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AN EXISTENCE THEOREM AND LATTICE APPROXIMATIONS FOR A VARIATIONAL PROBLEM ARISING IN COMPUTER VISION*

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Abstract. A variational method for the reconstruction and segmentation of images was recently proposed by Mumford and Shah [15]. In this paper we treat two aspects of the problem. The first concerns existence of solutions, and the second concerns discrete approximations. Discrete versions of this problem have been proposed and studied in [5,12,14,15]. However, it seems that these discrete versions do not properly approximate the continuous problem in the sense that their solutions may not converge to a solution of the continuous problem as the lattice spacing tends to zero. Thus, these discrete formulations in the limit fail to capture properties of the continuous formulation (such as rotation invariance).

Here we consider the use of an alternate lattice approximation for the boundaries of the image and Minkowski content as a cost term for the boundaries. Several properties of Minkowski content are derived. These are used to show that partially discrete versions of the variational problem possess some desirable convergence properties. Specifically, under suitable conditions, solutions to the discrete problem converge in the continuum limit to a solution of the continuous problem, thereby retaining (in the limit) the advantages of the continuous problem. We also present an existence result that is applicable to both discrete and continuous versions of the problem.

1. Introduction. A variational approach to the problem of reconstructing and segmenting an image degraded by noise was recently proposed by Mumford and Shah in [15] (see also Blake and Zisserman [4,5]). The method involves minimizing a cost functional over a space of boundaries with suitably smooth functions within the boundaries. Specifically, if g represents the observed image defined on $\Omega \subset \mathbf{R}^2$, then a reconstructed image f and its associated edges Γ are found by minimizing

$$E(f,\Gamma) = c_1 \iint_\Omega (f-g)^2\,dx\,dy + c_2 \iint_{\Omega\setminus\Gamma} \|\nabla f\|^2\,dx\,dy + c_3 L(\Gamma) \tag{1}$$

where c_1, c_2, c_3 are constants, $\|\cdot\|$ denotes the Euclidean norm and $L(\Gamma)$ denotes the length of Γ. An interesting special case of this problem is obtained if f is restricted to be constant within connected components of $\Omega\setminus\Gamma$. In this case, the optimal value of f on a connected component of $\Omega\setminus\Gamma$ is simply the mean of g over the connected component. Hence, the solution depends only on Γ and is obtained by minimizing

$$E(\Gamma) = c_1 \sum_{i=1}^{k} \iint_{\Omega_i} (g-\overline{g}_i)^2\,dx\,dy + c_3 L(\Gamma) \tag{2}$$

where $\Omega_1,\ldots,\Omega_k$ are the connected components of $\Omega\setminus\Gamma$, and $\overline{g}_i$ is the mean of g over Ω_i.

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Discrete versions of these problems have also been proposed [5,15]. In these discrete problems, the original image g is defined on a subset of the lattice $\frac{1}{n}\mathbf{Z}^2$ with lattice spacing $\frac{1}{n}$. The reconstructed image f is defined on the same lattice, while the boundary Γ consists of a subset of line segments joining neighboring points of the dual lattice. For the discrete problem, f and Γ are found by minimizing

$$E(f,\Gamma) = c_1 \sum_{i\in\Omega} \frac{1}{n^2}(f_i - g_i)^2 + c_2 \sum_{\substack{i,i'\in\Omega \\ \text{adjacent} \\ \overline{ii'}\cap\Gamma=\emptyset}} (f_i - f_{i'})^2 + c_3 L(\Gamma) \tag{3}$$

Similar discrete problems arise in the context of using Markov random fields for problems in vision as proposed by Geman and Geman [12] and studied by Marroquin [14] and others.

The continuous formulation has some distinct advantages over the discrete formulation. For example, the continuous problem is invariant under arbitrary rotations and translations. Also, results from the calculus of variations can be applied in the continuous case. In fact, such methods have yielded interesting results concerning the properties of the minimizing f and Γ [16,22,23]. However, since analytic solutions are not available, the problem must eventually be digitized to obtain numerical solutions. The discrete problem has the advantages of being more directly amenable to computer implementations, particularly with parallel algorithms or hardware. A desirable property of any discrete version of a continuous problem would be for solutions of the discrete problem to converge to solutions of the continuous problem in the continuum limit. In the examples above, one would like convergence of the discrete solutions as the lattice spacing tends to zero. It seems that this is not the case for the problems as defined above. Thus, these discrete formulations in the limit fail to capture properties of the continuous formulation. In particular, the discrete problem in the limit is generally not rotationally invariant, and the analytical results concerning solutions to the continuous problem are not applicable.

In this paper we consider modifications to both the cost functional and the discretization procedure which ensure convergence in the continuum limit. For the cost functional, we propose the use of Minkowski content as the penalty term for the boundaries instead of Hausdorff measure which has been previously used [1,2,17]. For the discretization procedure, we consider only digitizing the boundary. The observed and reconstructed images are still defined on continuous domains. Also, the discrete boundary consists of a union of closed lattice squares rather than a union of line segments. In Section 2 we introduce some preliminary definitions and results from geometric measure theory, and in Section 3 some additional properties of Minkowski content are derived. Section 4 gives an existence result applicable to the problems of interest and Section 5 contains results on the application of these ideas to the variational problem.

2. Metrics and Measures on the Space of Boundaries. In this section we introduce a variety of notions useful in dealing with the 'boundaries' or 'edges' of an image. The 'image' is usually a real valued function defined on a bounded open set $\Omega \subset \mathbf{R}^2$, although some of the results consider the more general case of $\Omega \subset \mathbf{R}^n$. A *boundary* generally refers to a closed subset of $\overline{\Omega}$. However, sometimes the boundary may be restricted to have certain additional properties such as having a finite number of connected components. A topology on the space of boundaries is required for the notion of convergence, and a measure of the 'cost' of a boundary is required for the variational problem.

For $A \subset \mathbf{R}^n$, the *δ-neighborhood* of A will be denoted by $A^{(\delta)}$ and is defined as

$$A^{(\delta)} = \{x \in \mathbf{R}^n : \inf_{y\in A} \|x - y\| < \delta\}$$

The notion of distance between boundaries which we will use is the Hausdorff metric $d_H(\cdot,\cdot)$ defined as

$$d_H(A_1, A_2) = \inf\{\rho : A_1 \subset A_2^{(\rho)} \text{ and } A_2 \subset A_1^{(\rho)}\}$$

It is elementary to show that $d_H(\cdot,\cdot)$ is in fact a metric on the space of all non-empty compact subsets of $\mathbf{R}^n$. An important property of this metric is that it induces a topology which makes the space of boundaries compact.

THEOREM 1. *Let $\mathcal{C}$ be an infinite collection of non-empty closed subsets of a bounded closed set $\overline{\Omega}$. Then there exists a sequence $\{\Gamma_n\}$ of distinct sets of $\mathcal{C}$ and a non-empty closed set $\Gamma \subset \overline{\Omega}$ such that $\Gamma_n \to \Gamma$ in the Hausdorff metric.*

Proof: See [10], Theorem 3.16. □

For the 'cost' of a boundary, the usual notion of length cannot be applied to highly irregular boundaries. Hence a measure on the space of boundaries which generalizes the usual notion of length is desired. A variety of such measures for subsets of $\mathbf{R}^n$ have been investigated. (e.g., see [10]). Perhaps the most widely used and studied are Hausdorff measures [10,11,19].

For a non-empty subset A of $\mathbf{R}^n$, the *diameter* of A is defined by $|A| = \sup\{\|x - y\| : x, y \in A\}$. Let

$$\omega_s = \frac{\Gamma(\frac{1}{2})^s}{\Gamma(\frac{s}{2}+1)}$$

where $\Gamma(\cdot)$ is the usual Gamma function. For integer values of s, ω_s is the volume of the unit ball in $\mathbf{R}^s$. For $s > 0$ and $\delta > 0$ define

$$\mathcal{H}^s_\delta(A) = 2^{-s}\omega_s \inf\{\sum_{i=1}^{\infty} |U_i|^s : A \subset \bigcup_{i=1}^{\infty} U_i,\ |U_i| \le \delta\}$$

The *Hausdorff s-dimensional measure* of A is then given by

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)$$

Note that the factor $2^{-s}\omega_s$ in the definition of $H^s_\delta(\cdot)$ is included for proper normalization. With this definition, for integer values of s Hausdorff measure gives the desired value on sets where the usual notions of length, area, and volume apply.

Many properties of Hausdorff measure can be found in [10,11,19]. The following definitions are required to state several useful properties. A *curve* $\Gamma \subset \mathbf{R}^n$ is the image of a continuous injection $g : [0,1] \to \mathbf{R}^n$. The *length* of a curve Γ is defined as

$$L(\Gamma) = \sup\{\sum_{i=1}^{m} \|g(t_i) - g(t_{i-1})\| \ : \ 0 = t_0 < t_1 < \cdots < t_m = 1\}$$

and Γ is said to be *rectifiable* if $L(\Gamma) < \infty$. Finally, a compact connected set is called a *continuum*.

THEOREM 2. *If* $\Gamma \subset \mathbf{R}^n$ *is a curve, then* $\mathcal{H}^1(\Gamma) = L(\Gamma)$.

Proof: See [10] Lemma 3.2. □

THEOREM 3. *If* Γ *is a continuum with* $\mathcal{H}^1(\Gamma) < \infty$, *then* Γ *consists of a countable union of rectifiable curves together with a set of* $\mathcal{H}^1$*-measure zero.*

Proof: See [10], Theorem 3.14. □

THEOREM 4. *If* $\{\Gamma_n\}$ *is a sequence of continuua in* $\mathbf{R}^n$ *that converges (in Hausdorff metric) to a compact set* Γ, *then* Γ *is a continuum and* $\mathcal{H}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n)$.

