REPRESENTATION THEORY FOR LINEAR INFINITE DIMENSIONAL CONTINUOUS TIME SYSTEMS

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0. INTRODUCTION AND SURVEY OF RESULTS

Representation theory for finite dimensional systems has been the subject of a great deal of discussion in recent years. In the case of linear systems defined over fields an account of the theory can be found in the books of BROCKETT and KALMAN-FALB-ARBIB and in the case of systems defined over rings in the book of EILENBERG, the papers of ROUCHALEAU-WYMAN-KALMAN, ROUCHALEAU-WYMAN and the thesis of JOHNSTON. For a lucid survey of results on systems defined over commutative rings, see SONTAG.

Over the past three years a body of results for linear infinite dimensional systems has been developed. Here two lines of enquiry have been pursued. On the one hand, BARAS, BROCKETT and especially FUHRMANN have developed a theory where the underlying input space is taken to be a Hilbert space (ℓ^2 or L^2) and the rich theory of restricted shift operators on ℓ^2 or L^2 is exploited to develop a theory of realization of infinite dimensional systems. On the other hand, AUBIN-BENSOUSSAN, BENSOUSSAN-DELFOUR-MITTER, KALMAN-MATSUO, KAMEN and others have tried to develop a theory of infinite dimensional which is very similar to the finite dimensional theory. For earlier work in this direction see BALAKRISHNAN and KALMAN-HAUTUS. The basic difference between these two approaches seems to be the following: If one requires that the realizations obtained be canonical (in the same sense as in the finite dimensional case, that is, reachable and completely distinguishable) and to obtain a state-space topological isomorphism theorem then it is necessary to work with a sufficiently large input function space (a space which contains sums of dirac impulses). However, it is not at all clear that the finite dimensional conceptual framework is the correct one for infinite dimensional systems. The other possibility is to fix the input space as ℓ^2 (in the discrete-time case) or L^2 and use the deep theory of Hardy spaces, invariant subspaces and canonical models. For excellent accounts of the possibilities using the latter approach see the papers by

Baras, Fuhrmann and DeWilde in this volume.

In this paper we try to show what can be done using the first approach. We present two possibilities.

Let U and Y be reflexive, separable Banach spaces which are the space of input values and output values respectively. An <u>input function</u> is a function from $(-\infty, 0)$ into U and an output function is a function from $(0, \infty)$ into Y. Suppose we are given a mapping f from the input function space to Y which is linear, continuous (in an appropriate topology) and time-invariant. Our problem of interest is to find a <u>canonical internal representation</u>, that is, a state space X (a topological vector space) and linear mappings $F:X \to X$, $G:U \to X$ and $H:X \to Y$ such that <u>the input-output map f is realized by</u>

$$\text{(A)} \qquad y(t) = Hx(t), \quad t \geq 0$$

and x(t) satisfies (in some appropriate sense) the differential equation

$$\text{(B)} \qquad \begin{cases} \dfrac{dx}{dt} = Fx + G\tilde{u}(t), \ t \in (-\infty, 0) \\ \dfrac{dx}{dt} = Fx, \quad t \geq 0. \end{cases}$$

Furthermore, it is required that the <u>reachability</u> operator defined by (B) be surjective and that the mapping $x(0) \longmapsto y(\cdot)$ from X into the output function space be injective (<u>complete distinguishability</u>).

In the first instance we consider a very large input function space. Let Φ be the space of real infinitely differentiable functions on $(-\infty, 0]$, $\Phi = \mathcal{E}(-\infty, 0]$, then the space of input functions will be $L_1(\Phi; U)$ the space of nuclear operators from $\Phi \to U$. More concretely, an input function $\tilde{u}$ will be described by

$$\tilde{u} = \sum_n \lambda_n \mu_n u_n$$

where μ_n is a bounded sequence of Φ' (which is identical to the space of distributions with compact support on $(-\infty, 0]$), u_n is a bounded sequence of U, and λ_n are real numbers such that $\sum_n |\lambda_n| < +\infty$. We will show that Φ' is a commutative algebra and $L_1(\phi; U)$ is a locally convex Hausdorff space and a topological ϕ'-module. The space $L_1(\Phi; U)$ is very large and contains functions in $L^1(-\infty, 0; U)$ with compact support and Dirac measures.

On $L_1(\Phi; U)$ a translation operator T(t) is defined, which is a strongly continuous semi group, whose infinitesimal generator is <u>bounded</u>. We define the <u>Hankel operator</u>

$$\text{(1)} \qquad H\tilde{u}(t) = f\left(T(t)\tilde{u}\right) \qquad t \geq 0$$

where $f \in L\left(L_1(\Phi; U); Y\right)$ and is the input-output map. H commutes under shifts.

The output function space will be $\Gamma = C^\infty(0, +\infty); Y)$, i.e. the space of infinitely differentiable functions from $[0, +\infty) \to Y$. It can be equipped with a structure of a Fréchet space and of a topological Φ'-module and $H \in L\left(L_1(\Phi;\ U);\Gamma\right)$ will be a homomorphism of Φ'-modules between $L_1(\Phi;\ U)$ and Γ.

A canonical internal representation can be constructed using Nerode equivalence. The state space is

$$X = L_1(\Phi;\ U)/\mathrm{Ker}\ H \tag{2}$$

and X is a locally convex Hausdorff space and a topological Φ'-module. We can define a strongly continuous semi-group on X by setting

$$\Gamma(t)[\tilde{u}] = [T(t)\tilde{u}] \tag{3}$$

and $\Gamma(t)$ has a <u>bounded</u> infinitesimal generator F. The mappings G and H are defined by

$$Gu = [u\delta_o],\ \text{where } \delta_o \text{ is the dirac measure concentrated at zero.} \tag{4}$$

$$H[\tilde{u}] = f(\tilde{u}). \tag{5}$$

(4) explains why we need to include dirac measures (with values in U) in the input function space.

Knowing $x(0) = [\tilde{u}]$, we can define the state of the system

$x(t)$ for $t \geq 0$ by

$x(t) = \Gamma(t)\ x(0).$

In this first part we also discuss the concept of dual systems. It will then be necessary to work with weak topologies, but we will get dual algebraic as well as topological properties. However, although the Hankel operators will be duals of each other, the canonical state spaces will not be exactly duals of each other (this is a consequence of infinite dimensionality).

We then discuss the case where the input function space is taken to be $\left(H^m(-\infty,\ 0;\ U')\right)'$ the dual of the Sobolev space of order m with values in U', which is a Hilbert space if U is a Hilbert space. The state space is then a Hilbert space. In this case, we can describe the evolution of the state in terms of an operational differential equation and also prove a state-space isomorphism theorem.

