Scattering Theory, Unitary Dilations and Gaussian Processes

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I. INTRODUCTION

The themes of modelling and representation of linear deterministic and stochastic systems have dominated much of systems theory in the last thirty years. In the critical period of this development attention was focussed on the reconciliation between the input-output (external) and state space (internal) points of view of systems. A central result in this development is the statement that a minimal (in the sense of dimension) state space realization of linear finite-dimensional system is unique (up to isomorphism) and corresponds to one which is both controllable and observable. In recent years Jan Willems has forcefully argued that this input-output-state space view is narrow and inadequate to deal with models of dynamical systems arising out of physics, econometrics as well as stochastic processes (notably Markov processes). For one thing, there is no natural identification of what is an input and what is an output in these systems. For another, the need to fix the initial state as an equilibrium state in

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conventional realization theory is unnatural and leads to conceptual difficulties. For a detailed exposition of this work see [23] .

The other important theme in systems theory is one of optimization and approximation. Models of systems obtained from physical principles or data are often of high dimension. When these models are used for prediction and feedback control it is necessary to obtain approximate models of much lower order so that the algorithms for prediction and/or control are computationally tractable. There is an important question here as to what is the appropriate representation on which the approximation (reduction) process should be carried out. Zames [24] has argued that this approximation should be done on an input-output basis and an internal representation of the approximate input-output map could then be obtained for the purpose of prediction and control. This argument rests on the notion that two systems may be near each other in an input-output sense (for example in L^∞-topology) and yet may drastically differ dimensionally in their internal representations. The two processes of approximation and internal representation do not, in general, commute and working with the input-output representation for the needs of approximation is a more stable operation.

There is an apparent contradiction here since we have just argued that the input-output view so prevalent in the early days of systems theory is not an appropriate one. Fortunately, Scattering Theory as developed by Adamjan and Arov [1] and Lax and Phillips [15] and the theory of Abstract Hankel Operators comes to the rescue here. The work of Adamjan, Arov and Krein (for example, [2]) and Ball and Helton [6] provide a mathematical framework for dealing with representation and approximation issues in a Hilbert space setting in a rigorous manner. In the systems theory context, Scattering Theory for Gaussian Processes was investigated in the doctoral thesis of Y. Avniel at M.I.T. [cf. 5] and Scattering Theory and Approximation of Linear Systems has been investigated by Willems himself in his own framework for dynamical Systems [22]. A state space view of Hankel approximation has been provided by Glover [13] in an important paper.

Intimately connected with the theory of Scattering is the theory of minimal unitary dilations of contraction semigroups contracting strongly to zero. Indeed, the theorem of Nagy asserts that every contraction semigroup contracting strongly to zero has a (unique) minimal unitary dilation. This theorem has a physical interpretation of coupling a dissipative system to a heat bath so that the resulting composite system is conservative [cf. 17]. The dual of this question, namely, how certain observables of an infinite-dimensional conservative system can exhibit dissipative behavior has been investigated by Picci by using the theory of stochastic realization [cf. 20].

This semi-expository paper consists of two parts. In the first part we describe in an essentially self-contained manner the Scattering Theory associated with stationary Gaussian processes. This is done in discrete time to avoid certain technical difficulties. The new contribution in this part of this paper is the result which states that for completely non-deterministic stationary Gaussian processes the spectral density can be recovered (up to unitary isomorphism) from the Hankel operator induced by the scattering function associated with the process. This result to some extent justifies using this Hankel operator for model reduction in stochastic systems. We then show the relationship of this scattering view to the theory of Unitary Dilations and Markovian representations. These latter ideas are all contained in the work of Lindquist and Picci [cf.18] and Foias and Frahzo [10]. The exposition serves to show that if we restrict ourselves to a ℓ^2-theory then representation questions for stochastic (and deterministic) systems are nothing else but a version of Scattering Theory à la Adamjan, Arov and Krein. It is worth mentioning that the Scattering Function plays an important role in the parametrization of the unit ball of the quotient space L^∞ / H^∞ and the corresponding extension problem for Hankel operators.

The second part of the paper is concerned with the theory of minimal unitary dilations of contraction semi-groups, its relation to positive definite functions on a group and the theory of open systems [8]. We also present a new construction of a unitary dilation of contraction semi-groups which makes evident the coupling to white noise (heat bath) which is implicit in the construction of the dilation.

2.1 NOTATION

Z stands for the set of integers, $\delta(n)$ for the indicator function of $\{0\} \subset Z$, C the complex numbers, and for $a \in C$, $\bar{a}$ denotes the complex conjugate of a. For a matrix $A = (a_{ij})_{i,j=1}^{p}$ we denote by A^* the Hermitian conjugate of $A : A^* = (b_{ij})_{i,j=1}^{p}$ $b_{ij} = + \bar{a}_{ji}$, and by A' its transposition. For a family of subsets $\{M_j\}_j$ of Hilbert space H, we denote by $\bigvee_j M_j$ the smallest closed linear manifold (subspace) that includes each M_j, and by $\bigwedge_j M_j$ the largest subspace contained in each of them (their intersection). $\overline{M}_j$ denotes the closure of M_j in H. For subspaces M, N, of H, $M \ominus N$ denotes the orthogonal complement of N in M. For a countable family $\{M_j\}$ of mutually orthogonal subspaces: $M_i \perp M_j$ $i \neq j$, we let $\sum_j \oplus M_j$ be their orthogonal sum. P_M stands for the orthogonal projection of H onto the subspace M. For a bounded linear operator $A : H_1 \rightarrow H_2$ of Hilbert space H_1 into H_2, we denote by [A] the matrix of A with respect to specified orthonormal bases in H_1, H_2.

$A^* : H_1 \rightarrow H_2$ denotes the adjoint of A. A|M stands for the restriction of A to the subspace $M \subset H_1$. $B(H_1, H_2)$ denotes the Banach space of all bounded linear operators from H_1 into H_2 with $B(H) = B(H,H)$.

By $\ell_2(-\infty,\infty; N)$ we denote the usual Hilbert space of sequences $\{hj\}_{j=-\infty}^{\infty}$ with values in (the Hilbert space) N for which $\sum_j \|h_j\|_N^2 < \infty$. $\ell_2(0,\infty; N)$, $\ell_2(-\infty, -1; N)$ are seen naturally as subspaces of $\ell_2(-\infty,\infty; N)$. L_2, L^∞ will denote respectively the Lebesgue spaces on the circle $T = \{e^{i\lambda} : \lambda \in [-\infty, \infty]\}$ (with respect to the normalized Lebesgue measure $\frac{d\lambda}{2\pi}$ of square integrable, essentially bounded complex valued functions). Each function can be viewed as defined on $[-\pi, \pi]$. Similarly for the spaces $L_2(C^p)$, $L_\infty(C^p)$ of functions f taking values in (C^p) for which $\|f(\pm)\|_{C^p} \in L_p$, $\|f(\pm)\|_{C^p} \in L_\infty$ respectively. $L_\infty(B(C^p))$ is defined analogously for weakly measurable, $B(C^p)$ valued functions f for which ess. sup $\{\| f(e^{i\lambda}) \| B(C^p) : \lambda \in [-\pi,\pi]\} < \infty\}$. $H_2^{\pm}$ are the subspaces of L_2 defined by

$$H_2^+ = \{f \in L_2 : \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\lambda})e^{-in\lambda} \quad d\lambda = 0, n = -1, -2, \dots\}$$

$$H_2^- = \{f \in L_2 : \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\lambda})e^{-in\lambda} \quad d\lambda = 0, n = 0, 1, 2, \dots\},$$

and we have the orthogonal decomposition $L_2 = H_2^+ \oplus H_2^-$. Each $f \in H_2^+$ having a Fourier series

$$f(e^{i\lambda}) \sim \sum_0^\infty a_n e^{in\lambda}$$

generates the function

$$g(z) = \sum_0^\infty a_n z^n$$

belonging to the Hardy class H_2 of functions g(z) holomorphic in $|z| < 1$ and such that

$$\|g\|_{H_2} = \sup_{0<r<1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\lambda})|^2 d\lambda \right]^{1/2} < \infty .$$

Moreover, the (a.e. existing) radial limit $g(e^{i\lambda})$ of g(z) equals $f(e^{i\lambda})$ a.e. and $\|f\|_{L_2} = \|g\|_{H_2}$. The function g(z) is seen as the analytic extension of $f \in H_2^+$ to the unit disc $|z| < 1$ and is denoted by f(z). We identify H_2^+ and H_2 and denote them commonly by H_2. Using the conjugation with respect to the unit circle ($z \to \frac{1}{\bar z}$) by the reflection principles for $f \in H_2 \subset L_2$ the function $\bar f$ defined by $\bar f(e^{i\lambda}) = \overline{f(e^{i\lambda})}$ has an analytic extension to $|z| > 1$: $f(\frac{1}{\bar z})$ which we again denote by $\bar f$. The space $\bar H_2 = f \in L_2 : \bar f \in H_2$ is the space of functions $f \in L_2$ having an analytic extension to the exterior of the disc $|z| < 1$ and we have

$$\|f\|_{L_2} = \sup_{\rho>1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\rho e^{i\lambda})|^2 d\lambda \right]^{1/2} .$$

$f \in \bar H_2$ are called conjugate analytic.