Proof: See [10], Theorem 3.18. □

Theorem 4 asserts that $\mathcal{H}^1$-measure is lower-semicontinuous on the set of connected boundaries with respect to the Hausdorff metric. In what follows we extend this result to a cost term for boundaries which depends on the number of connected components. Specifically, we define $\nu(\Gamma) = \mathcal{H}^1(\Gamma) + F(\#(\Gamma))$ where $\#(\Gamma)$ denotes the number of connected components of Γ, and F is any non-decreasing function such that $\lim_{n\to\infty} F(n) = \infty$.

THEOREM 5. $\#(\cdot)$ *and* $\nu(\cdot)$ *are lower-semicontinuous on the space of boundaries with respect to the Hausdorff metric.*

Proof: We will use the following notation,

$$\mathrm{r}(A_1, A_2) = \sup_{x\in A_1} \inf_{y\in A_2} \|x - y\|$$

Suppose $\Gamma_n \to \Gamma$. First we show $\#(\cdot)$ is a lower-semicontinuous function on the space of boundaries. Assume $\#(\Gamma) = c < \infty$. There exists an open cover of Γ consisting of c disjoint open sets $G_1, G_2, \ldots, G_c$ such that $\Gamma \cap G_i \neq \emptyset$, $\forall i$. Γ is closed so $\exists\delta > 0$ such that $\forall i$, $\mathrm{r}(\Gamma \cap G_i, \mathbf{R}^2\backslash G_i) > \delta$. Since $\mathrm{r}(\Gamma, \Gamma_n) \to 0$, for n sufficiently large $\Gamma_n \subset \cup_i G_i$ and $\Gamma_n \cap G_i \neq \emptyset$. Thus $\liminf_{n\to\infty} \#(\Gamma_n) \geq c$. If $\#(\Gamma) = \infty$ then we can repeat this argument for any c and the result follows.

Now we proceed to show $\nu(\Gamma) \leq \liminf_{n\to\infty} \nu(\Gamma_n)$. Assume (without loss of generality) that $\{\nu(\Gamma_n)\} \leq K$, for some $K < \infty$. It follows that $\#(\Gamma_n)$ is uniformly bounded, by $M < \infty$ say, and by the result above, $\#(\Gamma) \leq M$. Since the connected components of Γ are thus separated pairwise by some finite distance the result follows once we show it for connected Γ.

Assume Γ is connected. Let $\delta_n = d_H(\Gamma_n, \Gamma)$. Suppose Γ_n has more than one connected component and let C be one connected component of Γ_n. If for some $\epsilon > 0$, $d(C, \Gamma_n\backslash C) = 2(\delta_n + \epsilon)$, then $\{x : d(x, C) < \delta_n + \epsilon\}$ and $\{x : d(x, \Gamma_n\backslash C) < \delta_n + \epsilon\}$ are two disjoint open sets both containing points of Γ and whose union covers Γ. This contradicts the connectedness of Γ. Thus we can find $x \in C$ and $y \in \Gamma_n\backslash C$ such that $\|x - y\| \leq 2\delta_n$. Consider the straight line segment from x to y. It connects C to some other connected component of Γ_n. Since C was an arbitrary connected component of Γ_n we can find a similar straight line segment from each connected component of Γ_n joining it to some other component. Now if we add all the line segments to Γ_n the number of connected components is reduced to $M/2$ or fewer, the Hausdorff measure will increase by at most $2M\delta_n$ and we will have $d_H(\Gamma_n, \Gamma) \leq 2\delta_n$. Let p be the smallest integer such that $2^p \geq M$, then by repeating the above argument p times we get a modified, connected Γ_n such that its Hausdorff measure is at most $(2pM)\delta_n$ larger than before and $d_H(\Gamma_n, \Gamma) \leq 2^p\delta_n$. Thus the modified Γ_n still converge to Γ and since they are connected we can apply Theorem 4 to get, in terms of the original sequence

$$\mathcal{H}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n) + 2pM\delta_n = \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n).$$

The first result implies lower semicontinuity of F and together with the above result we get lower semicontinuity of ν. □

In view of the fact that the $\mathcal{H}^1$ measure can be discontinuous on the space of boundaries with the topology induced by the Hausdorff metric, a discretization procedure for the continuous problem which will be convergent is not immediately attainable. Here we consider the use of an alternate notion for the cost of boundaries and a modified discretization process (discussed in Section 5).

To measure the cost of the boundaries, we suggest the use of Minkowski content [11]. Let $\mu(\cdot)$ denote Lebesgue measure in $\mathbf{R}^n$. For any $A \subset \mathbf{R}^n$, $0 \leq s \leq n$, and $\delta > 0$, define

$$\mathcal{M}^s_\delta(A) = \frac{\mu(A^{(\delta)})}{\delta^{n-s}\omega_{n-s}}$$

As in the definition of Hausdorff measure, the term ω_{n-s} is included for proper normalization. Recall that $A^{(\delta)}$ is the δ-neighborhood of A — i.e. those points within distance δ of A. Equivalently, $A^{(\delta)}$ is the Minkowski set sum of A and the open ball of radius δ; or in the terminology of mathematical morphology [21] it is the dilation of A with the open ball of radius δ. In general, $\lim_{\delta\to 0} \mathcal{M}^s_\delta(A)$ may not exist (for an example see [11], Section 3.2.40). However, *lower* and *upper Minkowski contents* can be defined by

$$\mathcal{M}^s_*(A) = \liminf_{\delta\to 0^+} \mathcal{M}^s_\delta(A)$$

and

$$\mathcal{M}^{*s}(A) = \limsup_{\delta \to 0^+} \mathcal{M}^s_\delta(A)$$

respectively. If these two values agree (i.e if $\lim_{\delta\to 0} \mathcal{M}^s_\delta(A)$ exists) then the common value is simply called the *s-dimensional Minkowski content* and is denoted by $\mathcal{M}^s(A)$.

3. Properties of Minkowski Content. In this section we develop several properties of Minkowski content some of which will be used in Section 5. The results can roughly be categorized as properties of δ-neighborhoods, continuity and regularity properties of Minkowski content, and relationships between Minkowski content and Hausdorff measure.

First, we state two elementary properties. Two sets A_1, A_2 are said to be *positively separated* if

$$d(A_1, A_2) \equiv \inf\{\|a_1 - a_2\| : a_1 \in A_1, a_2 \in A_2\} > 0$$

The sets $A_1, A_2, \dots, A_m$ are called positively separated if $\min_{i\neq j} d(A_i, A_j) > 0$. The first property is that $\mathcal{M}^s$ is additive on positively separated sets, i.e. if $A_1, A_2, \dots, A_m$ are positively separated then $\mathcal{M}^s(\cup_{i=1}^m A_i) = \sum_{i=1}^m \mathcal{M}^s(A_i)$. This follows from the fact that for sufficiently small δ, the δ-neighborhoods of the A_i are disjoint. The second property is that for any set A, $A^{(\delta)} = \overline{A}^{(\delta)}$ and so $\mathcal{M}^s_\delta(A) = \mathcal{M}^s_\delta(\overline{A})$ for every $\delta > 0$, where $\overline{A}$ denotes the closure of A. Clearly $A^{(\delta)} \subset \overline{A}^{(\delta)}$. On the other hand, if $x \in \overline{A}^{(\delta)}$ then $\|x - y\| = \eta < \delta$ for some $y \in \overline{A}$. But $\|y - a\| < \delta - \eta$ for some $a \in A$, so that $\|x - a\| \leq \|x - y\| + \|y - a\| < \delta$. Hence, $x \in A^{(\delta)}$ and so the result follows.

The following two lemmas give properties of δ-neighborhoods which will be useful in showing continuity properties of Minkowski content. $B_r(x)$ and $\overline{B}_r(x)$ denote the open and closed balls, respectively, of radius r centered at x.

Lemma 1. *$\mu(\partial\Gamma^{(\delta)}) = 0$ for every $\Gamma \subset \mathbf{R}^2$.*

Proof: Let $\Gamma \subset \mathbf{R}^2$ and let $E = \partial\Gamma^{(\delta)}$. The Lebesgue density of E at x, $D_\mu(E, x)$, is defined as

$$D_\mu(E,x) = \lim_{r\to 0} \frac{\mu(E \cap B_r(x))}{\mu(B_r(x))}$$

when the limit exists. We will show that the Lebesgue density of E is less than 1 for all $x \in E$. Hence, $\mu(E) = 0$ will follow from the Lebesgue Density Theorem.

Let $x \in E = \partial\Gamma^{(\delta)}$. Then for each $r > 0$, there exists $c(r) \in \Gamma$ with $\|x - c(r)\| < \delta + r^2$. If $w \in B_\delta(c(r))$ then $w \notin E$, so that

$$\mu(E \cap B_r(x)) \leq \mu(B_r(x)) - \mu(B_r(x) \cap B_\delta(c(r)))$$

The circle of radius δ centered at $c(r)$ intersects the circle of radius r centered at x in two points which determine a chord C. Let S denote the segment of $B_r(x)$ determined by C, θ the central angle at x subtended by C, and a the distance from x to C. Then

$$\mu(B_r(x) \cap B_\delta(c(r))) \geq \mu(S) = \frac{1}{2}r^2(\theta - \sin\theta)$$

and

$$\lim_{r\to 0}\theta = \lim_{r\to 0} 2\cos^{-1}(\frac{d}{r}) = \lim_{r\to 0} 2\cos^{-1}(\frac{r^2 + 2\delta r^2 + r^4}{2r(\delta + r^2)}) = \pi$$

Therefore,

$$D_\mu(E,x) = \lim_{r\to 0}\frac{\mu(E\cap B_r(x))}{\mu(B_r(x))} \leq \lim_{r\to 0}\frac{\mu(B_r(x)) - \mu(S)}{\mu(B_r(x))} = \lim_{r\to 0}(1 - \frac{1}{2\pi}(\theta - \sin\theta)) = \frac{1}{2} \quad \square$$

Lemma 2. *If $\Gamma_n \to \Gamma$ in Hausdorff metric, then $\Gamma_n^{(\delta)} \to \Gamma^{(\delta)}$.*

Proof: Let $\epsilon > 0$. Since $\Gamma_n \to \Gamma$, $\exists\, N < \infty$ such that $d_H(\Gamma_n, \Gamma) < \epsilon\ \forall n > N$. If $x \in \Gamma^{(\delta)}$ then $x = a + \rho$ with $a \in \Gamma$ and $\|\rho\| < \delta$. For all $n > N$, there exists $a_n \in \Gamma_n$ with $\|a - a_n\| < \epsilon$. Then $x_n \equiv a_n + \rho \in \Gamma_n^{(\delta)}$ and $\|x - x_n\| = \|a - a_n\| < \epsilon$. Hence, $\Gamma^{(\delta)} \subset (\Gamma_n^{(\delta)})^{(\epsilon)}$. Similarly, $\Gamma_n^{(\delta)} \subset (\Gamma^{(\delta)})^{(\epsilon)}$. Thus, $d_H(\Gamma_n^{(\delta)}, \Gamma^{(\delta)}) < \epsilon\ \forall n > N$. $\square$

Two continuity properties of $\mathcal{M}^s_\delta$ may now be deduced. These follow directly from the corresponding continuity properties of Lebesgue measure on δ-neighborhoods.