1. <u>APPROACH USING DISTRIBUTION THEORY</u>

1.1 <u>The Input Function Space Φ</u>

We denote by Φ or $E(-\infty,\ 0]$ the space of infinitely differentiable functions on $(-\infty,\ 0]$. On Φ we define the countable family of semi-norms

$$||\phi||_{m,k} = \sup_{-k\leq t\leq 0}\ \sup_{p\leq m}\ |\phi^{(p)}(t)|\ ,\quad \phi \in \Phi \tag{1.1}$$

which define a topology of a Fréchet space for Φ.

Let us denote by E the space of infinitely differentiable functions on $(-\infty,\ +\infty)$ provided with the countable family of semi norms

$$||\phi||_{m,k} = \sup_{-k\leq t\leq k}\ \sup_{p\leq m}\ |\phi^{(p)}(t)|\ ,\quad \phi \in E \tag{1.2}$$

Let ρ denote the restriction of functions in E on $(-\infty,\ 0]$, i.e.,

$$\rho\phi(t) = \phi(t) \qquad t \in (-\infty,\ 0]. \tag{1.3}$$

Clearly ρ is a continuous mapping from $E \to \Phi$. Furthermore we have

<u>Proposition 1.1</u>: $\Phi \approx E/\mathrm{ker}\ \rho$. If $\phi \in \Phi$, there exists $\tilde{\phi}$, an extension of ϕ, $\tilde{\phi} \in E$ such that

$$||\tilde{\phi}||_{j,k} \leq K_j\ \sup\ (1,\ ||\phi||_{j,k}) \qquad \forall\ j,k \tag{1.4}$$

where the K_j's are constants.

Let Φ' be the dual of Φ (regarded as a space of distributions). It is well known that

Φ' = set of distributions with compact support in $(-\infty,\ 0]$.

Moreover

<u>Proposition 1.2</u>: The space Φ' is isomorphic to a closed, subspace of E'. From this fact it follows that the topology on Φ' is the same as that induced by E'. One can also show

<u>Proposition 1.3</u>: The space Φ is reflexive.

As a topological space Φ' is the dual of a Fréchet space. Since its elements are distributions with compact support we can provide Φ' with a nice algebraic structure.

For two distributions with compact support if we define multiplication as convolution, then it can be easily shown that Φ' is a <u>commutative algebra</u> (with unit, the dirac distribution at the origin) and also an <u>integral domain</u>. Furthermore since the topology of Φ' is that induced from E', <u>convolution is continuous</u>.

For applications, one should note that finite sums of Dirac measures are contained in Φ'.

Let $L_1(\Phi;\ U)$ denote the space of nuclear operators from Φ into U (cf. TREVES, Chapter 47, for definitions and other facts). $L_1(\Phi;\ U)$ is a linear subspace of $L(\Phi;\ U)$.

On $L_1(\Phi;\ U)$ we can define an external multiplication as follows: for $\mu \in \Phi'$ and $\tilde{u} \in L_1(\Phi;\ u)$ we define

$$\mu * \tilde{u}(\phi) = \tilde{u}(\phi * \tilde{\mu}),\ \text{where } \tilde{\mu} \text{ is the distribution} \tag{1.5}$$

$$\tilde{\mu}(\phi) = \mu(\tilde{\phi}) \qquad \left(\tilde{\phi}(s) = \phi(-s)\right)$$

which has meaning since $\phi * \tilde{\mu} \in \Phi$. It can be shown that with this definition of external multiplication

Proposition 1.4: The space $L_1(\Phi;\ U)$ is a unitary topological Φ'-module.

Remarks: a) If $U = \mathbb{R}^n$, then $L_1(\Phi;\ U) = (\Phi')^n = L(\Phi;\ \mathbb{R}^n)$ which is a free unitary finitely generated Φ'-module. b) If Ω is a bounded open subset of $(-\infty, 0)$, then $L^1(\Omega;\ U) \subset L_1(\Phi;\ U)$ with continuous injection.

By definition

$$L_1(\Phi;\ U) = \Big\{\tilde{u} = \sum_n \lambda_n \mu_n u_n \,\Big|\, \mu_n \in \Phi',\ u_n \in U,\ u_n,\ \mu_n \text{ bounded}, \sum_n |\lambda_n| < +\infty\Big\}$$

For $\phi \in \Phi$ and $\tilde{u} \in L_1(\Phi;\ U)$, we define

(1.6) $$\tilde{\phi} * \tilde{u} = \sum_n \lambda_n (\tilde{\phi} * \mu_n) u_n, \text{ where}$$

$\tilde{\phi} * \mu_n \in \Phi'$ is defined by

$$\tilde{\phi} * \mu_n (h) = \mu_n(\phi * h).$$

Proposition 1.5: The mapping $\phi \longmapsto \tilde{\phi} * \tilde{u}$ is linear and continuous from Φ into $L_1(\Phi;\ U)$.

The semi group of translations

Let $\theta(-t)$, $t \geq 0$ be the semi-group of translations on Φ, defined by

(1.7) $$\theta(-t)\ \phi(s) \equiv \phi(-t+s) \qquad t \geq 0,\ \ s \leq 0.$$

It follows from TREVES, p. 286 that the mapping

(1.8) $t \to \theta(-t)\phi$ is infinitely differentiable from $[0, +\infty)$ into Φ, and

$$\frac{d^p}{dt^p}\ \theta(-t)\phi(s) = (-1)^p \phi^{(p)}(-t+s) \qquad s \leq 0,\ t \geq 0.$$

Therefore $\theta(-t)$ has an infinitesimal generator which is $-\frac{d}{dt}$.

Remark: The family $\theta(-t)$ is not an equicontinuous semi group on Φ. This would however be true if $\Phi = S(-\infty, 0]$ instead of $\mathcal{E}(-\infty, 0)]$.

Since $-\frac{d}{dt} \in L(\Phi;\ \Phi)$ we can define its transpose $D \in L(\Phi';\ \Phi')$. Let $\sigma(t)$ be the semi group of translations on Φ' defined by

(1.9) $$\sigma(t)\ \mu(\phi) = \mu\big(\theta(-t)\phi\big) \qquad \forall\ \phi \in \Phi,\ t \geq 0.$$

We have

Proposition 1.6: The mapping $t \to \sigma(t)\mu$ is infinitely differentiable from $[0, +\infty)$

into Φ' and

(1.10) $$\frac{d}{dt}\ \sigma(t)\mu = D\ \sigma(t)\mu = \sigma(t)\ D\mu, \quad \forall\ t \geq 0.$$

On $L_1(\Phi;\ U)$ we may define a semi group $T(t)$, $t \geq 0$ by setting

(1.11) $$T(t)\ \tilde{u}(\phi) = \tilde{u}\big(\theta(-t)\phi\big) = \sum_n \lambda_n \big(\sigma(t)\mu_n\big)\ (\phi)\ u_n$$

hence $T(t)\tilde{u}$ has the representation

(1.12) $$T(t)\tilde{u} = \sum_n \lambda_n\ \big(\sigma(t)\ \mu_n\big)\ u_n.$$

Proposition 1.7: The mapping $t \to T(t)\tilde{u}$ is infinitely differentiable from $[0, +\infty)$ into $L_1(\Phi;\ U)$.