Analogously for the Banach space L_∞ we have the subspaces $H_\infty = H_\infty^+ \subset L_\infty$ of functions $f \in L_\infty$ having an analytic extension f(z) to $|z| < 1$ with

$$\|f\|_{L_\infty} = \sup_{|z|<1} |f(z)| = \|f\|_{H_\infty} .$$

Similarly, for the Hilbert space $L_2(C^p)$ we have the subspaces $H_2^+(C^p) = H_2(C^p$, $H_2^-(C^p)$ with the orthogonal decomposition $L_2(C^p) = H_2(C^p) \oplus H_2^-(C^p)$. In $L_\infty(B(C^p)$, again $H_\infty(B(C^p)$ is defined as the subspace of functions in $L_\infty(B(C^p)$ whose negatively indexed (matrix valued) Fourier coefficients vanish. For $\theta \in H_\infty(B(C^p)$

the function θ^* defined by $\theta^*(e^{i\lambda}) = [\theta(e^{i\lambda})]^*$ is identified with its analytic extension $\theta^*(\frac{1}{\bar z}) = [\theta(\frac{1}{\bar z})]^*$ to $|z| > 1$.

A function $f \in H$ is called *inner* if $|f(e^{i\lambda})| = 1$ a.e. . Similarly for $\theta \in H_\infty(B(C^p)$ if $\theta(e^{i\lambda})$ is unitary a.e. . $f \in H_2$ is called *outer* if $\bigvee_{n\geq 0} \{\chi^n f\} = H_2$ where χ denotes the function on T defined by $\chi(e^{i\lambda}) = e^{i\lambda}$. For $\varphi \in L_\infty(B(C^p)$ the Toeplitz operator $T_\varphi : H_2(C^p) \to H_2(C^p)$ whose matrix is block Toeplitz with respect to the standard basis $\{e^{ik\lambda}e_1, e^{ik\lambda}e_2, ..., e^{ik\lambda}e_p\}_{k \geq 0}$, $\{e_1, e_2, ..., e_p\}$ being the standard basis in C^p, is defined by $T_\phi f = \pi_+(\phi f)$ where π_+ is the Riesz projection of $L_2(C^p)$ onto $H_2(C^p)$. H_ϕ will denote the Hankel operator [with block Hankel matrix with respect to the standard bases in $H_2(C^p)$, $H_2^-(C^p)$], $H_\phi : H_2(C^p) \to H_2^-(C^p)$ defined by $T_\phi f = \pi_-(\phi f)$, π_- being the Riesz projection of $L_2(C^p)$ onto $H_2^-(C^p)$. The con-vention we employ regarding a Henkel operator as acting from $H_2(C^p)$ into $H_2^-(C^p)$ is not in accordance with the one employed in systems theory, where we act on $H_2^-(C^p)$ into $H_2(C^p)$: $H_\phi f = \pi_+(\phi f)$. It, however, conforms to that employed by Adamjan-Arov-Krein and enables us to use their results without modification, as well as to refer to them.

2.2 SCATTERING THEORY

Let H be a complex separable Hilbert space and let U be a unitary operator on H. A subspace D_+ is said to be *outgoing* for (U,H) if it satisfies

(i) $UD_+ \subset D_+$

$(2.2.1)_+$ (ii) $\bigwedge_{-\infty}^{\infty} U^n D_+ = \{0\}$

$$\text{(iii)} \quad \bigvee_{-\infty}^{\infty} U^n D_+ = H \quad .$$

A subspace D_- for which

$$(2.2.1)_- \quad \begin{aligned} &\text{(i)} \quad U^* D_- \subset D_- \\ &\text{(ii)} \quad \bigwedge_{-\infty}^{\infty} U^n D_- = \{0\} \\ &\text{(iii)} \quad \bigvee_{-\infty}^{\infty} U^n D_- = H \end{aligned}$$

is said to be *incoming* for (H,U).

2.2.1 DEFINITION. A quadruple (U,H,D_+,D_-) satisfying (2.2.1) is said to be a *scattering system*.

2.2.2 THEOREM (Translation Representation Theorem [19, Th. II.1.1]). *Let* (U,H,D_+) *be outgoing. Then there exists a Hilbert space* N_+ *and a unitary map* r_+ *of* H *onto* $\ell_2(-\infty,\infty;N_+)$ *such that*

$$(2.2.2) \quad \begin{aligned} &\text{(i)} \quad r_+ [D_+] = \ell_2(0,\infty; N_+) \quad , \\ &\text{(ii)} \quad U_+ = r_+ U r_+^{-1} \end{aligned}$$

is the right shift operator on $\ell_2(-\infty,\infty; N_+)$. This representation is unique up to automorphism of N_+.

Proof (Standard (cf. [15, p. 77]). We give the proof to establish various quantities introduced later. By $(2.2.1)_+$ - (ii) the operator $U|D_+$ is an isometry having no unitary. By Wold's decomposition theorem [19, Th. I.1.1] we may write uniquely

$$(2.2.3) \qquad D_+ = \sum_{n=0}^{\infty} \oplus\, U^n N_+ \quad , \qquad N_+ = D_+ \ominus UD_+ \quad .$$

Since for any $m > 0$

$$\begin{aligned} U^{-m} D_+ &= U^{-m}[D_+ \ominus U^m D_+) \oplus U^m D_+] = \\ &U^{-m} [(\sum_{k=0}^{m-1} \oplus\, U^k N_+) \oplus U^m D_+] = \\ &(\sum_{k=-1}^{-m} \oplus\, U^k N_+) \oplus D_+) \quad , \end{aligned}$$

we obtain by $(2.2.1 - \text{iii})_+$ that

$$(2.2.4) \qquad H = \sum_{-\infty}^{\infty} \oplus\, U^n N_+ \quad .$$

It follows that for arbitrary $h \in H$

$$h = \sum_{-\infty}^{\infty} \oplus\, U^n P_{N_+} U^{-n} h, \quad \| h \|_H^2 = \sum_{-\infty}^{\infty} \| P_{N_+} U^{-n} h \|_H^2 .$$

Hence the map

$$r_+ : H \to \ell_2(-\infty,\infty; N_+)$$

defined by

$$(2.2.5) \qquad r_+ h = \{P_{N_+} U^{-n} h\}_{n=-\infty}^{\infty}$$

is isometric. Since for $\{h_n\}_{-\infty}^{\infty} \in \ell_2(-\infty,\infty; N_+)$, $h = \sum_{-\infty}^{\infty} U^n h_n \in H$, the map r_+ is onto and thus unitary. By (2.2.3) we obtain (i). From (2.2.5)

$$r_+Uh = P_{N_+} \{U^{-(n-1)} h\}_{n=-\infty}^{\infty} = U_+(r_+h) \quad ,$$

and (ii) follows. By (2.2.4) U is a bilateral shift of multiplicity equal to dim N_+ and the uniqueness follows.

2.2.3 DEFINITION. The representation $(U_+, \ell_2(0,\infty; N_+), \ell_2(-\infty,\infty; N_+))$ is called an *outgoing transition representation.*

For (U,H,D_-) incoming we similarly obtain

(2.2.6) $$D_- = \sum_{n=-\infty}^{0} \oplus U^n N_- \quad , \quad N_- = D_- \ominus U^* D_-$$

and

(2.2.7) $$H = \sum_{-\infty}^{\infty} \oplus \; U^n N_- \quad .$$

For the corresponding map r_- of H onto $\ell_2(-\infty,\infty; N_-)$ we define

(2.2.8) $$r_- h = \{P_{N_-} U^{-(n+1)} h\}_{n=-\infty}^{\infty} \quad .$$

Thus

(i) $r_-[D_-] = \ell_2(-\infty, -1; N_-)$,

(ii) $U_- = r_- U r_-^{-1}$

is the right shift on $\ell_2(-\infty,\infty; N_-)$. The representation $(U_-, \ell_2(-\infty, -1; N_-), \ell_2(-\infty,\infty; N_-))$ is called an *incoming translation representation.*

2.2.4 DEFINITION ([1], [15]). The operator

$$S = r_- \, r_+^{-1} : \ell_2(-\infty,\infty; (C^p)) \to \ell_2(-\infty,\infty; (C^p))$$

is called the *abstract scattering operator.*

Clearly S is unitary. Denoting by V the right shift on $\ell_2(-\infty,\infty;(C^p))$, we readily obtain by the translation representation theorem

(2.2.9) $$SV = r_- r_+^{-1} V = r_- U r_+^{-1} = VS \; .$$

Let $F : \ell_2(-\infty,\infty; (C^p)) \to L_2(C^p)$ be the Fourier transform operator. The unitary operator

$$FSF^{-1} : L^2(C^p) \to L^2(C^p)$$

thus commutes by (2.2.9) with L_χ, the operator of multiplication by χ. It follows [19] that FSF^{-1} is a Laurent operator $L_S \in L_\infty(B(C^p))$, such that

$S(e^{i\lambda})$ a.e. is a unitary map on C^p.