Theorem 6. *If $\Gamma_n \to \Gamma$ in Hausdorff metric then $\mu(\Gamma_n^{(\delta)}) \to \mu(\Gamma^{(\delta)})$ and so $\mathcal{M}^s_\delta(\Gamma_n) \to \mathcal{M}^s_\delta(\Gamma)$. I.e., $\mathcal{M}^s_\delta(\Gamma)$ is continuous in Γ with respect to Hausdorff metric.*

Proof: Since $\Gamma_n \to \Gamma$, by Lemma 2 we have $\Gamma_n^{(\delta)} \to \Gamma^{(\delta)}$. Let $\epsilon > 0$. Then there exists $N < \infty$ such that $\Gamma_n^{(\delta)} \subset (\Gamma^{(\delta)})^{(\epsilon)}\ \forall n \geq N$. Therefore, $\sup_{n\geq N}\mu(\Gamma_n^{(\delta)}) \leq \mu(\Gamma^{(\delta+\epsilon)})$. As $\epsilon \downarrow 0$, $\Gamma^{(\delta+\epsilon)} \downarrow \overline{\Gamma^{(\delta)}}$ so that $\limsup_{n\to\infty}\mu(\Gamma_n^{(\delta)}) \leq \mu(\overline{\Gamma^{(\delta)}})$. Then by Lemma 1 it follows that $\limsup_{n\to\infty}\mu(\Gamma_n^{(\delta)}) \leq \mu(\Gamma^{(\delta)})$.

Let K be a compact subset of $\Gamma^{(\delta)}$. Since $\{B_\delta(x) : x \in \Gamma\}$ is an open cover of K, there exists a finite subcover $B_\delta(x_1), \dots, B_\delta(x_m)$. Let $\epsilon > 0$. Since $\Gamma_n \to \Gamma$, there exists $N < \infty$ such that $\forall n \geq N$ we can find $y_{n,1}, \dots, y_{n,m} \in \Gamma_n$ with $\|y_{n,i} - x_i\| < \epsilon$ for $i = 1, \dots, m$. Then $\mu(B_\delta(x_i) \setminus B_\delta(y_{n,i})) < f(\epsilon) = \mu(B_1 \setminus B_2) \leq 2\delta\epsilon$ where B_1 and B_2 are balls of radius δ whose centers are ϵ apart. Therefore,

$$\mu(\Gamma_n^{(\delta)}) \geq \mu(\bigcup_{i=1}^m B_\delta(y_{n,i})) > \mu(K) - mf(\epsilon) \quad \forall n \geq N$$

and so $\inf_{n\geq N}\mu(\Gamma_n^{(\delta)}) > \mu(K) - mf(\epsilon)$. Since $\epsilon > 0$ is arbitrary and $f(\epsilon) \to 0$ as $\epsilon \to 0$ we have $\liminf_{n\to\infty}\mu(\Gamma_n^{(\delta)}) \geq \mu(K)$. Finally, since this is true for every compact $K \subset \Gamma^{(\delta)}$, we have $\liminf_{n\to\infty}\mu(\Gamma_n^{(\delta)}) \geq \sup_{K\subset\Gamma^{(\delta)}}\mu(K) = \mu(\Gamma^{(\delta)})$.

Thus,

$$\liminf_{n\to\infty}\mu(\Gamma_n^{(\delta)}) = \limsup_{n\to\infty}\mu(\Gamma_n^{(\delta)}) = \lim_{n\to\infty}\mu(\Gamma_n^{(\delta)}) = \mu(\Gamma^{(\delta)}) \quad \square$$

PROPOSITION 1. *$\mathcal{M}^s_\delta(\Gamma)$ is continuous in δ for all $\delta > 0$.*

Proof: As $\eta \uparrow \delta$, we have $\Gamma^{(\eta)} \uparrow \Gamma^{(\delta)}$ so that $\mu(\Gamma^{(\eta)}) \uparrow \mu(\Gamma^{(\delta)})$. As $\eta \downarrow \delta$, we have $\Gamma^{(\eta)} \downarrow \overline{\Gamma^{(\delta)}}$. Then by Lemma 1, $\mu(\Gamma^{(\eta)}) \downarrow \mu(\overline{\Gamma^{(\delta)}}) = \mu(\Gamma^{(\delta)})$. Thus, $\lim_{\eta\to\delta} \mu(\Gamma^{(\eta)}) = \mu(\Gamma^{(\delta)})$. □

All the results given so far in this section were proved for $\Gamma \subset \mathbf{R}^2$. However, these results and proofs can easily be extended to $\mathbf{R}^n$.

We now state a result given in Federer [11] relating Minkowski content to Hausdorff measure. A subset Γ of $\mathbf{R}^n$ is called *m-rectifiable* if there exists a Lipschitzian function mapping a bounded subset of $\mathbf{R}^m$ onto Γ.

THEOREM 7. *If Γ is a closed m-rectifiable subset of $\mathbf{R}^n$ then $\mathcal{M}^m(\Gamma) = \mathcal{H}^m(\Gamma)$.*

Proof: See [11] Theorem 3.2.39. □

We will present a proof of Theorem 7 in the restricted case of 1-dimensional measure in $\mathbf{R}^2$ (i.e., $m = 1, n = 2$), which is stated as Theorem 8. The basic idea of our proof is contained in the proof of Proposition 4. This idea will be used again in the proof of Theorem 9 on the Γ-convergence of Minkowski content, which is true only for 1-dimensional measures, and this motivates including the proof.

The following two preliminary results give upper and lower bounds on $M^1_\delta(\Gamma)$ for rectifiable and connected sets respectively. These two results could be appropriately extended to s-dimensional measure in $\mathbf{R}^n$.

PROPOSITION 2. *If $\Gamma \subset \mathbf{R}^2$ is rectifiable then $\mu(\Gamma^{(\delta)}) \le 2\delta\mathcal{H}^1(\Gamma) + \pi\delta^2$ and so $\mathcal{M}^1_\delta(\Gamma) \le \mathcal{H}^1(\Gamma) + \frac{1}{2}\pi\delta$.*

Proof: Since Γ is rectifiable, $\Gamma = \{\gamma(t) : 0 \le t \le 1\}$ where $\gamma : [0,1] \to \mathbf{R}^2$ is rectifiable and $\mathcal{H}^1(\Gamma) = \sup\{\sum_{i=1}^m \|\gamma(t_i) - \gamma(t_{i-1})\| : 0 = t_0 < t_1 < \cdots < t_m = 1\}$. For $E = 1, 2, \ldots$ let $\{t_{ij}\}$ be a sequence of dissections such that $\max_i\{\|t_{ij} - t_{i-1,j}\|\} \to 0$ and $\mathcal{H}^1(\Gamma) = \lim_{j\to\infty} \sum_{i=1}^{m(j)} \|\gamma(t_{ij}) - \gamma(t_{i-1,j})\|$ Let $C_j = \cup_{i=1}^{m(j)} S_i$ where S_i is the straight line joining $\gamma(t_{i-1,j})$ and $\gamma(t_{ij})$. Then $\mu(S_{ij}^{(\delta)}) = 2\delta\|\gamma(t_{ij}) - \gamma(t_{i-1,j})\| + \pi\delta^2$, and

$$\begin{aligned}\mu(\cup_{i=1}^k S_{ij}^{(\delta)}) &= \mu(\cup_{i=1}^{k-1} S_{ij}^{(\delta)}) + \mu(S_{kj}^{(\delta)}) - \mu(S_{kj}^{\delta} \cap \bigcup_{i=1}^{k-1} S_{ij}^{(\delta)}) \\ &\le \mu(\cup_{i=1}^{k-1} S_{ij}^{(\delta)}) + \mu(S_{kj}^{(\delta)}) - \pi\delta^2 \\ &= \mu(\cup_{i=1}^{k-1} S_{ij}^{(\delta)}) + 2\delta\|\gamma(t_{kj}) - \gamma(t_{k-1,j})\|\end{aligned}$$

By induction on i, we get

$$\mu(C_j^\delta) \le \sum_{i=1}^{m(j)} 2\delta\|\gamma(t_{ij}) - \gamma(t_{i-1,j})\| + \pi\delta^2$$

Since $C_j \to \Gamma$ in Hausdorff metric, by Theorem 6

$$\mu(\Gamma^{(\delta)}) = \lim_{j\to\infty} \mu(C_j^{(\delta)}) \le 2\delta\mathcal{H}^1(\Gamma) + \pi\delta^2 \quad \square$$

PROPOSITION 3. *If $\Gamma \subset \mathbf{R}^2$ is connected, then $\mathcal{M}^1_\delta(\Gamma) \ge |\Gamma|$.*

Proof: Let $x, y \in \Gamma$, and let $\epsilon > 0$. Since Γ is connected, we can find $x = x_0, x_1, \ldots, x_k = y$ in Γ with $\|x_i - x_{i-1}\| < \epsilon$ for $1 \le i \le k$. Let $P(w)$ denote the point obtained by the orthogonal projection of w onto the straight line T through x and y, and let $p(w)$ be the coordinate of $P(w)$ considering T as the real line with origin at x and positive direction towards y. I.e.,

$$p(w) = \frac{\langle w - x, y - x\rangle}{\|y - x\|}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product. Note that $|p(x_i) - p(x_{i-1})| = \|P(x_i) - P(x_{i-1})\| \le \|x_i - x_{i-1}\| < \epsilon$. By deleting intermediate points and reordering the indices as necessary, we can assume that $0 = p(x_0) < p(x_1) < \cdots < p(x_k) = \|x - y\|$ and $p(x_i) - p(x_{i-1}) < \epsilon$.