Let us mention the useful formulas

(1.13) $$\sigma(t)\mu = \delta_{-t} * \mu$$

(1.14) $$T(t)\tilde{u} = \delta_{-t} * \tilde{u}$$

Remark: If $\Phi = S(-\infty, 0]$, then $\sigma(t)$ and $T(t)$ are equicontinuous semi groups of class C_0.

2. EXTERNAL REPRESENTATION OF LINEAR TIME-INVARIANT SYSTEMS

2.1 Notations and preliminary results

If $\Phi' \otimes U$ denotes the tensor product of Φ' and U, i.e., the subspace of $L(\Phi;\ U)$ of elements

$$\tilde{u} = \Sigma\ \mu_n\ u_n$$

where the sum is finite. Clearly $\Phi' \otimes U$ is dense in $L_1(\Phi;\ U)$. The topology induced on $\Phi' \otimes U$ is called the Π-topology, and provided with it, $\Phi' \otimes U$ is denoted $\Phi' \otimes_\Pi U$. An important result is the following (see TREVES, p. 438); the dual of $\Phi' \otimes_\Pi U$ is algebraically isomorphic to $B(\Phi';\ U)$, the space of bilinear continuous forms on $\Phi' \times U$. It follows from the density of $\Phi' \otimes_\Pi U$ into $L_1(\Phi;\ U)$ that

(2.1) $$\big(L_1(\Phi;\ U)\big)' \equiv B(\Phi';\ U).$$

Denoting by

(2.2) $$\Phi(-\infty, 0;\ U') = \{u_*(t) : (-\infty, 0) \to U' \mid t \longmapsto \langle u_*(t), u \rangle \in \Phi \quad \forall\ u \in U\}$$

we have

Theorem 2.1: $B(\Phi' : U)$ and $\Phi(-\infty, 0;\ U')$ are algebraically isomorphic.

With $L_1(\Phi;\ U)$ as our choice of input function space and Y being a separable reflexive Banach space (the space of output values), we get a representation for

$f \in \mathcal{L}\left(L_1(\Phi;\ U);\ Y\right)$. This is given in

Theorem 2.2: If $f \in \mathcal{L}\left(L_1(\Phi;\ U);\ Y\right)$, there exists a unique family $K(t)$ of operators from $U \to Y$ which satisfy

$$K(t) \in \mathcal{L}(U;\ Y), \quad \forall\, t \geq 0, \quad \|K(t)\| \leq C_k \quad \forall\, t \in [0,\ k] \tag{2.3}$$

$$\langle K(-t)u,\ y_*\rangle \in \Phi, \quad \forall\, u \in U,\ y_* \in Y', \tag{2.4}$$

such that

$$\langle f(\tilde{u}),\ y_*\rangle = \sum_n \lambda_n \int_{-\infty}^{0} \langle K(-t)u_n,\ y_*\rangle\, \mu_n(t)\, dt, \quad \forall\, y_* \in Y'. \tag{2.5}$$

Remarks: a) If $u(\cdot) \in L^1(\Omega;\ U)$, then

$$\langle f\left(u(\cdot)\right),\ y_*\rangle = \int_{-\infty}^{0} \langle K(-t)u(t),\ y_*\rangle\, dt \text{ and hence}$$

$$f\left(u(\cdot)\right) = \int_{-\infty}^{0} K(-t)u(t)\, dt. \tag{2.6}$$

b) If $\tilde{u} = \sum_n \delta_{t_n} u_n$ then

$$f\Big(\sum_n \delta_{t_n} u_n\Big) = \sum_n K(-t_n)\, u_n. \tag{2.7}$$

Definition 2.1: The operator $K(t)$ is called the kernel or impulse response of f.

2.2 Output functions space

Let Y be a reflexive separable Banach space, which will be called the space of output values, and will play a part parallel to U. We shall denote by

$$\Gamma = C^\infty[0,\ +\infty;\ Y)$$

the space of infinitely differentiable functions from $[0,\ +\infty)$ into Y. It is a Fréchet space for the topology defined by the family of semi norms

$$\|y(\cdot)\|_{mk} = \sup_{0 \leq t \leq k}\ \sup_{p \leq m} \|y^{(p)}(t)\|_Y. \tag{2.8}$$

We shall equip Γ with a structure of Φ'-module.

We first have

Proposition 2.1: If $y(\cdot) \in \Gamma$ and $\mu \in \Phi'$, there exists a unique $z(\cdot) \in \Gamma$ such that if $y_* \in Y'$ and if $\tilde{\phi}(t)$ denotes an element of $\mathcal{E}$ which is an extension of $\phi(t) = \langle y_*,\ y(t)\rangle$, then

$$\langle y_*,\ z(t)\rangle = \mu * \tilde{\phi}(t), \quad \forall\, t \geq 0. \tag{2.9}$$

Furthermore, for fixed u, the mapping

$$y(\cdot) \to z(\cdot) \qquad \text{from } \Gamma \to \Gamma$$

is continuous.

We shall set

$$z(\cdot) = \mu \cdot y(\cdot) \tag{2.10}$$

and call it the external product of $\mu \in \Phi'$ and $y(\cdot) \in \Gamma$. We have

Proposition 2.2: With the external multiplication (2.10), Γ becomes a unitary topological Φ'-module.

2.3 Structure of the external representation

The external representation of a linear invariant system is defined by a mapping $f \in \mathcal{L}\left(L_1(\Phi;\ U):\ Y\right)$. The mapping f can be extended to a mapping

$$\mathcal{H}: L_1(\Phi;\ U) \to \Gamma \quad \text{, by setting}$$

$$(\mathcal{H}\,\tilde{u})(t) = f\left(T(t)\tilde{u}\right), \qquad \forall\, t \geq 0. \tag{2.11}$$

The operator $\mathcal{H}$ will be called the Hankel operator of the system.