2.2.5 DEFINITION ([1], [15]). S is called the *scattering matrix.*

It is clear from the translation representation theorem that S is determined to within right and left multiplication by unitary transformations on C^p.

The unitary maps $F_- = Fr_-$, $F_+ = Fr_+$ are called the incoming and outgoing spectral representation. We have the following:

a) $F_-(D_-) = H^2(C^p)$.

(2.2.9) b) $F_+(D_+) = SH^2(C^p)$.

c) $F_-(Uh) = \chi Fh$, $h \in H$.

Moreover, the operator

(2.2.10) $P_-P_+: D_+ \to D_-$ is unitarily equivalent to the Hankel operator H_S, where $P_\pm = P_{D_\pm}$, and the operator

(2.2.11) $PD_-^{\perp} P_+: D_+ \to D_-^{\perp}$ is equivalent to the Toeplitz operator T_S.

2.3 COMPUTATION OF THE SCATTERING FUNCTION FOR REGULAR, MAXIMAL RANK, STATIONARY GAUSSIAN SEQUENCES

Let (Ω, A, P) be a fixed probability space and let

$$\{\underline{y}(n) : n \in Z\} \quad , \underline{y}(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ \cdot \\ \cdot \\ \cdot \\ y_p(n) \end{pmatrix}$$

be a centered stationary process with $y_j(n) \in L_2(\Omega, \mathcal{A}, P)$ j=1, ..., p. Let $f_{\underline{y}\underline{y}}(\lambda) = (f_{kj}(\lambda))_{k,j=1}^{p}$, $\lambda \in [-\pi, \pi]$ be its spectral density satisfying

$$(2.3.1) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det f_{\underline{y}\underline{y}}(\lambda)\, d\lambda > -\infty$$

i.e., the process is *regular* and of *maximal rank*. Let

$$H = H\underline{y} = \bigvee_{n \in Z} \{y_1(n), y_2(n), ..., y_p(n)\} \subset L_2(\Omega, \mathcal{A}, P)$$

be the space spanned by the process and let U be the unitary shift operator on H associated with the $\underline{y}$ process [21, p. 14]:

$$U y_j(n) = y_j(n+1) \qquad j = 1, ..., p, \qquad n \in Z \quad .$$

We consider the *past* and *future* of $\{\underline{y}(n)\}_{-\infty}^{\infty}$ defined by

$$D_- = H_{\underline{y}}^-(0) = \bigvee_{k \le 0} \{y_1(k), ..., y_p(k)\} \quad ,$$

$$D_+ = H_{\underline{y}}^+(0) = \bigvee_{k \ge 0} \{y_1(k), ..., y_p(k)\} \quad .$$

By (2.3.1) it follows [21, Th. II.6.1]

$$\bigwedge_{-\infty}^{\infty} U^n D_- = \{0\} = \bigwedge_{-\infty}^{\infty} U^n D_+ .$$

We readily obtain that (U, H, D_+, D_-) is a scattering system.

Now let (U, H, D_+, D_-) be the scattering system associated with the regular maximal rank $\underline{y}$ process. The subspace $N_- = D_- \ominus U^* D_-$ ($N_- = D_- \ominus U^* D_-$) is the *forward (backward)* innovation subspace at n = 0. Since for a scattering system $(U, H, D_+ D_-)$ we have

$$\dim N_- = \text{multiplicity } U = \dim N_+ \quad ,$$

we can arrange the maps r_+ to be onto $\ell^2(-\infty, \infty; (C^p))$.

We next compute the scattering matrix S for the $\underline{y}$ process. Let $\{v_1(0), ..., v_p(0)\}$ be an orthonormal basis for N_-. Let $v_j(n) = U^n v_j(0)$ and define

$$\underline{v}(n) = \begin{pmatrix} v_1(n) \\ \cdot \\ \cdot \\ \cdot \\ v_p(n) \end{pmatrix} \quad , \qquad n \in Z .$$

By (2.2.4) the process $\{v(n)\}_{-\infty}^{\infty}$ is a (centered) white noise process with covariance $R_{vv}(n) = \delta(n) I_{C^p}$ constituting the forward innovation process for the $\underline{y}$ process. It is determined up to a choice of basis in N_-. By (2.2.6) we may write

$$\underline{y}(0) = \sum_{-\infty}^{\infty} A(k)\underline{v}(k) \qquad A(k) = (a_{ij}(k))_{i,j=1}^{p} \qquad A(k) = [0],\ k > 0 .$$

(Wold's representation). It follows from (2.2.8)

$$(2.3.2) \qquad r_{-}y_j(0) = \left\{ \sum_{m=1}^{p} \alpha_{jm}(k+1)v_m(0) \right\}_{k=-\infty}^{\infty}$$

Identifying N_- with (C^p) we readily obtain the representation

$$r_{-}y_j(0) = \begin{pmatrix} \alpha_{j1}(k+1) \\ \cdots\cdots \\ \cdots\cdots \\ \alpha_{jp}(k+1) \end{pmatrix}_{k=-\infty}^{\infty} .$$

Consider the function

$$\Lambda(z) = \sum_{-\infty}^{\infty} A'(k)z^k .$$

Since

$$\sum_{k=-\infty}^{\infty} \sum_{i,j=1}^{p} |\alpha_{ij}(k)|^2 \le \sum_{j=1}^{p} \|y_j(0)\|_H^2 ,$$

$\Lambda(z)$ is analytic in $|z| > 1$. For $\Lambda(z)$ we have,

$$\frac{1}{2\pi} \Lambda^*(z)\Lambda(z) = f_{\underline{y}\underline{y}}(\lambda)$$

By the incoming properties

$$H_2^-(C^p) = \bigvee_{n<0} \{ e^{in\lambda}\Lambda(e^{i\lambda})\underline{a} : \underline{a} \in (C^p) \}$$

i.e., Λ is *coniugate outer* [14, p. 121). Thus

$$(2.3.3) \qquad (Fr_{-} y_1(0), \ldots, Fr_{-} y_p(0)) = \bar{\chi}\Lambda .$$

Since the translates (in $H_{\underline{y}}$) of $y_1(0),\ldots,y_p(0)$ and their linear combinations are dense in $H_{\underline{y}}$. Fr_- is determined by the above expression.

We now consider the outgoing representation. Let $\varepsilon_1(0),\ldots,\varepsilon_p(0)$ be an orthonormal basis in N^+. We similarly obtain

$$\underline{y}(0) = \sum_{-\infty}^{\infty} B(k)\underline{\varepsilon}(k) \qquad B(k) = (\beta_{ij}(k))_{i,j=1}^{p}, \qquad B(k) = [0],\ k < 0 .$$

This representation constitutes the representation of $y(0)$ in terms of the backward innovation process $\{\underline{\varepsilon}(n)\}_{-\infty}^{\infty}$, $\underline{\varepsilon}(n) = \begin{pmatrix} \varepsilon_1(n) \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_p(n) \end{pmatrix}$. We define

$$\Gamma(z) = \sum_{0}^{\infty} B'(k)z^k$$

which is analytic in $|z| < 1$. In a similar fashion we obtain by direct computation

$$\frac{1}{2\pi} \Gamma^*(z)\Gamma(z) = f_{\underline{y}\underline{y}}(\lambda), \qquad z = e^{i\lambda}$$

with Γ being outer. Also

$$(2.3.4) \qquad (Fr_{+}y_1(0), \ldots Fr_{+}y_p(0)) = \Gamma .$$

Combining (2.3.3) with (2.3.4), we obtain

$$S\Gamma = \bar{\chi}\Lambda$$

and thus

$$S = \bar{\chi}\Lambda\Gamma^{-1} .$$

One easily verifies that $S(e^{i\lambda})$ is unitary a.e. $\lambda \in [-\pi,\pi]$. We thus obtain

2.3.1 THEOREM. *For a regular maximal rank process* $\{\underline{y}(n)\}$ *we have*

$$S = \bar{\chi}\Lambda\Gamma^{-1}$$

where S *is determined up to left and right multiplication by constant unitary matrices.*

For the case $p = 1$ we have

2.3.2 COROLLARY. *For a regular process* $\{\underline{y}(n)\}_{-\infty}^{\infty}$

$$S = \bar{\chi}\frac{\bar{\Gamma}}{\Gamma} \quad ,$$

and S is determined up to multiplication by a constant of unit modulus.