For $u, v \in \mathbf{R}^2$ with $p(u) < p(v)$, let $R(u,v) = \{w \in \mathbf{R}^2 : p(u) < p(w) < p(v)\}$. Then

$$\Gamma^{(\delta)} \supset \cup_{i=0}^k B_\delta(x_i) \supset \cup_{i=1}^k B_\delta(x_i) \cap R(x_i, x_{i-1})$$

Since the $R(x_i, x_{i-1})$ for $i = 1, 2, \ldots, k$ are disjoint,

$$\begin{aligned}\mathcal{M}^1_\delta(\Gamma) &\ge \frac{\mu(\cup_{i=1}^k B_\delta(x_i) \cap R(x_i, x_{i-1}))}{2\delta} \\ &= \frac{1}{2\delta}\sum_{i=1}^k \mu(B_\delta(x_i) \cap R(x_i, x_{i-1})) \\ &\ge \frac{1}{2\delta}\sum_{i=1}^k 2\sqrt{\delta^2 - \epsilon^2}\,(p(x_i) - p(x_{i-1})) \\ &= \|x - y\|\sqrt{1 - \frac{\epsilon^2}{\delta^2}}\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have $\mathcal{M}^1_\delta(\Gamma) \ge \|x - y\|$. Finally, the result follows since $x, y \in \Gamma$ are arbitrary. □

Using the bounds of Propositions 2 and 3, the following proposition can be shown.

PROPOSITION 4. *If $\Gamma \subset \mathbf{R}^2$ is connected and consists of a countable union of rectifiable curves then $\mathcal{M}^1(\Gamma) = \mathcal{H}^1(\Gamma)$.*

Proof: First, we prove the result when Γ is a rectifiable curve which does not intersect itself. Let $\Gamma = \{\gamma(t) : 0 \le t \le 1\}$ where $\gamma : [0,1] \to \mathbf{R}^2$ is rectifiable and $\gamma(s) \ne \gamma(t)$ if $s \ne t$. Let $0 = t_0 < t_1 < \cdots < t_m = 1$, and for $i = 1, 2, \ldots, m$ let $\Gamma_i = \{\gamma(t) : t_{i-1} < t < t_i\}$. If $K_i \subset \Gamma_i$ $i = 1, 2, \ldots, m$ are continuua then they are positively separated. Therefore, for sufficiently small δ the $K_i^{(\delta)}$ are disjoint. From Proposition 3 we have

$$\mathcal{M}^1_\delta(\Gamma) \ge \sum_{i=1}^m \mathcal{M}^1_\delta(K_i) \ge \sum_{i=1}^m |K_i|$$

for all sufficiently small δ. Hence,

$$\liminf_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) \ge \sum_{i=1}^{m} \|\gamma(t_i) - \gamma(t_{i-1})\|$$

and since the dissection $\{t_i\}$ is arbitrary

$$\liminf_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) \ge \mathcal{H}^1(\Gamma)$$

On the other hand, from Proposition 2, $\mathcal{M}^1_\delta(\Gamma) \le \mathcal{H}^1(\Gamma) + \frac{1}{2}\pi\delta$ so that

$$\limsup_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) \le \mathcal{H}^1(\Gamma)$$

Thus,

$$\mathcal{M}^1(\Gamma) = \lim_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) = \mathcal{H}^1(\Gamma)$$

Now, suppose $\Gamma = \cup_{i=1}^{\infty} C_i$ is connected where the C_i are rectifiable curves. By decomposing the C_i as necessary, we can assume that they are not self-intersecting and that C_i intersects C_j in at most a finite number of points for $i \ne j$. Then $\mathcal{H}^1(\Gamma) = \sum_{i=1}^{\infty} \mathcal{H}^1(C_i)$. Let $E_k = \cup_{i=1}^{k} C_i$. Then by a dissection argument similar to that used above we get

$$\liminf_{\delta\to 0} \mathcal{M}^1_\delta(E_k) \ge \sum_{i=1}^{k} \mathcal{H}^1(C_i)$$

and so

$$\liminf_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) \ge \sup_k \liminf_{\delta\to 0} \mathcal{M}^1_\delta(E_k) \ge \mathcal{H}^1(\Gamma)$$

Also, from Proposition 2 and the fact that Γ is connected we have

$$\begin{aligned}\mu(E_k^{(\delta)}) &= \mu(E_{k-1}^{(\delta)}) + \mu(C_k^{(\delta)}) - \mu(E_{k-1}^{(\delta)} \cap C_k^{(\delta)})\\ &\le \mu(E_{k-1}^{(\delta)}) + \mu(C_k^{(\delta)}) - \pi\delta^2 \le \mu(E_{k-1}^{(\delta)}) + 2\delta\mathcal{H}^1(C_k)\end{aligned}$$

By induction we get

$$\mu(E_k^{(\delta)}) \le 2\delta \sum_{i=1}^{k} \mathcal{H}^1(C_k) + \pi\delta^2$$

for every integer k. Since $E_k^{(\delta)}$ is an increasing sequence of sets with $\Gamma^{(\delta)} = \cup_{i=k}^{\infty} E_k$ we have

$$\mu(\Gamma^{(\delta)}) = \lim_{k\to\infty} \mu(E_k^{(\delta)}) \le 2\delta\mathcal{H}^1(\Gamma) + \pi\delta^2$$

Thus

$$\limsup_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) \le \mathcal{H}^1(\Gamma)$$

and so the result follows. □

The next inequality gives bounds for s-dimensional Minkowski content in $\mathbf{R}^2$ which are valid for *every* subset of $\mathbf{R}^2$. This could also be appropriately extended to $\mathbf{R}^n$. Here, we use the notation

$$\mathcal{H}^s_{\delta,2\delta}(\Gamma) = 2^{-s}\omega_s \inf\{\sum_{i=1}^{\infty} |U_i|^s \;:\Gamma \subset \bigcup_{i=1}^{\infty} U_i,\ \delta \le |U_i| \le 2\delta\}$$

PROPOSITION 5. *For every $\Gamma \subset \mathbf{R}^2$ and $0 \le s \le 2$,*

$$\frac{2^{s-1}}{\omega_s\omega_{2-s}}\mathcal{H}^s_\delta(\Gamma) \le \mathcal{M}^s_\delta(\Gamma) \le \frac{16}{\omega_s\omega_{2-s}}\mathcal{H}^s_{\delta,2\delta}(\Gamma)$$

and so

$$\frac{2^{s-1}}{\omega_s\omega_{2-s}}\mathcal{H}^s(\Gamma) \le \mathcal{M}^s_*(\Gamma) \le \liminf_{\delta\to 0} \frac{16}{\omega_s\omega_{2-s}}\mathcal{H}^s_{\delta,2\delta}(\Gamma)$$

where $\mathcal{H}^s_{\delta,2\delta}(\Gamma) = 2^{-s}\omega_s \inf\{\sum_{i=1}^{\infty} |U_i|^s \;:\Gamma \subset \bigcup_{i=1}^{\infty} U_i,\ \delta \le |U_i| \le 2\delta\}$

Proof: Consider the closed lattice squares formed by the points $\frac{1}{\sqrt{2\delta}}\mathbf{Z}^2$. Form a cover $\{U_i\}$ of Γ by taking all lattice squares whose intersection with Γ is non-empty. Then $\{U_i\}$ is a δ-cover of Γ and $\bigcup_i U_i \subset \overline{\Gamma^{(\delta)}}$. Hence,

$$\begin{aligned}\frac{2^s}{\omega_s}\mathcal{H}^s_\delta(\Gamma) &\le \sum_i |U_i|^s = \frac{2}{\delta^{2-s}} \sum_i (\frac{\delta}{\sqrt{2}})^2 = \frac{2}{\delta^{2-s}}\mu(\bigcup_i U_i)\\ &\le \frac{2}{\delta^{2-s}}\mu(\overline{\Gamma^{(\delta)}}) = \frac{2}{\delta^{2-s}}\mu(\Gamma^{(\delta)}) = 2\omega_{2-s}\mathcal{M}^s_\delta(\Gamma)\end{aligned}$$

To show the second part of the first inequality, let $\{U_i\}$ be any cover of Γ with $\delta \le |U_i| \le 2\delta$. Without loss of generality, we assume that $U_i \cap \Gamma$ is non-empty for each i. Select $x_i \in \Gamma \cap U_i$. Then $\cup_i \overline{B}_{|U_i|}(x_i) \supset \cup_i U_i \supset \Gamma$ so that $\cup_i \overline{B}_{2|U_i|}(x_i) \supset \Gamma^{(\delta)}$ since $|U_i| \ge \delta$. Therefore,

$$\mathcal{M}^s_\delta(\Gamma) \le \frac{\mu(\cup_i \overline{B}_{2|U_i|}(x_i))}{\delta^{2-s}\omega_{2-s}} \le \frac{\sum_i 4\pi|U_i|^2}{\delta^{2-s}\omega_{2-s}} \le \frac{4\pi}{\omega_{2-s}} \sum_i \frac{|U_i|^2}{(\frac{|U_i|}{2})^{2-s}} = \frac{2^{4-s}}{\omega_{2-s}} \sum_i |U_i|^s$$

and so

$$\mathcal{M}^s_\delta(\Gamma) \le \frac{2^{4-s}}{\omega_{2-s}} \inf\{\sum_{i=1}^{\infty} |U_i|^s \;:\Gamma \subset \bigcup_{i=1}^{\infty} U_i,\ \delta \le |U_i| \le \delta\} = \frac{16}{\omega_s\omega_{2-s}}\mathcal{H}^s_{\delta,2\delta}(\Gamma) \quad \square$$

Note that the definition of $\mathcal{H}^s_{\delta,2\delta}$ is similar to Hausdorff measure, except that the diameter of the covering sets is bounded below as well as above. Hence, its value may be quite different from Hausdorff measure. As an aside, one consequence of the above proposition is the known result that the *Minkowski dimension* of a set is greater than or equal to its *Hausdorff dimension* [9,13].

