.Control,* **44**, no2, 1986, 565-596.

Francis B.A.

[F1]   Notes on optimal sensitivity theory:the single-input/single-output case ; Technical Report, Department of Electrical Engineering, University of Waterloo. 1981.

[F2]   "A course in H$^\infty$ control theory".Springer-Verlag, 1987, Berlin, FRG.

Fuhrmann P.A.

[Fu]   "Linear systems and operators in Hilbert space".Mc Graw-Hill, 1981, New York.

Garnett J.B.

[G]   "Bounded analytic functions".Academic Press, 1981.

Hoffman K.

[H]   "Banach space of analytic functions". Prentice-Hall, 1962, Englewood Cliffs, N.Y.

Jonckheere E., Verma M.

[JV] A spectral characterization of H^∞-optimal feedback performance. The multivariable case; Technical Report, Department of Electrical Engineering Systems, University of Southern California, 1986.

Nagy S.G., Foias C.

[NF] "Harmonic analysis of operators on Hilbert space". North Holland, Amsterdam, 1970.

Sarason D.

[S1] Generalized interpolation in H^∞;*Trans.AMS.* **127**, 1967,179-203.

[S2] Operator-theoretic aspects of the Nevanlinna-Pick interpolation problem; *Opierator and Function Theory* , 1985, 279-314.

Vidyasagar M.

[V] "Control systems synthesis: a factorization approach". MIT Press, 1985, Cambridge,MA.

Zames G.

[Z] Feedback and optimal sensitivity: model reference transformation, multiplicative seminorms, and approximate inverses; *IEEETrans.Aut.Control* **AC-26**(2), 1981, 301-320.

Zames G., Francis B.A.

[ZF] Feedback minimax sensitivity and optimal robustness; *IEEE Trans.Aut.Control*, **AC-28**(5), 1983, 585-600.

Zames G., Mitter S.K.

[ZM] A Note on Essential Spectra and Norms of Mixed Hankel-Toeplitz Operators. *Systems and Control Letters.*, **10**, n.3, 1988, 159-165.

20