Proposition 2.3:

$$\mathcal{H} \in \mathcal{L}\left(L_1(\Phi;\ U);\ \Gamma\right). \tag{2.12}$$

Proposition 2.4: The operator $\mathcal{H}$ is an homomorphism of Φ'-modules between $L_1(\Phi;\ U)$ and Γ.

3. INTERNAL REPRESENTATION OF THE SYSTEM

An internal representation of the system corresponding to the Hankel operator $\mathcal{H}$ is a triple $(F,\ G,\ H)$, where $F: X \to X$, $G: U \to X$ and $H: X \to Y$ are linear operators, with G and H being continuous, such that F is the infinitesimal generator of a semi-group $\Gamma(t)$ on X which satisfies

$$K(t) = H\Gamma(t)G, \qquad \forall\, t \geq 0$$

(recall that $K(t)$ is the kernel (impulse response) of f). In this section we construct a canonical internal representation.

3.1 The state space

Let

$$X = L_1(\Phi;\ U)/\mathrm{Ker}\ \mathcal{H} \tag{3.1}$$

which will be called the state space. Since $\mathcal{H}$ is an homomorphism of Φ'-modules, and since $\mathcal{H}$ is continuous, $\mathrm{Ker}\ \mathcal{H}$ is a topological submodule of $L_1(\Phi;\ U)$. Therefore X

admits the structure of a Φ'-module with a multiplication defined by

$$(3.2)\qquad \mu\cdot[\tilde{u}] = [\mu * \tilde{u}]$$

where $[\tilde{u}]$ denotes an element of X, for which $\tilde{u}$ is a representative. Moreover X can be given the structure of a locally convex Hausdorff space, the topology of which is defined by the following basis of continuous semi norms

$$(3.3)\qquad \omega_\alpha([\tilde{u}]) = \inf_{\tilde{u}_1 \in [\tilde{u}]} \Pi_\alpha(\tilde{u}_1)$$

where Π_α ranges over the basis of continuous semi norms of $L_1(\Phi; U)$.

Hence X is a unitary topological Φ'-module.

3.2 Internal structure

From the definition of X, it follows that there exists a canonical factorisation of H, as follows

$$(3.4)\qquad \begin{tikzcd} L_1(\Phi; U) \arrow[rr, "H"] \arrow[dr, "b"'] & & \Gamma \\ & X \arrow[ur, "a"'] & \end{tikzcd}$$

where b is surjective, and a is injective. More precisely we have

$$(3.5)\qquad b\,\tilde{u} = [\tilde{u}]$$

$$(3.6)\qquad a\,[\tilde{u}] = H\,\tilde{u}$$

It is easy to show that a and b are continuous and homomorphisms of Φ'-modules.

We then define mappings $F:X \to X$, $G:U \to X$, $H:X \to Y$, by setting

$$(3.7)\qquad F[\tilde{u}] = b(\Delta\,\tilde{u}), \text{ where } \Delta \text{ is the infinitesimal generator of } T(t)$$

$$(3.8)\qquad G\,u = b\,(u\delta_0)$$

$$(3.9)\qquad H[\tilde{u}] = a[\tilde{u}](0) = f(\tilde{u}).$$

From properties of a and b, it follows that F, G, H are linear and continuous. Furthermore F is the infinitesimal generator of the semi group $\Gamma(t)$ on X defined by

$$(3.10)\qquad \Gamma(t)[\tilde{u}] = b\,T(t)\,\tilde{u}$$

From properties of T(t) it follows that

$$t \to \Gamma(t)[\tilde{u}]$$

is infinitely differentiable from $[0, +\infty)$ into X.

We shall introduce some notation. Let x(t) be a mapping from $[0, +\infty)$ into X, which is infinitely differentiable, and let $\mu \in \Phi'$. If there exists an element $\xi \in X$ such that

$$(3.11)\qquad \langle \xi, x_* \rangle = \int_{-\infty}^{0} \langle x(-t), x_* \rangle\,\mu(t)\,dt, \qquad \forall x_* \in X'$$

we will write (note that ξ is unique)

$$(3.12)\qquad \xi = \int_{-\infty}^{0} x(-t)\,\mu(t)\,dt.$$

We have

Proposition 3.1:

$$(3.13)\qquad K(t) = H\,\Gamma(t)\,G \qquad \forall t \geq 0$$

$$(3.14)\qquad b\,\tilde{u} = \sum_n \lambda_n \int_{-\infty}^{0} \Gamma(-t)\,G\,u_n\,\mu_n(t)\,dt.$$

3.3 Evolution of the state

We shall call $b\,\tilde{u}$ the state reached at time 0 and denote it x(0). Hence from (3.14)

$$(3.15)\qquad x(0) = \sum_n \lambda_n \int_{-\infty}^{0} \Gamma(-s)\,G\,u_n\,\mu_n(s)\,ds.$$

We shall now define what we mean by the state of the system at any time $t \gtrless 0$. First, for $t \geq 0$ we write, by definition

$$(3.16)\qquad x(t) = \sum_n \lambda_n \int_{-\infty}^{0} \Gamma(t-s)\,G\,u_n\,\mu_n(s)\,ds = b\big(T(t)\tilde{u}\big), \qquad t \geq 0.$$

For $t < 0$, we cannot define x(t) by formula (3.16), since T(t) has no meaning for $t < 0$. We shall instead define x(t) in the sense of distributions by giving a meaning to

$$\int_{-\infty}^{0} x(t)\,\phi(t)\,dt$$

for any $\phi \in \Phi$.

For $\phi \in \Phi$, we set if $\tilde{u} \in L_1(\Phi; U)$

$$(3.17)\qquad x_\phi = b\,(\check{\phi} * \tilde{u})$$

where $\check{\phi} * u$ has been already defined in section 1. From this, it follows that for fixed $\tilde{u}$, the mapping

$$\phi \to x_\phi \in L(\Phi; X).$$

We have

Proposition 3.2:

$$(3.18)\qquad \frac{dx}{dt} = Fx(t) \qquad \forall t \geq 0$$

(3.19) $x_{-\phi'} = Fx_\phi + G\,\tilde{u}(\phi) \qquad \forall\, \phi \in \Phi.$

Setting

(3.20) $y(t) = (\mathcal{H}\,\tilde{u})(t)$

we can summarize the following relationships already obtained

(3.21) $y(t) = Hx(t) \qquad t \geq 0$

(3.22) $\dfrac{dx}{dt} = Fx(t) \qquad t \geq 0$

(3.23) $x(0) = \displaystyle\int_{-\infty}^{0} \Gamma(-s)\; G\, u_n\, \mu_n(s)\; ds$

(3.24) $\displaystyle\int_{-\infty}^{0} \frac{dx}{dt}\; \phi(t)\; dt = F \int_{-\infty}^{0} x(t)\; \phi(t)\; dt + G\,\tilde{u}(\phi) \qquad \forall\, \phi \in \Phi.$

(3.25) The sets $L_1(\Phi, U)$ (input functions), Γ (output functions) and X (state space) are unitary topological Φ'-modules.