We can now prove the following special case of Theorem 7.

THEOREM 8. *If $\Gamma \subset \mathbf{R}^2$ is a compact set with a finite number of connected components then $\mathcal{M}^1(\Gamma) = \mathcal{H}^1(\Gamma)$.*

Proof: Since the connected components of Γ are compact, disjoint, and finite in number, they are positively separated. By additivity of both $\mathcal{M}^1$ and $\mathcal{H}^1$, we need only consider the case in which Γ has one connected component. Hence, we assume that Γ is a continuum. If $\mathcal{H}^1(\Gamma) = \infty$ then $\mathcal{M}^1(\Gamma) = \infty$ from Proposition 5. Therefore, we can assume that $\mathcal{H}^1(\Gamma) < \infty$.

Then from Lemma 3.12 of [10], Γ is arcwise connected. Since Γ is compact, we can define a sequence of curves C_j inductively as follows (as in the proof of Lemma 3.13 of [10]). Let C_1 be a curve in Γ joining two of the most distant points of Γ. Given $C_1, C_2, \ldots, C_j$, let $x \in \Gamma$ be at a maximum distance from $\cup_{i=1}^{j} C_i$ and let d_j denote this maximum distance. If $d_j = 0$ then the procedure terminates and we let $C_i = \emptyset$ for $i \geq j+1$. Otherwise, let C_{j+1} be a curve in Γ joining x and $\cup_{i=1}^{j} C_i$ that is disjoint from $\cup_{i=1}^{j} C_i$ except for an endpoint.

Let $E_k = \cup_{j=1}^{k} C_j$. It is shown in [10] (proof of lemma 3.13) that $\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\cup_{i=1}^{\infty} E_k)$. Also,

$$\sum_{j=1}^{\infty} d_j \leq \sum_{j=1}^{\infty} \mathcal{H}^1(C_j) = \mathcal{H}^1(\Gamma) < \infty$$

so that $d_j \to 0$. This implies that $E_k = \cup_{j=1}^{k} C_j \to E$ in Hausdorff metric as $k \to \infty$ and so $\overline{\cup_{k=1}^{\infty} E_k} = \Gamma$. Hence, from Proposition 10 and using the fact that $\mathcal{M}^1(A) = \mathcal{M}^1(\overline{A})$ for any A, we get

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\cup_{k=1}^{\infty} E_k) = \mathcal{M}^1(\cup_{k=1}^{\infty} E_k) = \mathcal{M}^1(\overline{\cup_{k=1}^{\infty} E_k}) = \mathcal{M}^1(\Gamma) \quad \square$$

Note that $\mathcal{M}^1$ and $\mathcal{H}^1$ do not agree on all compact sets. An example of a compact set on which they disagree is given in [11] (Section 3.2.40).

The final result shown in this section is that Minkowski content possesses a useful type variational convergence property known as Γ-convergence (or epi-convergence). This notion of convergence, introduced by De Giorgi [6,7,8] and independently by Attouch [3], is useful in problems involving the convergence of functionals. The result on Γ-convergence will be used in Section 5 to prove some convergence properties of solutions to certain variational problems. Given a topological space (X, τ), and functions $F_n, F : X \to \mathbf{R} \cup \{-\infty, +\infty\}$, the sequence $\{F_n\}$ is said to be Γ-*convergent* (or *epi-convergent*) to F at $x \in X$ if the following two conditions hold:

(i) for every sequence $\{x_n\}$ converging to x in (X, τ), $F(x) \leq \liminf_{n\to\infty} F_n(x_n)$, and

(ii) there exists a sequence $\{x_n\}$ converging to x in (X, τ) such that $F(x) \geq \limsup_{n\to\infty} F_n(x_n)$.

We will show that for every sequence $\delta_n \to 0$, $\mathcal{M}^1_{\delta_n}$ is Γ-convergent to $\mathcal{M}^1$ on the space of compact subsets of $\mathbf{R}^2$ with a bounded number of connected components and with the topology induced by the Hausdorff metric.

First, we need the following lemma as stated in [10].

LEMMA 3. *Let $\mathcal{C}$ be a collection of balls contained in a bounded subset of $\mathbf{R}^n$. Then there exists a finite or countably infinite disjoint subcollection $\{B_i\}$ such that*

$$\bigcup_{B \in \mathcal{C}} B \subset \bigcup_i B_i'$$

where B_i' is the ball concentric with B_i and of three times the radius.

Proof: See [10], Lemma 1.9.

Now the the Γ-convergence of Minkowski content can be shown.

THEOREM 9. *For every sequence $\delta_n \to 0^+$, $\mathcal{M}^1_{\delta_n}$ is Γ-convergent to $\mathcal{M}^1$ on the space of compact subsets of $\mathbf{R}^2$ with a bounded number of connected components and with the topology induced by the Hausdorff metric. I.e., let $\Gamma \subset \mathbf{R}^2$ be compact with $\#(\Gamma) \leq M < \infty$, and let $\delta_n > 0$ satisfy $\lim_{n\to\infty} \delta_n = 0$. Then the following two conditions hold:*

(i) *For every sequence of compact sets $\Gamma_n \subset \mathbf{R}^2$ with $\Gamma_n \to \Gamma$ in Hausdorff metric and $\#(\Gamma_n) \leq M$ $\forall n$ we have*

$$\mathcal{M}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n)$$

(ii) *There exists a sequence of compact sets $\Gamma_n \subset \mathbf{R}^2$ with $\Gamma_n \to \Gamma$ in Hausdorff metric and $\#(\Gamma_n) \leq M$ $\forall n$ such that*

$$\mathcal{M}^1(\Gamma) \geq \limsup_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n)$$

Proof: Since $\#(\Gamma) \leq M$ we have $\Gamma = \cup_{i=1}^{k} F_i$ where $k \leq M$ and $F_1, F_2, \ldots, F_k$ are the connected components of Γ. Since the F_i are compact and disjoint, they are positively separated, i.e there exists $\eta > 0$ such that $F_i^{(\eta)} \cap F_j^{(\eta)} = \emptyset$ for $i \neq j$. Then $\mathcal{M}^1(\Gamma) = \sum_{i=1}^{k} \mathcal{M}^1(F_i)$, and for sufficiently large n, $\mathcal{M}^1_{\delta_n}(\Gamma_n) = \sum_{i=1}^{k} \mathcal{M}^1_{\delta_n}(\Gamma_n \cap F_i^{(\eta)})$. Thus, it is sufficient to prove the result under the assumption that Γ is connected.

Suppose $\mathcal{H}^1(\Gamma) = \infty$. Form a δ-covering of Γ by placing a closed ball of radius δ about each point of Γ. Then by Lemma 3, we can find a disjoint subcollection (necessarily finite) of balls such that concentric balls of radius 3δ cover $\Gamma^{(\delta)}$. Let $N(\delta)$ be the number of balls in this finite disjoint subcollection. Then $6\delta N(\delta) \geq \mathcal{H}^1_{6\delta}(\Gamma) \to \infty$ as $\delta \to 0$. Let $\epsilon > 0$. Since $\Gamma_n \to \Gamma$, for sufficiently large n we have $\Gamma_n \cap B_{\frac{\delta}{2}}(x_i) \neq \emptyset$. Also, since $\#(\Gamma_n) \leq M$, there is a connected component of $\Gamma_n \cap B_\delta(x_i)$ with diameter greter than or equal to $\frac{\delta}{2}$ for at least $N(\delta) - M$ values of i. Using Proposition 3 and the fact that the balls are positively separated, we have for sufficiently large n

$$\mathcal{M}^1_{\delta_n}(\Gamma_n) \geq \mathcal{M}^1_{\delta_n}(\Gamma_n \cap \cup_{i=1}^{N(\delta)} B_\delta(x_i)) = \sum_{i=1}^{N(\delta)} \mathcal{M}^1_{\delta_n}(\Gamma_n \cap B_\delta(x_i))$$

$$\geq \sum_{i=1}^{N(\delta)} |\Gamma_n \cap B_\delta(x_i)| \geq \frac{\delta}{2}(N(\delta) - M)$$

Since $\delta N(\delta) \to \infty$ as $\delta \to 0$, $\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n) = \infty$, and so the result follows.

Now, suppose $\mathcal{H}^1(\Gamma) < \infty$. From Theorem 3 we have $\Gamma = S \cup (\bigcup_{i=1}^{\infty} C_i)$ where $\mathcal{H}^1(S) = 0$, and C_i are rectifiable curves. From the construction used in the proof of this result (see [10], also part of the proof is reproduced in the proof of Proposition 4), $\mathcal{H}^1(\Gamma) = \sum_{i=1}^{\infty} \mathcal{H}^1(C_i)$ and if $x \in C_i \cap C_j$ then x is an endpoint of at least one of C_i or C_j.

Consider $\cup_{i=1}^k C_i$. By decomposing the C_i, we can assume that they are simple curves which meet each other only at endpoints. The C_i are rectifiable curves, so that $C_i : [0,1] \to \mathbf{R}^2$ and

$$\mathcal{H}^1(C_i) = \mathcal{M}^1(C_i) = \sup\{\sum_{j=1}^{m(i)} \|C_i(t_{i,j-1} - C_i(t_{ij})\| : 0 = t_{i0} < t_{i1} < \ldots < t_{i,m(i)} = 1\}$$

For each $i = 1, 2, \ldots, k$, let $0 = t_{i0} < t_{i1} < \cdots < t_{i,m(i)} = 1$, and consider the points $x_{ij} = C_i(t_{ij})$.