Proof. The outer function $\bar{\Lambda}$ satisfies $|\bar{\Lambda}| = |\Gamma|$ on T and thus $\bar{\Lambda} = \gamma\Gamma$ a.e. where γ is a constant of unit modulus.

2.3.3 REMARK. The scattering matrix S was defined by outer and conjugate outer factors of the density $f_{\underline{y}\underline{y}}$. Since those are determined up to left multiplication by a constant unitary matrix, we may wish to make a canonical choice (which amounts to choosing specific orthonormal bases in N_+, N_-) in the following fashion: For $\Gamma(0)$ we consider its polar decomposition $\Gamma(0) = KP$ (K unitary, $P > 0$) and define $\Gamma_1(z) = K^{-1}\Gamma(z)$. For Γ_1 we have $\Gamma_1(0) > 0$. This Γ_1 is unique. Similarly for Λ. In this way, the density $f_{\underline{y}\underline{y}}$ will have a unique S associated with it. From the viewpoint of seeing S as the phase function associated with f_{yy}, this may be appealing.

2.4 COMPLETELY NON-DETERMINISTIC STATIONARY SEQUENCES, THEIR SCATTERING FUNCTIONS, AND INDUCED HANKEL OPERATORS

The scattering function of stationary sequences plays the role of the Heisenberg S-matrix in quantum mechanics. The physics of quantum systems is believed to be contained in the S-matrix and this object can in principle be determined experimentally. A natural question then is whether the scattering function of stationary Gaussian sequences, which measures the interaction between the past and future of the process, determines the spectral density of the process. To answer this question we introduce the class of completely non-deterministic processes.

2.4.1 DEFINITION [7]. The process y is said to be completely non-deterministic if

$$H^-_{\underline{y}}(0) \cap H^+_{\underline{y}}(1) = \{0\}. \tag{2.4.1}$$

This condition states that no value in $H^+_{\underline{y}}(1)$ can be predicted without error based on the information $H^-_{\underline{y}}(0)$. This condition is more restrictive than regularity (cf.

BLOOMFIELD-JEWELL-HAYASHI, loc.cit., for an example of a regular process which is completely non-deterministic).

2.4.2 THEOREM. *The scattering matrix S determines the spectral density $f_{\underline{y}\underline{y}}$ up to the form $K^*f_{\underline{y}\underline{y}}K$ where K is a constant $p \times p$ non-singular matrix iff $\underline{y}$ is completely non-deterministic.*

2.4.3 REMARK. For p=1, this result was obtained by Levinson and McKean [16].

2.4.4 LEMMA. *The scattering matrix S determines the density $f_{\underline{y}\underline{y}}(\lambda)$ up to the form*

$$K^* f_{\underline{yy}}(\lambda) K \tag{2.4.2}$$

where K is a constant pxp non-singular matrix, iff

$$\dim \operatorname{Ker} T_S = p . \tag{2.4.3}$$

Proof. First note that for any representation of S

$$S = \bar{\chi}YX^{-1}$$

with the columns of X in $H_2(C^p)$ and those of $\bar{\chi}Y$ in $H_2^-(C^p)$, the columns of X belong to Ker T_S. Moreover (on T) F_+

$$Y^*Y = (SX)^*SX = X^*X \quad .$$

Assume (2.4.3) holds. It thus follows that

$$X(e^{i\lambda}) = \Gamma(e^{i\lambda})K \tag{2.4.4}$$

where K is a pxp full rank constant matrix. Thus,

$$\frac{1}{2\pi}X^*(z)\,X(z) = \frac{1}{2\pi}K^*\Gamma^*(z)\Gamma(z)K = K^* f_{\underline{yy}}(\lambda)\,K \qquad z = e^{i\lambda}$$

proving the 'if' part.

Now assume (2.4.3) not to hold, i.e., dim Ker $T_S > p$. We can thus find a pxp matrix $X(e^{i\lambda})$ of full rank a.e. λ such that the columns of X belong to Ker T_S and (2.4.4) does not hold. If we define

$$Y = \bar{\chi} S X$$

then the columns of $\bar{\chi}Y$ are in $H_2^-(C^p)$ and $S = \bar{\chi}YX^{-1}$ with $Y^*Y = X^*X$. The result follows.

We next characterise condition (2.4.3) on a process level.

2.4.5 LEMMA. *We have*

$$F_+^*[\text{Ker } T_S] = H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0) \quad .$$

Proof. Let $0 \neq f \in$ Ker T_S . From the well-known identity $H_S^*H_S + T_S^*T_S = 1$, it follows that

$$H_S^*H_S f = f$$

i.e.,

$$\|H_S f\| = \|f\| .$$

From (2.2.10) and (2.2.11) we obtain for $\xi = F_+^* f \in H_y^+(0)$

$$\|P_- \xi\| = \|\xi\| ,$$

and $\xi \in H_{\underline{y}}^-(0)$. Thus

$$F_+^*[\text{Ker } T_S] \subset H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0) .$$

Now let $\xi \in H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0)$. It follows from (2.2.10) and (2.2.11)

$$H_S(F_+ \xi) = F_- \xi \ .$$

Let $f = F_+ \xi \in H_2(C^p)$. We obtain

$$\|H_S f\| = \|F_- \xi\| = \|\xi\| = \|F_+ \xi\| = \|f\|$$

and

$$H_S^* H_S f = f \ .$$

Thus $f \in$ Ker T_S which implies

$$F_+[H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0)] \subset \text{Ker } T_S .$$

By the unitarity of F_+

$$\dim \text{Ker } T_S = \dim H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0) ,$$

and since $\underline{y}$ is regular and of full rank we readily conclude

$$\dim H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(0) = p \quad \text{iff} \quad \dim H_{\underline{y}}^-(0) \wedge H_{\underline{y}}^+(1) = 0 .$$

Proof of Theorem 2.4.2. Combine Lemmas 2.4.4. and 2.4.5.

The converse question, namely, when is a function $S \in L_\infty(B(C^p))$ the scattering matrix of some full rank, p-dimensional completely non-deterministic process is of interest.

We first observe that any $S \in L_\infty(B(C^p))$ which is unitary valued a.e. on T is the scattering matrix of the canonical scattering system [1]

$$U = L\chi \ , \ H = L_2(C^p) \ , \ D_+ = SH_2(C^p) \ , \ D_- = H_2^-(C^p) \ .$$

The above question amounts to characterizing all scattering systems (U, H, D_+, D_-) for which there exists a set $\{\varepsilon_1, ..., \varepsilon_p\}$ of linearly independent vectors such that

$$H = \overline{\text{span}\{U^n \xi_j : j=1, \dots, p, n=0, \pm 1, \dots\}}$$

$$D_+ = \overline{\text{span}\ \{U^n \xi_j : j=1, \dots, p,\ n \geq 0\}}$$

$$D_- = \overline{\text{span}\ \{U^n \xi_j : j=1, \dots, p,\ n \leq 0\}}$$

and such that any other linearity independent set satisfying the above is of cardinality p. The corresponding process will be $\{\underline{\xi}(n)\}_{-\infty}^{\infty}$ where $\underline{\xi}(0) = \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_p \end{pmatrix}$, $\underline{\xi}(n) = \begin{pmatrix} U^n\xi_1 \\ \cdot \\ \cdot \\ \cdot \\ U^n\xi_p \end{pmatrix}$, and the spectral density is obtained by

$f_{\underline{\xi}\underline{\xi}}(\lambda) = \left(\frac{d(E_\lambda \xi_i, \xi_j)_H}{d\lambda}\right)_{i,j=1,\dots,p}$ $\{E_\lambda : \lambda \in [-\pi, \pi]\}$ being the resolution of the identity for U. The answer is given in the following.

2.4.6 THEOREM. *Let* $S \in L_\infty(B(C^p))$ *be such that*

(i) $S(e^{i\lambda})$ *is* a.e. λ *a unitary map on* (C^p) ,

(ii) $\dim \operatorname{Ker} T_S = p$.

Then there exists a p-dimensional full rank completely non-deterministic process $\underline{y}$ whose scattering matrix is S .