The space of input functions and the state space are locally convex Hausdorff spaces. The space of output functions is a Frechet space, F, G, H are linear and continuous; F is the infinitesimal generator of the semi group $\Gamma(t)$.

(3.26) The mapping $\tilde{u} \to x(0): L_1(\Phi;\, U) \to X$ is continuous, surjective, and is an homomorphism of Φ'-modules.

(3.27) The mapping $x(0) \to y(\cdot): X \to \Gamma$ is continuous, injective and is an homomorphism of Φ'-modules.

Using standard terminology we will call the set (U, Y, X, F, G, H) a canonical internal representation. To complete this paragraph, we will point out that, using the module structure of the internal representation we have

(3.28) $\mathcal{H}\,\tilde{u} = \sum_n \lambda_n\, \mu_n \cdot \mathcal{H}(u_n\, \delta_0)$

(3.29) $x(0) = \sum_n \lambda_n\, \mu_n \cdot b(u_n\, \delta_0).$

4. DUALITY

4.1 Bilinear form associated with the external representation

In a previous section we have introduced the space $\Phi(-\infty, 0;\, U')$, which is isomorphic to the strong dual of $L_1(\Phi;\, U)$. We provide it with the topology of the strong dual. We will similarly consider $\Phi(-\infty, 0;\, Y)$. We now introduce the space

$$\Phi(Y) = C^\infty\big((-\infty, 0];\; Y\big)$$

provided with the topology of a Frechet space defined by the basis of continuous

semi norms

(4.1) $\|y(\cdot)\|_{jk} = \sup_{-k\leq t\leq 0}\; \sup_{p\leq j}\; \|y^{(p)}(t)\|.$

We have

Proposition 4.1: The injection of $\Phi(Y)$ into $\Phi(-\infty, 0;\, Y)$ is continuous.

Let us now introduce the operator

(4.2) $(\check{\mathcal{H}}\,\tilde{u})(t) = (\mathcal{H}\,\tilde{u})(-t) \qquad t \leq 0.$

Clearly we have

(4.3) $\check{\mathcal{H}} \in \mathcal{L}\big(L_1(\Phi;\, U);\; \Phi(Y)\big).$

We next define on the pair $L_1(\Phi;\, U) \times L_1(\phi;\, Y')$ the bilinear form

(4.4) $K(\tilde{u}, \tilde{y}_*) = [\check{\mathcal{H}}\,\tilde{u}(\cdot),\; \tilde{y}_*]$

From proposition 4.1, it follows that K is separately continuous. The bilinear form K can be explicitly written as follows

(4.5) $K(\tilde{u}, \tilde{y}_*) = \sum_n \sum_m \alpha_m\, \lambda_n \displaystyle\int_{-\infty}^{0} \nu_m(t)\; dt\;\; \Big[\int_{-\infty}^{0} (\langle K(-t-s)u_n,\; y_m^* \rangle)\; u_n(s)\; ds\Big]$

$\qquad = \sum_n \sum_m \alpha_m\, \lambda_n\, \nu_m * \mu_n\; (\langle K(-\cdot)\; u_n,\; y_m^* \rangle)$

where

$$\tilde{u} = \sum_n \lambda_n\, u_n\, \mu_n$$

$$\tilde{y}_* = \sum_m \alpha_n\, y_m^*\, \nu_m.$$

We shall use the following notation

(4.6) $\sigma\big(L_1(\Phi;\, U);\; \phi(-\infty, 0;\, U')\big) = \text{space } L_1(\Phi;\, U)$

provided with the weak topology.

We have

(4.7) $\check{\mathcal{H}} \in \mathcal{L}\Big(\sigma\big(L_1(\Phi;\, U);\; \Phi(-\infty, 0: U')\big);\; \sigma\big(\Phi(-\infty, 0;\, Y): L_1(\Phi;\, Y')\big)\Big)$

and

(4.8) K is separately continuous on $L_1(\Phi;\, U)$ and $L_1(\Phi;\, Y')$ for the weak topologies

$\sigma\big(L_1(\Phi;\, U);\; \Phi(-\infty, 0,\, U')\big)$ and $\sigma\big(L_1(\Phi;\, Y');\; \Phi(-\infty, 0;\, Y)\big).$

We denote by $S(t)$ the semi group of translations on $L_1(\Phi;\ Y')$. It readily follows from (4.5) that

$$K\left(T(t)\tilde{u},\ \tilde{y}_*\right) = K\left(\tilde{u},\ S(t),\ \tilde{y}_*\right). \tag{4.9}$$

Furthermore the mappings

$$t \to T(t)\ \tilde{u}$$

and

$$t \to S(t)\ \tilde{y}_*$$

are infinitely differentiable on $\sigma\left(L_1(\Phi;\ U);\ \Phi(-\infty,\ 0;\ U;)\right)$ and $\sigma\left(L_1(\Phi;\ Y');\ \Phi(-\infty,\ 0;\ Y)\right)$ (since they already are for the strong topology). The corresponding infinitesimal generators will be denoted by Δ and Ξ.

4.2 The dual system

We consider the transpose

$$\check{H}^* \in L\left(\sigma\left(L_1(\Phi;\ Y'),\ \Phi(-\infty,\ 0;\ Y)\right);\ \sigma\left(\Phi(-\infty,\ 0;\ U');\ L_1(\Phi;\ U)\right)\right)$$

and define a linear mapping from $L_1(\Phi;\ Y') \to U'$ by setting

$$g(\tilde{y}_*) = \check{H}\ \tilde{y}_*(0). \tag{4.10}$$

Then

$$g \in L\left(\sigma\left(L_1(\Phi;\ Y');\ \Phi(-\infty,\ 0;\ Y)\right);\ \sigma(U';\ U)\right). \tag{4.11}$$

Let us notice that we do not have necessarily

$$g \in L\left(L_1(\Phi;\ Y');\ U'\right).$$

Definition 4.1: The triple $(Y',\ U',\ g)$ defines the external representation of a system, called the dual system of $(U,\ Y,\ f)$.