The connected components of $\cup_{i=1}^k C_i \setminus \{x_{ij}\}$ are given by $G_{ij} = \{C_i(t) : t_{i,j-1} < t < t_{ij}\}$ for $1 \le i \le k$, $1 \le j \le m(i)$. For each i, j, let K_{ij} be a compact subset of G_{ij}. Then the K_{ij} are positively separated since they are a finite collection of disjoint compact sets. Therefore, for some $\eta > 0$, the $\overline{K_{ij}^{(\eta)}}$ are disjoint. Since $\Gamma_n \to \Gamma$ and $\#(\Gamma_n) \le M$, for n sufficiently large $\Gamma_n \cap \overline{K_{ij}^{(\eta)}}$ has a connected component whose diameter approaches the diameter of K_{ij} except for at most M values of i, j. I.e., except for at most M values of i, j, there is a connected component T_{nij} of $\Gamma_n \cap \overline{K_{ij}^{(\eta)}}$ such that for every $\epsilon > 0$ there exists $N > 0$ with $|T_{nij}| > |K_{ij}| - \epsilon$ and $\delta_n < \eta$ for all $n \ge N$. Hence, by Proposition 3, for all $n \ge N$

$$\begin{aligned}\mathcal{M}^1_{\delta_n}(\Gamma_n) &\ge \mathcal{M}^1_{\delta_n}(\Gamma_n \cap \bigcup_{i,j} \overline{K_{ij}^{(\eta)}}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{m(i)} \mathcal{M}^1_{\delta_n}(\Gamma_n \cap \overline{K_{ij}^{(\eta)}}) \\ &\ge \sum_{i=1}^{k} \sum_{j=1}^{m(i)} (|K_{ij}| - \epsilon) - M(\max_{i,j}\{|K_{ij}|\})\end{aligned}$$

and so

$$\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n) \ge \sum_{i=1}^{k} \sum_{j=1}^{m(i)} |K_{ij}| - M(\max_{i,j}\{|K_{ij}|\})$$

Taking the sup over the compact sets K_{ij} gives

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n) &\ge \sup_{K_{ij}}\{\sum_{i=1}^{k} \sum_{j=1}^{m(i)} |K_{ij}| - M(\max_{i,j}\{|K_{ij}|\})\} \\ &= \sum_{i=1}^{k} \sum_{j=1}^{m(i)} \|C_i(t_{i,j-1}) - C_i(t_{ij})\| - M(\max_{i,j}\{\|C_i(t_{i,j-1}) - C_i(t_{ij})\|\})\end{aligned}$$

Then, taking the sup over the t_{ij} gives

$$\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n) \ge \sum_{i=1}^{k} \mathcal{H}^1(C_i)$$

since $M < \infty$ and $\max_{i,j}\{\|C_i(t_{i,j-1}) - C_i(t_{ij})\|\} \to 0$ as $\max_{i,j}\{\|t_{i,j-1} - t_{ij})\|\} \to 0$. Finally, letting $k \to \infty$ gives

$$\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Gamma_n) \ge \mathcal{H}^1(\Gamma) = \mathcal{M}^1(\Gamma)$$

which proves (i).

To show (ii), take $\Gamma_n = \Gamma$. From Theorem 8, $\mathcal{M}^1(\Gamma) = \mathcal{H}^1(\Gamma)$ so that in particular $\lim_{\delta\to 0} \mathcal{M}^1_\delta(\Gamma) = \mathcal{M}^1(\Gamma)$ exists. Hence, for every sequence $\delta_n \to 0$, condition (ii) is satisfied by taking $\Gamma_n = \Gamma$. $\square$

Note that Theorem 9 is not true in general if the bound on the number of connected components is dropped. For example, let $r_1, r_2, \ldots$ denote an enumeration of the rationals between 0 and 1. Take $\Gamma_n = \{(r_i, 0) : 1 \le i \le n\}$ and $\delta_n = 1/n^2$. Then $\Gamma_n \to \Gamma = \{(x, 0) : 0 \le x \le 1\}$, but $\mathcal{M}^1_{\delta_n}(\Gamma_n) \le \frac{1}{2}\pi n \delta_n \to 0$ while $\mathcal{M}^1(\Gamma) = 1$. However, we conjecture that the restriction on the number of connected components can be dropped if we impose the additional assumption that $d_H(\Gamma_n, \Gamma)/\delta_n \to 0$ as $n \to \infty$.

4. An Existence Theorem. In this section we will treat the question of the existence of a minimizing pair (f, Γ) for E. We have already developed some results for the cost associated strictly with the boundary so in this section we will be focusing on the function f. Since it may be desirable to introduce other costs associated with the boundary, we will state assumptions required on the boundaries in order to treat the remainder of the problem rather than quote results from the last section. We mention here however that these assumptions are satisfied by the definitions given in Section 2. Also, we will generalize the functional E. We will use the following set of assumptions on the space of boundaries.

A1 The space of boundaries is contained in the set of nonempty closed sets in $\mathbf{R}^2$.

A2 With respect to the topology induced by the Hausdorff metric on the space of boundaries $\nu(\cdot)$ is a nonnegative lower semicontinuous, coercive functional. (I.e. ν bounded sets are compact.)

We now generalize the functional E somewhat in anticipation of other applications. Henceforth E is defined by,

$$E(f, \Gamma) = \int_{\Omega\setminus\Gamma} \Phi(g, f, D^{\alpha_1} f, D^{\alpha_2} f, \ldots, D^{\alpha_s} f) + \nu(\Gamma)$$

and, for convenience we introduce the notation,

$$J(f, \Gamma) = \int_{\Omega\setminus\Gamma} \Phi(g, f, D^{\alpha_1} f, D^{\alpha_2} f, \ldots, D^{\alpha_s} f)$$

$g \in L^\infty(\Omega)$. s is a positive integer. Each α_i is a fixed multi-index, using the notation of [20]. f belongs to the subspace of functions in $L^{p_0}(\Omega\backslash\Gamma)$ whose distributional derivative $D^{\alpha_i}f$ exists as an $L^{p_i}(\Omega\backslash\Gamma)$ function, where each p_i satisfies $1 \le p_i < \infty$ for all $1 \le i \le s$. We will denote this space of functions by $\mathcal{D}(\Omega\backslash\Gamma)$. The following describes the assumptions on Φ.

A3 Φ is a nonnegative real function on $\mathbf{R}^{2+s}$ such that for any fixed domain $\Omega' \subset \Omega$ and fixed $g \in L^\infty(\Omega)$ the functional $\int_{\Omega'} \Phi(g, f, v_1, v_2, \ldots, v_s)$ is a lower semicontinuous, coercive functional on $L^{p_0}(\Omega') \times L^{p_1}(\Omega') \times \ldots \times L^{p_s}(\Omega')$ with respect to the weak (product) topology. Furthermore $\int_\Omega \Phi(g, 0, 0, \ldots, 0) < \infty$.

We note that $(g-f)^2 + v_1^2 + v_2^2$ is such a function with $p_0 = p_1 = p_2 = 2$. The formulation presented in the introduction satisfies these conditions with $\mathcal{D}(\Omega\backslash\Gamma) = W^{1,2}(\Omega\backslash\Gamma)$.

We now introduce a notion of convergence on sequences of pairs $\{(f_n, \Gamma_n)\}$. $(f_n, \Gamma_n) \to (f, \Gamma)$ will imply $\Gamma_n \to \Gamma$ in the topology induced by the Hausdorff metric. Now, for each n if $f_n \in L^p(\Omega\backslash\Gamma_n)$ let $\widehat{f_n} \in L^p(\Omega\backslash\Gamma)$ be defined by extending f_n to Ω, setting it to zero on Γ_n and then restricting it to $\Omega\backslash\Gamma$. By $(f_n, \Gamma_n) \to (f, \Gamma)$ we mean $\Gamma_n \to \Gamma$ in the topology induced by the Hausdorff metric and $\widehat{f_n} \to f$ weakly in $L^{p_0}(\Omega\backslash\Gamma)$ and $\widehat{D^{\alpha_i} f_n} \to D^{\alpha_i} f$ weakly in $L^{p_i}(\Omega\backslash\Gamma)$ for each $1 \le i \le s$.

Lemma 4. *Under assumptions A1, A2 and A3 we can for any E bounded sequence $\{(f_n, \Gamma_n)\}$ extract a subsequence (also denoted $\{(f_n, \Gamma_n)\}$) such that for some boundary Γ and some $f \in \mathcal{D}(\Omega\backslash\Gamma)$, $(f_n, \Gamma_n) \to (f, \Gamma)$.*

Proof: Assume the conditions of the Lemma and suppose we are given an E bounded sequence. We can assume there is some Γ such that $\Gamma_n \to \Gamma$ since otherwise by assumption A2 we can first extract a subsequence and find a boundary with this property. Since the sequence is E bounded we can conclude from A3 that the sequence $\{\int_{\Omega\backslash\Gamma} \Phi(g, \widehat{f_n}, \widehat{D^{\alpha_1} f_n}, \ldots, \widehat{D^{\alpha_s} f_n})\}$ is bounded. Hence, by A3, we can find functions $f \in L^{p_0}(\Omega\backslash\Gamma), v_1 \in L^{p_1}(\Omega\backslash\Gamma), \ldots, v_s \in L^{p_s}(\Omega\backslash\Gamma)$ and a subsequence (which we still denote the same way) such that, $\widehat{f_n} \to f$ weakly in $L^{p_0}(\Omega\backslash\Gamma)$ and $\widehat{D^{\alpha_i} f_n} \to v_i$ weakly in $L^{p_i}(\Omega\backslash\Gamma)$ for each $1 \le i \le s$. We claim that $f \in \mathcal{D}$ and $D^{\alpha_i} f = v_i$.