Proof. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ span the kernel of T_S and define

$$\Gamma = [\,\Gamma_1 \mid \Gamma_2 \mid \dots \mid \Gamma_p\,] .$$

Let

$$\Lambda = S\Gamma .$$

Since $\Lambda_j = (S\Gamma_j) + \pi_+(S\Gamma_j) = \pi_-(S\Gamma_j)$, $j=1,\dots,p$, the columns of $\Lambda = [\Lambda_1 \mid \Lambda_2 \mid \dots \mid \Lambda_p]$ are in $H_2^-(C^p)$ and by (i)

$$\Lambda^*(z)\Lambda(z) = \Gamma^*(z)\Gamma(z) \qquad z = e^{i\lambda} .$$

If we define

$$f_{\underline{y}\underline{y}}(\lambda) = \frac{1}{2\pi}\,\Gamma^*(e^{i\lambda})\,\Gamma(e^{i\lambda})$$

the theorem follows provided we show that Γ is outer and $\chi\Lambda$ conjugate outer. Let $U = L_\chi$ and define :

$$\hat{D}_- = \bigvee_{n\leq -1}\{\chi^n\Lambda_1, \dots, \Lambda_p\} \subset H_2^-((C^p)) ,\quad \hat{D}_+ = \bigvee_{n\geq 0}\{\chi^n S\Gamma_1, \dots, \chi^n S\Gamma_p\} \subset SH_2(C^p) .$$

Let

(2.4.5) $$\hat{H} = (\bigvee_{n\in Z} U^n \hat{D}_-) \vee (\bigvee_{n\in Z} U^n \hat{D}_+) .$$

It is easily verified that $(2.2.1)_\pm$ - (i), (ii) holds for $(U, D_\pm)$. In [1] Adamjan-Arov show [1, Th. 2.5] that a quadruple (U, H, D_+, D_-) satisfying $(2.2.1)_\pm$ - (i), (ii) and (2.4.5) has a scattering matrix S which is unitary valued a.e. on T iff

$$\bigvee_{n\in Z} U^n D_+ = H = \bigvee_{n\in Z} U^n D_-$$

and, moreover, from their generalized functional model [1, Th. 2.1] we need have

$$D_- = H_2^-(C^p),\quad D_+ = \hat{S}\, H_2(C^p) .$$

A straightforward computation gives

$$\hat{S} = S$$

and the result follows.

From the theorem of Nehari [3],and its vector generalization we know that for a bounded Hankel operator H with symbol $\varphi \in L_\infty$, there exists a function $\varphi_\mu \in L^\infty$ such that

$$\|H_\varphi\| = \|\varphi_\mu\|_\infty.$$

The function φ_μ is called a minifunction for H_φ. In general φ_μ is not unique. We however have

2.4.7 THEOREM. *The Hankel operator* H_S *determines* S *uniquely. Indeed,* S *is its unique minifunction.*

Proof. From Lemma 2.4.4 we note that $f \in \text{Ker } T_S$ is an eigenvector of $H_S^* H_S$ corresponding to the eigenvalue $\|H_S\| = 1$. Since $S = \chi\Lambda\Gamma^{-1}$ every column of Γ belongs to this kernel. Thus, the projection of the above eigenspace on the first coordinate in $l_2(0,\infty;(C^p))$ spans $\Gamma(0)$. Now observe that for $\Gamma(0)$ we have, because of its outer property in $H2(B(C^p))$,

$$\log \frac{|\det \Gamma(0)|}{(2\pi)^{p/2}} = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log \det f_{\underline{yy}}(\lambda)\, d\lambda > -\infty ,$$

so that $\Gamma(0)$ is of full rank. We conclude that the aforementioned projection is *onto* the firtst coordinate space. According to a result of Adamjan-Arov-Krein [2, Corollary 3.1] for Hankel operator H_ϕ to have a unique minifunction, it is sufficient that the projection of the eigenspace of $H_\phi^* H_\phi$ corresponding to $\|H_\phi\|$ on the first coordinate space be onto. The result follows.

2.4.8 REMARK. It is of interest to observe that since for a completely non-deterministic process, the eigenvectors of $H_S^* H_S$ corresponding to the eigenvalue $\|H_S\|$ are only the columns of Γ, the projection of this eigenspace on the first coordinate is not only onto, but also 1-1. In [2, Sec. 2] it is shown that for any Hankel operator $H : H_2(C^p) \to H_2^-(C^p)$ satisfying this condition its unique minifunction is of the form ρS, $\rho = \|H\|$.

Thus, up to a constant multiple $\rho > 0$ all minifunctions of such Hankel operators are in 1-1 correspondence with regular, full rank, completely nondeterministic processes.

The case of Rational Functions. Let $\{y(n)\}_{-\infty}^{\infty}$ have rational density

$$f_{\underline{yy}}(\lambda) = \frac{|P(z)|^2}{|Q(z)|^2} \qquad z = e^{i\lambda},$$

where the polynomials P,Q have no zeros in $|z| < 1$ and are relatively prime. Since $f_{\underline{yy}} \in L_1$, the polynomial Q has its zeros in $|z| > 1$. Write

$$P = P_1 P_2$$

where P_1 of degree k has its zeros on T and P_2 in $|z| > 1$. For $P_1(z) = \prod_{j=1}^{k}(z-\alpha_j)$, $\{\alpha_j\}_{j=1}^{k} \subset T$ we have

$$\frac{\overline{P_1(e^{i\lambda})}}{P_1(e^{i\lambda})} = e^{-ik\lambda}(-1)^k \prod_{j=1}^{k} \bar{\alpha}_j .$$

Thus

$$(2.4.6) \qquad S = \gamma\bar{\chi}^{k+1}\frac{\bar{\psi}_e}{\psi_e} \qquad \psi_e = \frac{P_2}{Q}, \qquad \gamma = (-1)^k \prod_{j=1}^{k}\bar{\alpha}_j \quad ,$$

where $|\gamma| = 1$ and ψ_e is outer. In [3] Adamjan-Arov-Krein show that (2.4.6) is the general form of unimodular minifunctions and that in this case k+1 is the dimension of the eigenspace corresponding to the singular value $1 = \|H_S\|$. *We conclude that a regular process with rational spectral density is completely nondeterministic iff it has no zeros on T.*

2.5 MARKOV PROCESSES AND UNITARY DILATIONS

In this section we show how a Markov structure is intrinsically associated with unitary dilations and the resulting Scattering theory of Lax and Phillips. The results in this section are due to Lindquist and Picci [18] and Foias and Frahzo

[10]. In some sense however the results of this section are essentially contained in Adamjan-Arov [1]. This is demonstrated in this section.

In a Hilbert space setting, a centered stationary process $\{\underline{x}(n)\}_{-\infty}^{\infty}$ is said to be Markov if for all $n \geq s$

$$P_{H_2^-(s)} P_{X(s)} h \qquad h \in H_2^+(n) \quad ,$$

where $X(s) = \text{span}\ \{x_j(s): j=1,..., m\}$. In our setting, all stationary processes will be generated by the shift U (on $H_{\underline{y}}$) associated with the $\underline{y}$ process. Thus, for a stationary process $\{\underline{x}(n)\}_{-\infty}^{\infty}$ (in $H_{\underline{y}}$) we will have $\underline{x}(n) = U^n \underline{x}(0)$. It readily follows from above that one can define the notion of a Markov subspace $X \subset H_{\underline{y}}$ (for U) if X satisfies (see [18]) $\underline{y}$

$$\text{(2.5.1)} \qquad P_{\overset{s}{\underset{-\infty}{\vee}} U^m x} U^n x = P_{U^s x} U^n x \quad , \qquad n \geq s,\ x \in X \quad .$$

Thus X is a Markov subspace (for U) iff the process $\{U^n X\}$ has the (weak) Markov property. In what follows a Markov process $\{U^n X\}$ will invariably arise in this fashion.

Markov subspaces $X \subset H_{\underline{y}}$ which are *representations* for the process $\underline{y}$, i.e., for which

$$\{y_1(0),..., y_p(0)\} \subset X \ ,$$

satisfy

$$H_{\underline{y}} = \overset{\infty}{\underset{-\infty}{\vee}} U^n X \quad ,$$

nd are said to be of full range. There is a direct relationship between Markov rocesses of full range and unitary dilations (see also [10]). Recall [19] that a nitary operator U on a Hilbert space H is said to be the minimal unitary (power) ilation of a contraction A on $X \subset H$ if

$$A^n = P_X U^n | X \qquad n \geq 0 \text{ and } H = \overset{\infty}{\underset{-\infty}{\vee}} U^n X \ \text{(minimality)}.$$

2.5.1 PROPOSITION. $X \subset H_{\underline{y}}$ *is a Markov subspace of full range iff* U *(on* $H_{\underline{y}}$*) is the minimal unitary (power) dilation of the state operator*

$$A = P_X U | X : X \to X$$

Proof. From (2.5.1) we obtain for $x,x' \in X$ and $m,n \geq 0$

$$(U^{-m}x, U^n x') = (U^{-m}x, P_X U^n x).$$

Denoting $A(n) = P_X U^n | X$, we obtain

$$(x, A(m+n)x') = (x, U^{m+n}x') = (U^{-m}x, U^n x') = (U^{-m}x, P_X U^n x')$$

$$= (x, P_X U^{-m} P_X U^n x') = (x, A(m)A(n)x') \ .$$

We infer that $A(m+n) = A(m)A(n)$ and

$$A(n) = A^n(1) = A^n \ .$$

Since X is of full range, we conclude that U in $H_{\underline{y}}$ is the minimal unitary dilation of A (in X). This proves the 'only if' part. The 'if' part follows by reversing the argument.