Proposition 4.2: The Hankel operator G of the dual system satisfies

$$(G\ \tilde{y}_*)(t) = (\check{H}^*\tilde{y}_*)(-t),\quad t \geq 0,\ \forall\ \tilde{y}_* \in L_1(\Phi;\ Y'). \tag{4.12}$$

4.3 Canonical realizations of the dual systems

Let us set

$$X = L_1(\Phi;\ U)/\mathrm{Ker}\ \check{H} \tag{4.13}$$

$$Z = L_1(\Phi;\ Y')/\mathrm{Ker}\ \check{H}^* \tag{4.14}$$

provided with the quotient topology (when $L_1(\Phi;\ U)$ and $L_1(\Phi;\ Y')$ are provided with

the weak topologies

$$\sigma\left(L_1(\Phi;\ U);\ \Phi(-\infty,\ 0;\ U')\right);\ \sigma\left(L_1(\Phi;\ Y');\ \Phi(-\infty,\ 0;\ Y)\right).$$

Thus X and Z are locally convex Hausdorff T.V.S.

Let us set

$$\begin{aligned} M &= \mathrm{Ker}\ \check{H} \subset L_1(\Phi;\ U) \\ N &= \mathrm{Ker}\ \check{H} \subset L_1(\Phi;\ Y') \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} M^\perp &= \{u_*(\cdot)\ \mid\ [u_*(\cdot),\ \tilde{u}] = 0 \qquad \forall\ \tilde{u} \in M\} \\ N^\perp &= \{y(\cdot)\ \mid\ [y(\cdot),\ \tilde{y}_*] = 0, \qquad \forall\ \tilde{y}_* \in N\}. \end{aligned} \tag{4.16}$$

It is known (cf. BOURBAKI) that there exists an algebraic isomorphism between the dual of X (respectively Z) and $M^\perp$ (respectively $N^\perp$). Furthermore the topologies on X and Z coincide with the topologies $\sigma(X,\ M^\perp)$ and $\sigma(Z,\ N^\perp)$. One can define a duality pairing between X and Z by setting

$$\phi([\tilde{u}],\ [\tilde{y}_*]) = K\ (\tilde{u},\ \tilde{y}_*) \tag{4.17}$$

and mappings

$$\begin{aligned} \alpha &\in L\left(\sigma(X,\ M^\perp);\ \sigma(N^\perp,\ Z)\right) \\ \beta &\in L\left(\sigma(Z,\ N^\perp);\ \sigma(M^\perp,\ X)\right) \end{aligned} \tag{4.18}$$

such that

$$\begin{aligned} \left[\alpha[\tilde{u}],\ [\tilde{y}_*]\right] &= \phi([\tilde{u}],\ [\tilde{y}_*]) \\ \left[\beta[\tilde{y}_*],\ [\tilde{u}]\right] &= \phi([\tilde{u}],\ [\tilde{y}_*]). \end{aligned} \tag{4.19}$$

the mappings α and β are injective. Hence

(4.20) X is algebraically isomorphic to a subspace of $N^\perp$ (dual of Z) $= \mathrm{Im}\ \alpha$, dense in $\sigma(N^\perp,\ Z)$

Z is algebraically isomorphic to a subspace of $M^\perp$ (dual of X) $= \mathrm{Im}\ \beta$, dense in $\sigma(M^\perp,\ Z)$.

Considering now the canonical factorisations

$$\begin{tikzcd} L_1(\Phi;\ U) \arrow[r, "\check{H}"] \arrow[d, "b"'] & \Phi(-\infty,\ 0;\ Y) \\ X \arrow[r, "\alpha"] & Z' = N^\perp \arrow[u, "a"'] \end{tikzcd} \tag{4.21}$$

$$\begin{tikzcd} L_1(\Phi;\, Y') \arrow[rr, "\check{H}^*"] \arrow[d, "b_*"'] & & \Phi(-\infty,\, 0;\, U') \\ Z \arrow[r, "\beta"] & X' = M^{\perp} \arrow[ur, "a_*"'] & \end{tikzcd}$$

We have from (4.19) and (4.21)

(4.22) $a_* = b^* \beta$

$a^* = \beta\, b_*$, (upper star denotes transpose with respect to the first diagram)

Let us then define

(4.23) $F[\tilde{u}] = [\Delta\, \tilde{u}] = b(\Delta\, \tilde{u})$

$G\, u = [u\, \delta_0]$

$H[\tilde{u}] = f(\tilde{u})$

(4.24) $A[\tilde{y}_*] = [\Xi\, \tilde{y}_*] = b_*(\Xi\, \tilde{y}_*)$

$B\, y_* = [y_*\, \delta_0]$

$C\, \tilde{y}_* = g\, (\tilde{y}_*)$

The preceding mappings have the following properties

(4.25) $F \in L\left(\sigma(X,\, M^{\perp});\;\; \sigma(X,\, M^{\perp})\right)$

$G \in L\left(\sigma(U,\, U');\;\; \sigma(X,\, M^{\perp})\right)$

$H \in L\left(\sigma(X,\, M^{\perp});\;\; \sigma(Y,\, Y')\right)$

(4.26) $A \in L\left(\sigma(Z,\, N^{\perp});\;\; \sigma(Z,\, N^{\perp})\right)$

$B \in L\left(\sigma(Y,\, Y);\;\; \sigma(Z,\, N^{\perp})\right)$

$C \in L\left(\sigma(Z,\, N^{\perp});\;\; \sigma(U',\, U)\right)$

(4.27) $\beta\, A = F^*\, \beta$

$C = G^*\, \beta$

$\beta\, B = H^*$.

The evolution of the dual systems is described by the equations

(4.28) $\dfrac{dx}{dt} = Fx(t) \qquad t \geq 0$

$\dfrac{dz}{dt} = Az(t) \qquad t \geq 0$

(4.29) $x_{-\phi'} = Fx_{\phi} + G\tilde{u}(\phi) \qquad \forall\, \phi \in \Phi$

$z_{-\phi'} = Az_{\phi} + B\tilde{y}_*(\phi) \qquad \forall\, \phi \in \Phi$

where the derivatives in (4.28) must be taken in $\sigma(X,\, M^{\perp})$ and $\sigma(Z,\, N^{\perp})$ and in (4.29), x_{ϕ} and z_{ϕ} are vector distributions with values in $\sigma(X,\, M^{\perp})$ and $\sigma(Z,\, N^{\perp})$.

4.3 Algebraic Properties of Dual Systems

The space $L_1(\Phi;\, U)$ can be provided with an external multiplication as we have seen before. It can be shown that

$$\tilde{u} \longmapsto \mu * u \in L\left(\sigma\left(L_1(\Phi;\, U),\, \Phi(-\infty,\, 0;\, U')\right);\; \sigma\left(L_1(\Phi;\, U),\, \Phi(-\infty,\, 0;\, U')\right)\right)$$

and therefore $L_1(\Phi;\, U)$ remains a unitary Φ'-module with respect to the weak topology.