Let g be any test function in $\Omega\backslash\Gamma$, i.e. $g \in C_o^\infty(\Omega\backslash\Gamma)$. Consider the subsequence extracted above. Since $d(\text{supp}(g), \Gamma) > 0$ (using A1) it follows that for n sufficiently large $g \in C_o^\infty(\Omega\backslash\Gamma_n)$ and $\widehat{f_n} = f_n$ on supp(g) for any f_n defined on $\Omega\backslash\Gamma_n$. Thus along the subsequence we have,

$$\begin{aligned}\int_{\Omega\backslash\Gamma} v_i g &= \lim_{n\to\infty} \int_{\Omega\backslash\Gamma} \widehat{D^{\alpha_i} f_n} g \quad = \lim_{n\to\infty} \int_{\Omega\backslash\Gamma_n} D^{\alpha_i} f_n g \\ &= -\lim_{n\to\infty} \int_{\Omega\backslash\Gamma_n} f_n D^{\alpha_i} g = -\lim_{n\to\infty} \int_{\Omega\backslash\Gamma} \widehat{f_n} D^{\alpha_i} g \\ &= -\int_{\Omega\backslash\Gamma} f D^{\alpha_i} g\end{aligned}$$

We conclude from this that $D^{\alpha_i} f = v_i$ and hence $f \in \mathcal{D}(\Omega\backslash\Gamma)$. □

Corollary. *If the space of boundaries is the space of closed sets in $\overline{\Omega}$ then for any J bounded sequence $\{(f_n, \Gamma_n)\}$ we can extract a subsequence (also denoted $\{(f_n, \Gamma_n)\}$) such that for some boundary Γ and some $f \in \mathcal{D}(\Omega\backslash\Gamma)$, $(f_n, \Gamma_n) \to (f, \Gamma)$.*

Proof: For this case Theorem 1 substitutes for A2, yeilding a Γ and a subsequence such that $\Gamma_n \to \Gamma$. The rest of the proof is the same. □

Lemma 5. *Let $\{(f_n, \Gamma_n)\}$ be any E bounded sequence such that $(f_n, \Gamma_n) \to (f, \Gamma)$, then under assumptions A1, A2 and A3*

$$E(f, \Gamma) \le \liminf_{n\to\infty} E(f_n, \Gamma_n)$$

Proof: Let Γ^ϵ be a closed ϵ neighbourhood of Γ, i.e. a closed neighbourhood of Γ such that $r(\Gamma^\epsilon, \Gamma) \le \epsilon$ and define,

$$E_\epsilon(f, \Gamma') = \int_{\Omega\backslash\Gamma^\epsilon\cup\Gamma'} \Phi(g, f, D^{\alpha_1} f, \ldots, D^{\alpha_s} f) + \nu(\Gamma')$$

For n sufficiently large ($\ge N$ say), $\Gamma_n \subset \Gamma^\epsilon$ and since $\Gamma \subset \Gamma^\epsilon$ we get $\widehat{D^{\alpha_i} f_n}|_{\Omega\backslash\Gamma^\epsilon\cup\Gamma_n} = D^{\alpha_i} f_n|_{\Omega\backslash\Gamma^\epsilon}$. Hence the sequence $\{D^{\alpha_i} f_n|_{\Omega\backslash\Gamma^\epsilon\cup\Gamma_n}\}_{n\ge N}$ converges weakly to $D^{\alpha_i} f|_{\Omega\backslash\Gamma^\epsilon}$ in $L^{p_i}(\Omega\backslash\Gamma^\epsilon)$ and similarly $\{f_n|_{\Omega\backslash\Gamma^\epsilon\cup\Gamma_n}\}_{n\ge N}$ converges weakly to $f|_{\Omega\backslash\Gamma^\epsilon}$ in $L^{p_0}(\Omega\backslash\Gamma^\epsilon)$. We can now write

$$\begin{aligned}\liminf_{n\to\infty} E_\epsilon(f_n, \Gamma_n) &\ge \liminf_{n\to\infty} \int_{\Omega\backslash\Gamma^\epsilon\cup\Gamma_n} \Phi(g, f_n, D^{\alpha_1} f_n, \ldots, D^{\alpha_s} f_n) + \liminf_{n\to\infty} \nu(\Gamma_n) \\ &\ge \int_{\Omega\backslash\Gamma^\epsilon} \Phi(g, f, D^{\alpha_1} f, \ldots, D^{\alpha_s} f) + \nu(\Gamma) \\ &= E_\epsilon(f, \Gamma)\end{aligned}$$

where the second inequality follows from A_2 and lower semicontinuity of $\int \Phi$ in the weak topology on $\mathcal{D}(\Omega\backslash\Gamma^\epsilon)$. From the nonnegativity of Φ and the fact that Γ is closed we conclude $\sup_{\epsilon>0} J_\epsilon(\cdot) = J(\cdot)$ and hence

$$\liminf_{n\to\infty} E(f_n, \Gamma_n) \ge E(f, \Gamma) \qquad \square$$

Theorem 10. *Under assumptions A1, A2 and A3 (and in particular letting ν be defined as in Section 2), there exists a minimizing pair (f, Γ) for the functional E.*

Proof: Apply Lemma 4 to a minimizing sequence, then apply Lemma 5. □

5. Application to Variational Problems in Vision. In this section we apply some results of the previous sections to the variational problem discussed in the introduction. As before, g represents an observed image defined on a bounded open set $\Omega \subset \mathbf{R}^2$, f is the reconstructed image, and Γ are the boundaries of the image. In the variational approach, f and Γ are obtained by minimizing the cost functional (1) or (2). Normally, g is assumed to be in $L^\infty(\Omega)$, Γ is a closed subset of

$\overline{\Omega}$, and f is in the Sobolev space $W^{1,2}(\Omega\backslash\Gamma)$. Under certain regularity assumptions, a number of interesting results concerning the nature of the minimizing f and Γ have been obtained [5,16,18,23]. Also, the existence of a minimizing pair (f,Γ) for various versions of the problem has been shown [1,2,17]. We have included the essence of [17] in Section 4.

Here we are concerned with the behavior of solutions to discrete versions of the problem as the lattice spacing tends to zero. Specifically, we are interested in whether or not solutions to the discrete problem converge to a solution of the continuous problem. It seems that this may not necessarily be the case for the discrete problem of (3). For example, consider the segmentation problem (2) where f is required to be piecewise constant. Take $\Omega = (0,1)\times(0,1)$, $g(x,y) = 0$ for $x < y$ and $g(x,y) = 1$ otherwise, and $4\sqrt{2}c_3 < c_1 < 8c_3$. Then the optimal solution to the discrete problem with sufficiently small lattice spacing seems to be $\Gamma = \emptyset$, while the optimal solution to the continuous problem seems to be $\Gamma = \{(x,x) : 0 \le x \le 1\}$.

This problem appears to be a result of the possible strict lower semicontinuity of the length of curves with respect to the Hausdorff metric. E.g., in this case, if $\Gamma = \{(x,x) : 0 \le x \le 1\}$ and Γ_n is the discrete approximation to Γ with lattice spacing $1/n$, then $\Gamma_n \to \Gamma$ but $L(\Gamma) = \sqrt{2}$ while $\lim_{n\to\infty} L(\Gamma_n) = 2$. The notion of length in the discrete case does not coincide in the continuum limit with the usual measure of length.

As previously mentioned, it may be possible to resolve this problem by modifying one or more of the topology on the space of boundaries, the cost functional, and the discretization process. Here we consider the use of Minkowski content for the cost of the boundaries and propose a modified discrete version of the problem. Specifically, given an observed image $g \in L^\infty(\Omega)$ we consider the problem of minimizing

$$E_\delta(f,\Gamma) = c_1 \iint_\Omega (f-g)^2\,dx\,dy + c_2 \iint_{\Omega\backslash\Gamma} \|\nabla f\|^2\,dx\,dy + c_3\mathcal{M}_\delta(\Gamma)$$

with Γ a closed subset of $\overline{\Omega}$ and $f \in W^{1,2}(\Omega\backslash\Gamma)$. For the discrete version of the problem with lattice spacing $\frac{1}{n}$, we simply restrict Γ to be composed of a union of closed lattice squares whose corners lie on $\frac{1}{n}\mathbb{Z}^2$. However, we still take g and f to be defined on the continuous domains Ω and $\Omega\backslash\Gamma$ respectively. Hence, we have only incorporated a partial discretization, i.e. we have only discretized Γ. However, the primary difficulty in numerical solutions is to properly deal with the boundary. For a fixed Γ, the minimization reduces to a standard variational problem whose Euler-Lagrange equations can be solved by standard algorithms for partial differential equations.

We now give some results concerning the problem of minimizing E_δ.

Theorem 11. *For every $\delta > 0$, there exists a pair (f_δ,Γ_δ) which minimizes E_δ.*

Proof: Since we have shown that $\mathcal{M}_\delta$ is continuous (Theorem 6), the existence proof given in Section 4 can be applied. □

Note that for any bounded Γ, $\mathcal{M}_\delta(\Gamma) < \infty$. Hence, a minimizing boundary may quite possibly have nonzero Lebesgue measure.

The next theorem establishes the desirable property of discrete to continuous convergence for E_δ with a fixed $\delta > 0$. We will use the same notion of convergence as used in Section 4. For $f \in W^{1,2}(\Omega\backslash\Gamma_n)$, f and its weak first order derivatives $D_{x_i}f$, $i = 1,2$, can be considered as functions in $L^2(\Omega\backslash\Gamma)$ by defining them to be zero on Γ_n and restricting. By $(f_n,\Gamma_n) \to (f,\Gamma)$ we mean that $\Gamma_n \to \Gamma$ in Hausdorff metric and that for the modified functions $\widehat{f_n} \to f$, and $\widehat{D_{x_i}f_n} \to D_{x_i}f$, $i = 1,2$ weakly in $L^2(\Omega\backslash\Gamma)$.