Having made the connection between a Markov process $\{U^n K\}$ and the dilation property characterizing it, the work of Adamjan-Arov [1] on the duality between dilation theory and the scattering operator model is directly applied. First, note that the process $\{U^n X\}$ is regular, i.e., satisfies

$$\underset{n\geq 0}{\wedge} \underset{k\leq n}{\vee} U^k X = \{0\} = \underset{n\geq 0}{\wedge} \underset{k\geq n}{\vee} U^k X$$

iff

$$A^n \to 0 \quad , \qquad A^{*n} \to 0 \quad (n \to \infty) \ .$$

Second, those Markov processes which in addition to being regular represent $\underline{y}$ (and are thus of full range) correspond to scattering systems according to a result of Adamjan-Arov [1, Th. 3.4] :

2.5.2 THEOREM. *Let* $X \subset H_{\underline{y}}$ *be a regular Markov subspace of full range. Then* $H_{\underline{y}}$ *decomposes and, moreover, uniquely into the orthogonal sum*

$$H_{\underline{y}} = D_- \oplus X \oplus D_+,$$

where $(U, H_{\underline{y}}, D_+, D_-)$ *is a scattering system.*

2.5.3 DEFINITION. A scattering system $(U, H, D_+, D_-)_X$ for which

$$D_- \subset D_+$$

is called a Lax-Phillips (L-P) scattering system.

Let $\{U^n X\}$ be an arbitrary regular Markov process of full range, and $(U, H_{\underline{y}}, D_+, D_-)_X$ its associated L-P. scattering system. Let $\Theta_X(e^{i\lambda})$ be the corresponding scattering matrix. For the induced incoming spectral representation F^-_X we obtain

$$F^-_X [D_-] = H^-_2(C^p) \quad ; \quad F^-_X [D_-] = \Theta_X H^2(C^p) .$$

Since $D_+ \perp D_-$

$$\Theta_X \in H_\infty(\ C^p)) .$$

To each regular full range Markov [illegible]cess there is thus associated an inner function Θ_X, which is the scattering mat[illegible] of the corresponding L-P system $(U, H_{\underline{y}}, D_+, D_-)_X$. From [1, Th. 3.3] it follows that the scattering matrices Θ associated with *regular full range Markov processes* are precisely the inner functions $\Theta \in H_\infty B(C^p))$ which are purely contractive [19, p. 188], i.e.; for which

$$\|\Theta(0)\| < 1 \quad .$$

2.6 FACTORIZATION OF THE SCATTERING MATRIX AND MODELLING

The 1-1 correspondence

$$X \leftrightarrow (U, H_{\underline{y}}, D_+, D_-)_X$$

enables us to translate the *realization problem of finding all regular Markovian representaions* for y to a covering problem in $L_2(C^p)$ via the outgoing spectral representation for $(U, H_{\underline{y}}, D_+, D_-)$. Let $X \subset H_{\underline{y}}$ be a regular Markov subspace reprsenting y, $(U, H_{\underline{y}}, D_+, D_-)_X$ its L-P scattering system, ΘX its scattering matrix, and F^+_X the corresponding outgoing spectral representation. Since $\{y_1(0),\dots,y_p)0)\} \subset X$ it follows

$$\text{(2.6.1)} \qquad F^+_X [H^-_y(0)] \in F^+_X \left[\bigvee_{k\le n} U^n X \right] = F^+_X [D_- \oplus X] = H^-_2(C^p) \ ,$$

and $F^+_X [H^-_y(0)]$ is a full range left shift invariant subspace of $H^-_2(C^p)$. Let V be the corresponding inner function obtained from the Beurling-Lax theorem, i.e.,

$$F^+_X [H^-_y(0)] = V^* H^-_2(C^p) .$$

Be the unitary of F^+_X

$$F^+_X(y(0)) = \bar{\chi} V^* \Lambda .$$

Thus

$$\{y_1(0), \dots, y_p(0)\} \subset X$$

translate under F^{+}_{X} to the equivalent condition

$$\bar{\chi}V^{*}\Lambda \in H_2^{-}(C^p) \ominus \Theta^{*}H_2^{-}(C^p) \;. \qquad (2.6.2)$$

Conversely, if Θ, V are inner functions for which (2.6.2) holds, the mapping

$$y(0) \to \bar{\chi}V^{*}\Lambda$$

induces in a natural fashion a spectral representation $F_{\Theta,V}$ for which

$$X = F^{-1}_{\Theta,V}\,[H_2^{-}(C^p) \ominus \Theta^{*}H_2^{-}(C^p)\,]$$

is a Markovian representation for y, and its corresponding L-P scattering system has its scattering matrix Θ_X coinciding with Θ. We have therefore proved the following

2.6.1 THEOREM. *Finding all models (realizations) of y is equivalent to finding all inner functions* Θ_1 *such that*

$$\bar{\chi}V^{*}\Lambda \in H_2^{-}(C^p) \ominus \Theta_1^{*}H_2^{-}(C^p)$$

for some inner function V. *Each model corresponds to a pair* (Θ_1, V).

2.6.2 COROLLARY. *All regular Markovian representations of* y *are paramterized by precisely those inner functions* Θ_1 *for which*

$$V^{*}S = \Theta_1^{*}\Theta_2 \qquad \Theta_2 \quad H_\infty(B(C^p)) \qquad (2.6.3)$$

for some inner function V.

Proof. (2.6.2) holds iff $\Theta_1\bar{\chi}\Lambda \in H_2(C^p)$ iff $\Theta_1 V^{*}S\Gamma \in H_2(C^p)$. Since Γ is outer the latter holds iff $\Theta V^{*}S \in H_\infty(B(C^p))$, i.e., iff (6.3) holds.

2.6.3 COROLLARY. *All regular Markov subspaces* $X \subset H_y^{-}(0)$ *representing* y *are parametrized by those and only those inner functions* Θ_1 *for which*

$$S = \Theta_1^{*}\Theta_2 \qquad \Theta_2 \in H_\infty(B(C^p)). \qquad (2.6.4)$$

Proof. $X \in H_y^{-}(0)$ implies $D_+ \oplus X = \bigvee_{n\le 0} U^n X \subset H_y^{-}(0)$ combining with (6.1) we conclude that V is a constant unitary matrix.

The possibility of writing the scattering matrix X in the form (2.6.4) has an interpretation on a process level. By the Beurling-Lax theorem, (2.6.4) holds iff (the invariant subspace for the left shift):

$$H_2^{-}(C^p) \ominus (\text{range } H_S) \text{ is of full range (for } \chi)$$

which is equivalent to

$$H_y^{-}(0) \ominus P_{H_y^{-}(0)}H_y^{+}(0) \quad \text{is of full range (for U).} \qquad (2.6.5)$$

$\Lambda^n L_\infty(B(C^p))$ function satisfying (2.6.4) is called [9] *strictly non-cyclic*, and the corresponding process - having a strictly non-cyclic scattering matrix - is called strictly monocyclic. We thus obtain [18, Lemma 7.3 and Th. 7.6]:

2.6.4 COROLLARY. *Let*

$$S = Q_1^{*}Q_2 = P_2P_1^{*}$$

be respectively the left, right coprime factorization of S. *Then all minimal regular Markov subspaces representing* y *are parametrized by those and only those inner functions* $\Theta 1$ *such that*

$$V^{*}S = \Theta_1^{*}\Theta_2 \qquad (2.6.6.)$$

where Θ_1, Θ_2 *are left coprime and* V *is an arbitrary left divisor of* P_2. *Moreover we have*

$$\det \Theta_1 = \det Q_1 .$$

Proof. Combine Corollary (2.6.2) with [14, Lemma III. 5-8].