Our output function space will be taken to be $\Phi(-\infty,\, 0;\, Y)$ and $\Phi(-\infty,\, 0;\, U')$ equipped with the weak topologies. It can be shown that these spaces can be provided with the structure of a unitary topological Φ'-module.

If $\check{H}$ and $\check{H}^*$ are the Hankel operators of the dual systems then it can be shown that $\check{H}: L_1(\Phi;\, U) \longrightarrow \Phi(-\infty,\, 0; Y)$ and $\check{H}^*: L_1(\Phi;\, Y') \longrightarrow \Phi(-\infty,\, 0;\, U')$ are Φ'-module homomorphisms.

5. APPROACH USING SOBOLEV SPACES

Introduction

We now show that working with a different input function space we can obtain a canonical internal representation where the evolution of the state can be described by an operational differential equation involving unbounded operators on a Hilbert space. The algebraic structure is however completely lost.

5.1 Notation and Assumptions

For $m \geq 1$, integer, let us set

(5.1) $$\Phi = H^m(-\infty,\, 0) = \left\{\phi \in L^2(-\infty,\, 0) \;\Big|\; \frac{d^j\phi}{dt^j} \in L^2(-\infty,\, 0),\;\; j = 1,\, 2,\, \ldots m\right\}$$

In (5.1) the $\dfrac{d^j\phi}{dt^j}$ are taken in the sense of distributions. The space $H^m(-\infty,\, 0)$ is a Hilbert space for the norm

(5.2) $$\left(\|\phi\| = |\phi|^2_{L^2(-\infty,\, 0)} + \sum_{j=1}^{m} \left|\frac{d^j\phi}{dt^j}\right|^2_{L^2(-\infty,\, 0)}\right)^{\frac{1}{2}}$$

and is called the Sobolev space of order m.

Let U and Y be separable Hilbert spaces. We denote by $\Phi(U')$ the space $H^m(-\infty,\, 0;\, U')$, that is

$$\Phi(U') = \{u_*:(-\infty, 0) \to U' \mid u_*, \frac{d^j u_*}{dt^j} \in L^2(-\infty, 0; U'),\ 1 \le j \le m\} \tag{5.3}$$

The space $\Phi(U')$ is a Hilbert space for the norm

$$||u_*|| = \left(|u_*|^2_{L^2(-\infty, 0:U')} + \sum_{j=1}^{m} \left| \frac{d^j u_*}{dt^j} \right|^2_{L^2(-\infty, 0;U')} \right)^{\frac{1}{2}} \tag{5.4}$$

The dual space $\Phi(U')'$ will be taken to be the input function space. It can be shown

Proposition 5.1:

$$\Phi(U') = L_2(\Phi'; U') \tag{5.5}$$

$$\left(\Phi(U')\right)' = L_2(\Phi; U) \tag{5.6}$$

where $L_2(\Phi; U)$ $\left(\text{resp. } L_2(\Phi'; U')\right)$ denotes the space of Hilbert Schmidt Operators from $\Phi \to U$ (resp. $\Phi' \to U'$).

Comparing with the distribution theory approach previously presented, we see that the space $L_1(\Phi; U)$ has been extended to $L_2(\Phi; U)$ with the major advantage that it is now a Hilbert space.

In view of the above proposition every $\tilde{u} \in \left(\Phi(U')\right)'$ has a representation of the form

$$\tilde{u} = \sum_n \alpha_n \mu_n u_n, \text{ where} \tag{5.7}$$

$\sum_n \alpha_n^2 < +\infty$, μ_n is an orthonormal basis of Φ' and u_n an orthonormal basis of U'. For our purposes a different representation is more useful. With the aid of the mapping

$$z \longmapsto \left(z, \frac{dz}{dt}, \ldots, \frac{d^m z}{dt^m}\right),$$

the space $H^m(-\infty, 0; U')$ can be identified with a closed subspace of $L^2(-\infty, 0; U')^m$. By the Hahn Banach theorem, every element $\tilde{u} \in \left(H^m(-\infty, 0; U')\right)'$ can be represented (in a non-unique way) by

$$< \tilde{u}, z > = \int_{-\infty}^{0} < u_o(\sigma), z(\sigma) > d\sigma + \sum_{i=1}^{m} \int_{-\infty}^{0} < u_i(\sigma), \frac{d^i z}{d^i}(\sigma) > d\sigma \tag{5.8}$$

where $u_o, u_1, \ldots, u_m$ belongs to $L^2(-\infty, 0; U')$.

On $\left(\Phi(U')\right)'$, we define a semi-group of translations by transposition, by setting

$$< T(t)u, z > = < \tilde{u}, \left(T(t)\right)' z >, \qquad t \ge 0 \tag{5.9}$$

where

$$T(t)'z(s) = z(-t+s), \qquad s \le 0. \tag{5.10}$$

It can be verified that $T(t)'$ is an equi-continuous semi-group of Class C° on $\Phi(U')$ and since $\Phi(U')$ is a Hilbert space, $T(t)$ is also an equi-continuous semi-group of class C° (cf. YOSIDA, p. 233 and 212).

5.2 External Representation of a Linear Time Invariant System

Let $C(0,\infty; Y)$ denote the space of continuous and bounded mappings from $[0, \infty) \to Y$ which is a Banach space for the sup norm.

A linear time invariant system is given by

$$f \in L\left(\left(\Phi(U')\right)'; Y\right). \tag{5.11}$$

We define the Hankel operator by

$$(H\,\tilde{u})(t) = f\left(T(t)\tilde{u}\right), \qquad t \ge 0 \tag{5.12}$$

and $H \in L\left(\left(\Phi(U')\right)'; C(0, \infty; Y)\right)$.

Let

$$f^*:Y' \to \Phi(U') \tag{5.13}$$

denote the transpose of f.