Theorem 12. *Let $(f^*_{\delta,n},\Gamma^*_{\delta,n})$ denote a minimizing pair for $E_{\delta,n}$, i.e. for the discrete problem E_δ with lattice spacing $\frac{1}{n}$. Then there exists a subsequence (still denoted $(f^*_{\delta,n},\Gamma^*_{\delta,n})$) and a pair (f_δ,Γ_δ) such that $(f^*_{\delta,n},\Gamma^*_{\delta,n}) \to (f_\delta,\Gamma_\delta)$ and (f_δ,Γ_δ) minimizes E_δ.*

Proof: The existence of a pair (f_δ,Γ_δ) with $(f^*_{\delta,n},\Gamma^*_{\delta,n}) \to (f_\delta,\Gamma_\delta)$ follows from the corollary to lemma 4. We only need show that (f_δ,Γ_δ) minimizes E_δ.

Let $(f^*_\delta,\Gamma^*_\delta)$ minimize E_δ. For each n, let Λ_n be obtained from Γ^*_δ by taking the smallest cover of Γ^*_δ using the closed lattice squares of the lattice with spacing $\frac{1}{n}$. Let h_n be the restriction of f^*_δ to $\Omega\backslash\Lambda_n$. From Theorem 6, $\lim_{n\to\infty} E_\delta(h_n,\Lambda_n) = E_\delta(f^*_\delta,\Gamma^*_\delta)$. Then, by the lower-semicontinuity of E_δ and the optimality of $(f^*_{\delta,n},\Gamma^*_{\delta,n})$ for the discrete problem with lattice spacing $\frac{1}{n}$, we have

$$\begin{aligned} E_\delta(f_\delta,\Gamma_\delta) &\le \liminf_{n\to\infty} E_\delta(f_{\delta,n},\Gamma_{\delta,n}) \le \liminf_{n\to\infty} E_\delta(h_n,\Lambda_n) \\ &= \lim_{n\to\infty} E_\delta(h_n,\Lambda_n) = E_\delta(f^*_\delta,\Gamma^*_\delta) \end{aligned}$$

Therefore, $E_\delta(f_\delta,\Gamma_\delta) = E_\delta(f^*_\delta,\Gamma^*_\delta)$ so that (f_δ,Γ_δ) minimizes E_δ. □

A natural question at this point concerns the behavior of $(f^*_\delta,\Gamma^*_\delta)$ as $\delta \to 0$. One would like $(f^*_\delta,\Gamma^*_\delta)$ to converge to a minimizing solution of the original cost functional E. We can show a convergence result if the number of connected components of the admissible boundaries is uniformly bounded. Following Section 2, we let the cost term for the boundaries be

$$\nu_\delta(\Gamma) = \mathcal{M}^1_\delta(\Gamma) + F(\#(\Gamma))$$

where $F(k) = 0$ for $k \le M < \infty$ and $F(k) = \infty$ for $k > M$. Let E^M_δ denote the cost functional with the above boundary term, and let E^M denote the cost functional whose boundary term is

$$\nu(\Gamma) = \mathcal{M}^1(\Gamma) + F(\#(\Gamma))$$

By Theorem 8, $\mathcal{M}^1(\Gamma)$ in the equation for $\nu(\Gamma)$ could equivalently be replaced by $\mathcal{H}^1(\Gamma)$. For these variational problems, we have the following convergence result, which essentially follows from the result on the Γ-convergence of Minkowski content (Theorem 9).

THEOREM 13. *Let $(f^*_\delta, \Gamma^*_\delta)$ denote a minimizing pair for E^M_δ, and let $\delta_n \to 0^+$. Then there is a subsequence (which we still denote by δ_n) such that $(f^*_{\delta_n}, \Gamma^*_{\delta_n}) \to (f,\Gamma)$ for some (f,Γ) which minimizes E^M. Furthermore, $E^M_{\delta_n}(f^*_{\delta_n}, \Gamma^*_{\delta_n}) \to E^M(f,\Gamma)$.*

Proof: The existence of a pair (f,Γ) with $(f^*_{\delta_n}, \Gamma^*_{\delta_n}) \to (f,\Gamma)$ follows from corollary to lemma 4. We need to show that (f,Γ) minimizes E_δ and that $E^M_{\delta_n}(f^*_{\delta_n}, \Gamma^*_{\delta_n}) \to E^M(f,\Gamma)$.

This follows from Theorem 9 on the epi-convergence of Minkowski content in the case of a bounded number of connected components. Specifically,

$$E^M(f,\Gamma) \le \liminf_{n\to\infty} E^M_{\delta_n}(f_{\delta_n}, \Gamma_{\delta_n}) \le \liminf_{n\to\infty} E^M_{\delta_n}(f^*, \Gamma^*) = E^M(f^*,\Gamma^*)$$

so that (f,Γ) minimizes E^M.

Also, we have

$$E^M(f,G) = \limsup_{n\to\infty} E^M_{\delta_n}(f,\Gamma) \ge \limsup_{n\to\infty} E^M_{\delta_n}(f_{\delta_n}, \Gamma_{\delta_n})$$

Thus,

$$\limsup_{n\to\infty} E^M_{\delta_n}(f,\Gamma) \le E^M(f,\Gamma) \le \liminf_{n\to\infty} E^M_{\delta_n}(f,\Gamma)$$

and so

$$E^M(f,\Gamma) = \lim_{n\to\infty} E^M_{\delta_n}(f,\Gamma) \quad \square$$

Finally, we give a result concerning the convergence of solutions when the lattice spacing and δ are simultaneously allowed to go to zero. The following theorem guarantees convergence of a subsequence to a solution of the continuous problem if $\delta \to 0$ at a rate slower than the lattice spacing.

THEOREM 14. *Let $\delta_n > 0$ with $\delta_n \to 0$ and let $(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n})$ denote a minimizing pair for $E^M_{\delta_n,n}$, i.e. for the discrete problem $E^M_{\delta_n}$ with lattice spacing $\frac{1}{n}$. If $n\delta_n \to \infty$ as $n \to \infty$ then there exists a subsequence (still denoted $(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n})$) and a pair (f,Γ) such that $(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n}) \to (f,\Gamma)$ and (f,Γ) minimizes E^M.*

Proof: As before, the existence of a pair (f,Γ) with $(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n}) \to (f,\Gamma)$ follows from corollary to lemma 4 and so we need to show that (f,Γ) minimizes E^M.

Let (f^*,Γ^*) minimize E^M, and for each n let (h_n, Λ_n) be obtained from (f^*,Γ^*) as in the proof of Theorem 12. Namely, Λ_n is the smallest cover of Γ^* using lattice squares of the lattice with spacing $\frac{1}{n}$, and h_n is the restriction of f^* to $\Omega\backslash\Lambda_n$. Then using Theorem 9 and the optimality of $(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n})$ we have

$$E^M(f,\Gamma) \le \liminf_{n\to\infty} E^M_{\delta_n}(f^*_{\delta_n,n}, \Gamma^*_{\delta_n,n}) \le \liminf_{n\to\infty} E^M_{\delta_n}(h_n, \Lambda_n)$$

Since Λ_n is the minimal cover of Γ^* on the lattice with spacing $\frac{1}{n}$, we have $\Lambda_n \subset (\Gamma^*)^{(\frac{\sqrt{2}}{n})}$ so that

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{M}^1_{\delta_n}(\Lambda_n) &\le \liminf_{n\to\infty} \frac{\mu((\Gamma^*)^{(\delta_n + \frac{\sqrt{2}}{n})})}{2\delta_n} \\
&= \liminf_{n\to\infty} \frac{\mu((\Gamma^*)^{(\delta_n + \frac{\sqrt{2}}{n})})}{2(\delta_n + \frac{\sqrt{2}}{n})} \frac{\delta_n + \frac{\sqrt{2}}{n}}{\delta_n} \\
&= \lim_{n\to\infty} \frac{\mu((\Gamma^*)^{(\delta_n + \frac{\sqrt{2}}{n})})}{2(\delta_n + \frac{\sqrt{2}}{n})}\left(1 + \frac{\sqrt{2}}{n\delta_n}\right) = \mathcal{M}^1(\Gamma^*)
\end{aligned}$$

It follows that

$$\liminf_{n\to\infty} E^M_{\delta_n}(h_n, \Lambda_n) \le E^M(f^*,\Gamma^*)$$

Therefore, $E^M(f,\Gamma) \le E^M(f^*,\Gamma^*)$ and (f,Γ) minimizes E^M. $\square$

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EXTENSION PROBLEMS UNDER THE DISPLACEMENT STRUCTURE REGIME*

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Abstract. This paper presents a unified approach to certain function-theoretic and matrix extension problems, which is based on the recently developed concept of matrices with a generalized displacement structure. We show that a variety of function extension problems, including Pade approximation and Caratheodory extension, are equivalent to the problem of extending a finite matrix with a given displacement structure into a larger (possibly infinite) matrix with the same structure. Moreover, such matrix extension problems can be efficiently solved by the same layer-peeling procedure that is used to determine the triangular factorization of matrices with a generalized displacement structure.

In general, the matrix extension problem mentioned above has many solutions. The set of all feasible extensions can be conveniently characterized in terms of the cascade model that is constructed by the layer peeling procedure: a particular extension is obtained by terminating the cascade model (which is the same for all extensions) with an arbitrary termination. Most often, the desired solution corresponds to a matrix extension that has finite rank. This is so, for instance when the Pade approximation is required to have minimal degree, or when the Schur-Caratheodory extension is required to have minimal H^∞ norm (but not so for maximum-entropy extension problems). We show that such finite-rank extensions correspond to choosing lossless terminations for the cascade model, and that the resulting extended matrix is intimately connected with the recently developed notion of generalized Bezoutians.

1. Introduction. Two classical extension problems frequently arise in a variety of signal processing and linear system theory applications. These are:

(i) *The Caratheodory-Fejer problem*: Given $c_1, c_2, \ldots, c_m$ construct a power-series

$$c(z) := 1 + \sum_{t=1}^{\infty} c_t z^t$$

such that $Re\ c(z) \geq 0$ for all $|z| < 1$.

(ii) *The Padé (partial realization) problem*: Given $h_0, h_1, \ldots, h_m$ construct a power-series

$$h(z) := \sum_{t=0}^{\infty} h_t z^t$$

such that $h(z)$ is rational of degree r, i.e., $h(z) = b(z)/a(z)$ with $deg\ b(z) \leq deg\ a(z) = r$.

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