The general Fuhrmann degree theory for strictly non-cyclic functions [12, Ch. iii.5] now arises naturally - S playing a central role. All regular Markovian subspaces $X \subset H_{\underline{y}}$ representing y are parametrized by an inner function $\Theta_1 \in H_\infty(B(C^p))$ will be of degree

$$d(\Theta_1) = \det \Theta_1 ,$$

an inner function in H_∞, and

$$d(Q_1) \text{ divides } d(\Theta_1).$$

Thus, the degree of the minimal subspace is the lowest, in the sense that $d(Q_1)$ is the weakest among the degrees of all other regular Markovian subspaces representing y. Applying [12,Th. II. 14.11] we infer that two minimal regular Markov subspaces representing y are quasi-similar.

2.6.5 COROLLARY. *For a* $p = 1$ *dimensional process* y *the minimal Markovian representation* of y *is parametrized by the inner function* q_1 *for which*

$$s = \bar{q}_1 q_2$$

is a coprime factorization.

III. UNITARY DILATIONS OF IRREVERSIBLE EVOLUTIONS

In the previous section we have shown how we can associate a Markov semigroup with a Gaussian process via the associated Scattering System. In this section, we consider the dual view of associating a Unitary Dilation with a Markov semigroup.

The evolution of a Hamiltonian (conservative) system is reversible while the evolution of real physical system is not. The real system returns to a state of thermal equilibrium at a temperature determined by its surroundings. This is the physical interpretation of the Poincaré Recurrence Theorem. We may argue that the construction of a unitary dilation of a Markov semigroup is the abstract interpretation of coupling a physical system of a finite number of degrees of freedom to a heat bath thereby producing a Hamiltonian system of infinite

number of degrees of freedom. The ideas of this section are due to Ford, Kac, Mazur [11], Lewis and Thomas [17], Evans and Lewis [9] and the first author.

We undertake the development of this section in continuous time since this is its natural setting.

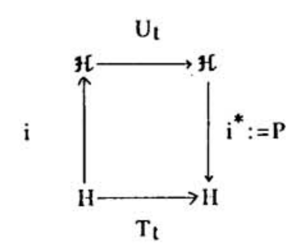
3.1 NOTATION AND PRELIMINARIES

Let H be a separable Hilbert space with scalar product, $<\cdot,\cdot>_H$ and norm $\|\cdot\|_H$. When there is no confusion the subscript H will be dropped. **R** denotes the set of real numbers , $\mathbf{R}^+$ the non-negative real numbers and $\mathbf{R}^-$ the nonpositive real numbers. For $I \subset \mathbf{R}$, an interval, $L^2(I)$ denotes the space of Borel measurable square-integrable complex-valued functions and $L^2(I;H)$ the space of Borel-measurable H-valued square integrable functions. $W^{1,2}(\mathbf{R};H)$ denotes the Sobolev space of H-valued functions and $W^{1,2}(I;H)$ the Sobolev space obtained by restriction of $W^{1,2}(\mathbf{R};H)$. Let $(T_t)_{t\in R^+}$ denote a one-parameter, strongly continuous, contractive semigroup on H, with $T_0 = I$, and let -A be the infinitesimal generator of $(T_t)_{t\in R^+}$. We know $T_t := e^{-tA}$ is a contractive semi-group iff, $\forall\, x \in D(-A)$, $\exists\, x^* \in H$ with $\|x^*\| = 1$, $(x^*,x) = \|x\|$ and $\mathrm{Re} <x^*, Ax> \geq 0$.

A semigroup of contractions $(T_t)_{t\in R^+}$ is said to contract strongly to zero if $\forall\, h \in H$, we have $\lim_{t\to\infty} \|T_t h\| = 0$.

3.2 MINIMAL UNITARY DILATION OF A CONTRACTIVE SEMIGROUP

3.2.1 DEFINITION. Given a contractive semi-group $(T_t)_{t\in R^+}$ on H, we say that a strongly continuous one-parameter group $(U_t)_{t\in R}$ on $\mathcal{H}$ is a unitary dilation of (T_t, H) if there exists an isometry $i: H \to \mathcal{H}$ such that the following diagram commutes:



The unitary dilation is said to be minimal if

$$\mathcal{H} = \overline{\cup \left\{ U_t(iH) \mid t \in \mathbf{R} \right\}} .$$

3.2.2 THEOREM [LAX-PHILLIPS]. *Let* (T_t, H) *be a strongly continuous, contractive, semigroup contracting strongly to zero. Then there exists a unitary dilation* $(U_t, \mathcal{H})$. *The dilation* $(U_t, \mathcal{H})$ *has the canonical representation with* $\mathcal{H} = L^2(\mathbf{R}; N)$, N *a Hilbert space and* $(U_t | t \in \mathbf{R})$ *being the unitary group of right translations on* $L^2(\mathbf{R}; N)$:

(3.2.1) $$(U_t f)(s) = f(s-t).$$

Proof. Since $(T_t)_{t \in \mathbf{R}^+}$ is a contraction,

(3.2.2) $$Q(h) \; < Ah, h > + < h, Ah > \geq 0 \; \forall \; h \in D(A).$$

Let $N_0 = \text{Ker}[Q(h)]$ and let P be the canonical projection of D(A) onto the quotient space $D(A)/N_0$. On $D(A)/N_0$ there exists a scalar product $< \cdot , \cdot >_\Lambda$ such that

(3.2.3) $$< Ph, Pk >_\Lambda = < Ah, k > + < k, Ah > , \; \forall \; h,k \in D(A).$$

Let N denote the Hilbert space completion of $D(A)/N_0$ with respect to the norm induced by (3.2.3). Therefore

(3.2.4) $$\int_{-t}^{0} \|PT_{-s}h\|_\Lambda^2 ds = \|h\|^2 - \|T_t h\|^2, \; \forall \; h \in D(A) \; t \geq 0.$$

If we let $t \to \infty$, since T_t contracts strongly to zero, there exists an isometric embedding i: $H \to L^2(\mathbf{R}; N)$, such that on D(A),

$$(ih)(s) = PT_{-s}h \, , \; \forall \; s \leq 0.$$

Regarding $L^2(\mathbf{R}^-; N)$ as a subspace of $L^2(\mathbf{R}; N)$, we have for $\forall \; h \in D(A)$ and $t \geq 0$

$$(U_t ih)(s) = \begin{cases} PT_{t-s}h & s \leq t \\ 0 & s > t \end{cases}$$

$$= (iT_t h)(s) + \eta_t(s)$$

where $\eta_t(s) \in L^2(\mathbf{R}^+; N) \subset i(H)^\perp$. Hence, $\forall \; t \leq 0$

$T_t = i^* U_t i$, and therefore U_t is a unitary dilation of T_t on $\mathcal{H} = L^2(\mathbf{R}; N)$.

The unitary dilation we have constructed is in fact minimal. This is done by constructing a linear stochastic differential equation involving an operator-valued Brownian motion. We first introduce positive definite kernels and consider their decomposition.

3.2.3 DEFINITION. A map K: $\mathbf{R} \times \mathbf{R} \to B(H)$ is said to be a kernel. The set of all such maps is denoted by $K(\mathbf{R}; H)$. A kernel K is said to be positive definite if $\forall \; h_1, ..., h_n$ in H and $x_1, ... , x_n$ in $\mathbf{R}$.

(3.2.5) $$\sum_{i,j=1}^{n} < K(x_i, x_j)h_j, h_i > \geq 0.$$

3.2.4 DEFINITION. Let $K \in K(\mathbf{R}; H)$. Let H' be a Hilbert space and let V: $\mathbf{R} \to B(H;H')$ be such that $K(x,y) = V(x)^* V(y)$. Then V is said to be a Kolmogoroff decomposition of K. This decomposition is minimal if $H' = \cup\{V(x)h | x \in \mathbf{R}, h \in H\}$. One can prove that every positive definite kernel has a minimal Kolmogoroff decomposition. This is done with the aid of the reproducing kernel Hilbert space associated with K.

With the notation of Theorem 3.2.2, let us introduce an operator-valued Brownian motion as follows:

Let W: $\mathbf{R} \to B(N; H)$ be the map given by

(3.2.6) $$(W_t \eta)(s) = \begin{cases} \chi_{[0,t]}(s)\eta, & t \geq 0 \\ -\chi_{[t,0]}(s)\eta, & t < 0, \end{cases}$$

where $\eta \in N$ and $\chi(.)$ is the characteristic function.

Consider the positive-definite kernel:

$(s,t) \to (s \wedge t)I_N$, where I denotes the identity operator. Then

$$(s \wedge t)I_N = W_t^* W_s.$$

In the sequel we denote by $(D(A), |.|)$ the Hilbert space D(A), with the graph norm.

3.2.5 THEOREM. *Let* $(U_t, \mathcal{H})$ *be the dilation of* (T_t, H) *given in Theorem* 3.2.2. *Then there exists a bounded linear operator* B: (D(A), |.|) → N *and an operator-valued Brownian motion* $W_t : R \to B(N; M)$, *where*

$M = \cup\{W_s\eta \mid s \in \mathbf{R}, \eta \in N\}$ and W_t *satisfies* (3.2.6) *such that*

$$(U_t i - U_s i)h = -\int_s^t U_r iAh\,dr + (W_t - W_s)Bh, \ \forall\ h \in D(A). \tag{3.2.7}$$

Proof. The proof is constructed by verifying equation (3.2.7) for $h \in D(A^2)$ and then by density for $h \in D(A)$. For $h \in D(A^2)$ one can show that a solution is given by

$$U_t ih = e^{-A(t-s)}U_s ih + \int_s^t W(dr)Be^{-A(t-r)}h$$

where the last term is a Wiener integral, which can be defined by an integration by parts formula. The fact that U_t is a minimal unitary dilation follows from the fact that w_t is a minimal Kolmogoroff decomposition.

3.2.6 REMARK. The stationary solution of the equation is given by

$$U_t ih = \int_{-\infty}^t W(ds)Be^{-A(t-s)}h.$$

We may verify that this U_t defines a regular stationary Gaussian process and there is a Lax-Phillips structure associated with it. We may also obtain an ordinary stochastic differential equation for the Markov semigroup attached to this Lax-Phillips system.

3.2.7 A NEW REPRESENTATION OF THE DILATION.

Let us assume that the semigroup $(T_t)_{t\in R+}$ on H is self-adjoint with generator -A. Then A is a positive self-adjoint operator which we assume to be injective. In this case. one can show that there exists a minimal unitary dilation $(U_t, \mathcal{H})$, where $\mathcal{H} = H \oplus L^2(\mathbf{R}; H)$.

Let us write a vector $\varphi \in L^2(\mathbf{R}; H)$ as $\varphi = \varphi^+ + \varphi^-$ with $\varphi^+ \in L^2(\mathbf{R}^+; H)$ and $\varphi^- \in L^2(\mathbf{R}^-; H)$. Then one can write the unitary dilation for $t \in \mathbf{R}^+$ as

$$U_t = \begin{pmatrix} T_t & \mathcal{A}_t \\ B_t & S_t + C_t \end{pmatrix}, \text{ where} \tag{3.2.8}$$

$\mathcal{A}_t: L^2(\mathbf{R}; D(A^{1/2})) \to H: \varphi \to (2A)^{1/2} \int_0^t T_{t-s}\varphi(s)ds$

$B_t: D(A^{1/2}) \to L^2(\mathbf{R}; H): h \to (B_t h)(s) = \chi_{[0,t]}(s)(-2A)^{1/2}T_{t-s}h$

$C_t: L^2(\mathbf{R}; D(A)) \to L^2(\mathbf{R}; H): \varphi \to (C_t\varphi)(s) = \chi_{[0,t]}(s)(-2A)^{1/2}\mathcal{A}_{t-s}\varphi$

$S_t: L^2(\mathbf{R}; H) \to L^2(\mathbf{R}; H): \varphi \to (S_t\varphi)(s) = \varphi(s-t)$

$\mathcal{A}_t$, B_t, C_t are densely defined contractions.

Moreover writing $U_t = e^{it\mathcal{K}}$, on physical grounds the Hamiltonian written as

$$\mathcal{K} = \mathcal{K}_s \oplus \mathcal{K}_c \oplus \mathcal{K}_R \tag{3.2.9}$$

$\mathcal{K}_R$ the Hamiltonian of the reservoir is the generator of the shift of motion. $\mathcal{K}_s$, the Hamiltonian of the system is zero. $\mathcal{K}_c$, the Hamilto coupling is of the form

$$\mathcal{K}_c = \begin{pmatrix} 0 & -iC^* \\ iC & 0 \end{pmatrix} \tag{3.2.10}$$

where C: $D(A^{1/2}) \to L^2(\mathbf{R}; H): h \to \delta_0 \otimes (2A)^{1/2}h$, δ_0 being the Dirac m formal and needs to be justified) and

$$C^*: L^2(\mathbf{R}; H) \to D(A^{1/2}): \varphi \to (2A)^{1/2}\varphi(o)$$

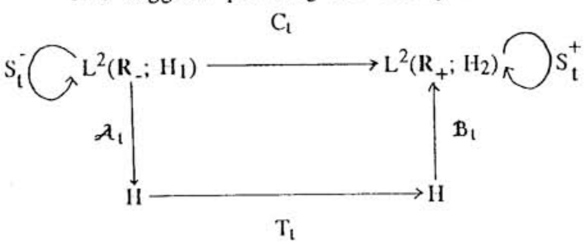
(this is also formal).

We now give some indications on how the new construction is arr Since $(T_t)_{t\in R+}$ is contractive, the quadratic form $F(x,x) = (Ax,x)$ + $\lim_{\varepsilon\downarrow 0} \frac{\|x\|^2 - \|T_\varepsilon x\|^2}{\varepsilon} \le 0$ for $x \in D(A)$. We claim that there exists an operato → H_1 where H_1 is a Hilbert space equal to $\overline{C\overline{D(A)}}$, such that $\|Cx\|^2$ $x \in D(A)$. In a similar manner there exists a pair (C', H_2) for $t \in \mathbf{R}_-$, such tl $(A^*x,x) + (x,A^*x)\ \forall\ x \in D(A^*)$. The operators C and C' are to be th coupling operators. In the self-adjoint case $C = (2A)^{1/2}$.

The idea is to construct the dilation on the space $\mathcal{H} = L^2(\mathbf{R}_-; H_1) \oplus$ H H_2). Now since (T_t, H) is a Markov semigroup, we must have (Us $\circ P_0^\perp \circ$

orthogonal to i(H) where $i : H \to \mathcal{H}$ is the injection $\xi \to \begin{pmatrix} 0 \\ \xi \\ 0 \end{pmatrix}$, $P_0 : \mathcal{H} \to H$ is the orthogonal projection and $P_0^{\perp}$ is the orthogonal projection on the orthogonal complement of H.

This suggests picturing the unitary dilation as follows:



The operators $\mathcal{A}_t^*$ and $\mathcal{B}_t$ "couple" (T_t, H) to $(S_t^-, L^2(R_-; H_1))$ and $(S_t^+, L^2(R_+; H_2))$ where S_t^- and S_t^+ are right shifts. For example $\mathcal{B}_t$ is given by

$$(B_t\xi)(s) = \begin{cases} -C+T_{t-s}\xi & \text{if } s \in [0,t] \\ 0 & \text{if } s \notin [0,t]) \end{cases}, \xi \in D(A)$$

It can be shown that $\mathcal{B}_t$ is a contraction. In the self-adjoint case there is a simplification and it is enough to couple H to $L^2(\mathbf{R}; H)$ and we try to give an intuitive justification of (3.2.8).

In a physical setting the shifts will correspond to the random behaviour of the heat bath and will be the flow of Brownian motion. We expect the coupling between the system and the heat bath to be instantaneous and this coupling will take place via the coupling operator $(2A)_{1/2}$. For $t \geq 0$, we therefore expect that a vector $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \begin{pmatrix} D(A) \\ L^2(R_+;H) \end{pmatrix}$ to be transformed into $\begin{pmatrix} -Axdt \\ db_t \otimes (-(2A)^{1/2}x) \end{pmatrix}$ in time dt, where b_t denotes standard Brownian motion and $db_t \otimes (-(2A)^{1/2}x)$ is an element of $L^2(\mathbf{R}^+; H) = L^2(\mathbf{R}^+) \otimes H$ (tensor product). The second component $db_t \otimes (-(2A)^{1/2}x)$ in integrated form is essentially $\mathcal{B}_t$ in (3.2.8).

Finally, we can explain the form of of K_s, K_R and K_c on physical grounds. Since the time evolution on H is self-adjoint, it does not contain a unitary part and

we expect H_s to be zero. The fact that K_c should be of the form (3.2.10) follows from the same argument given above.

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On the Theory of Nonlin

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Abstract A review of some of the re rol theory is presented. A central theme of state feedback and its use for alter a system. The relations between vario led invariance and a general decomposition p

1. Introduction

One of the most important new developmen the last decade has been, without any doubt and controlled invariant distributions, ial) geometric approach, as it is often calle ant and effective approach for solving v his regard, we mention the Disturbance Dec put Decoupling Problem. Moreover, this ap and sophisticated theoretical picture fo rol systems, and therefore is valuable in th tem theoretic concepts like observability, and invertibility.

Invariant and controlled invariant dist ory play the same role as - and in fact g led