Let us set

$$K^*(-t)y_* = (f^* y^*)(t) \qquad t \le 0. \tag{5.14}$$

Now $f^* y^* \in H^m(-\infty, 0; U')$ and

$$||K^*(\cdot)y_*||_{H^m(-\infty, 0; U')} \le c||y_*||_{Y'} \tag{5.15}$$

Since $u\delta_{-t}$ (δ_{-t} is the Dirac measure concentrated at $-t$) belongs to $\left(H^m(-\infty, 0;U')\right)'$, we can show that

$$K(t) \in L(U; Y) \text{ and}$$

$$||K(t)|| \le c \tag{5.16}$$

$$< K(-t)u, y_* > \in H^m(-\infty, 0) \qquad \forall\, u, y_*. \tag{5.17}$$

K(t) is called the kernel or the impulse response of the system. Using the representation for the input functions, we obtain a representation for f as

$$< f(\tilde{u}), y_* > = \int_{-\infty}^{0} < u_o(t), K^*(-t)y_* > dt + \sum_{i=1}^{m} \int_{-\infty}^{0} < u_i(t), \frac{d^i}{dt^i} K^*(-t)y_* > dt. \tag{5.18}$$

5.3 Canonical Internal representation

We introduce the state space

(5.19) $$X = \left(\Phi(U')\right)'/\mathrm{Ker}\, H$$

which is a Hilbert space for the quotient topology. Let us next define as usual

(5.20) $$Gu = [u\delta_0]$$

(5.21) $$H[\tilde{u}] = f(\tilde{u})$$

(5.22) $$\Gamma(t)\,[\tilde{u}] = [T(t)\tilde{u}]$$

then $G \in L(U;\, X)$, $H \in L(X;\, Y)$, and $\Gamma(t)$ is an equicontinuous semi group of class C_0 on X. We then get

(5.23) $$K(+t) = H\,\Gamma(t)\,G \qquad t \geq 0.$$

We will set as usual

(5.24) $$b\tilde{u} = [\tilde{u}]$$

(5.25) $$a[\tilde{u}] = H\tilde{u}$$

and $b \in L\left(\left(\Phi(U')\right)';\, X\right)$, $a \in L\left(X;\, C[0,\, +\infty;\, Y)\right)$. b and a are called the reachability and observability operators respectively.

We can prove the following theorem:

Theorem 5.1: Under the assumptions of the previous sections, the canonical internal representation of the linear time-invariant system given by f as in (5.11) is given by (U, Y, X, F, G, H), where U, Y, X are Hilbert spaces, $G \in L(U;\, X)$, $H \in L(X,\, Y)$, F is an unbounded operator on X which is the infinitesimal generator of an equicontinuous semi-group $\Gamma(t)$ of class C° on X. The Hankel Operator H admits the factorization

$$H = a \circ b, \text{ where}$$

$a \in L\left(X;\, C(0,\, \infty;\, Y)\right)$ is injective and $b \in L\left(\left(\Phi(U')\right)':\, X\right)$ is surjective.

The evolution of the system is described by

(5.26) $$\frac{dx}{dt} = Fx(t) \qquad \forall\, t \geq 0, \quad \text{if } x(0) = b\tilde{u} \in D(F); \quad \text{for } t < 0,$$

(5.27) $$\int_{-\infty}^{0} \frac{dx}{dt}\,\phi\, dt = F\int_{-\infty}^{0} x\,\phi\, dt + G\tilde{u}(\phi) \qquad \forall\, \phi \in H^m(-\infty,\, 0)$$

in the sense of vector distributions with values in X, and $\int_{-\infty}^{0} x\,\phi\, dt \in D(F)$, $\int_{-\infty}^{0} \frac{dx}{dt}\,\phi\, dt \in X$, when $x_\phi(\tilde{u})$ is given by

(5.28) $$x_\phi(\tilde{u}) = \int_{-\infty}^{0} ds\, \Gamma(-s)G\left[\int_{-\infty}^{0} u_0(t)\phi(t-s)\,dt + \sum_{i=1}^{m}\int_{-\infty}^{0} u_i(t)\phi^{(i)}(t-s)\,dt\right],$$

$$\phi \in D(-\infty,\, 0);$$

(5.29) $$(H\tilde{u})(t) = y(t) = Hx(t), \qquad t \geq 0.$$

5.4 The State Space Isomorphism Theorem

In this case we can prove a state space isomorphism theorem. The reason for being able to do this is that the reachability operator being onto is right invertible and since we are in a Hilbert space a right inverse (non-unique) is also continuous. This allows us to construct the required topological isomorphism.

Theorem 5.2: Let $\left(\Gamma(t),\, G,\, H\right)$ and $\left(\Gamma_1(t),\, G_1,\, H_1\right)$ be two reachable and observable realizations realizing the same kernel K(t). Then there exists a boundedly invertible transformation $\chi : X \to X_1$ (X and X_1 being the corresponding state spaces) such that the following diagram is commutative

$$\begin{tikzcd}
& U \arrow[dl, "G"'] \arrow[dr, "G_1"] & \\
X \arrow[rr, "\chi"] \arrow[d, "\Gamma(t)"'] & & X_1 \arrow[d, "\Gamma_1(t)"] \\
X \arrow[rr, "\chi"] \arrow[dr, "H"'] & & X_1 \arrow[dl, "H_1"] \\
& Y &
\end{tikzcd}$$

5.5 Inputs in $L^2(-\infty,\, 0;\, U')$

It is known that

$$L^2(-\infty,\, 0;\, U') \subset \left(H^m(-\infty,\, 0;\, U')\right)'$$

the canonical injection being continuous and having dense image. Therefore setting

(5.30) $$\tilde{X} = \{b(\tilde{u}) \mid \tilde{u} \in L^2(-\infty,\, 0;\, U')\},$$

$\tilde{X}$ is a dense subspace of X.

We can then write

(5.31) $$b(\tilde{u}) = \int_{-\infty}^{0} \Gamma(-t)G\tilde{u}(t)\,dt = x(0).$$

Then we can write (5.27) in a stronger sense. Indeed, we have

(5.32) $$\int_{-\infty}^{0} \frac{dx}{dt}\,\phi\, dt = F\int_{-\infty}^{0} x\,\phi\, dt + \int_{-\infty}^{0} G\tilde{u}(t)\,\phi(t)\, dt \qquad \forall\, \phi \in \Phi.$$

$$\int_{-\infty}^{0} x\,\phi\,dt = \int_{-\infty}^{0} \phi(t)\left[\int_{-\infty}^{t} \Gamma(t-s)G\tilde{u}(s)\,ds\right]$$

$$x(t) = \int_{-\infty}^{t} \Gamma(t-s)G\tilde{u}(s)ds \qquad t \leq 0.$$

llows from (5.32), $\forall x_* \in D(F^*)$

$$\frac{d}{dt} < x_*, x(t) > \; = \; < F^* x^*, x(t) > + < x^*, G\tilde{u}(t) > \quad \text{a.e.t.}$$

t) $\in D(F)$ a.e., then $x(t) \in D(F)$, $\forall t \leq 0$ and

$$\frac{dx}{dt} = Fx + G\tilde{u} \quad \text{a.e.} \quad t < 0.$$

(5.34) and (5.35) are operational differential equations involving un- erators which have been extensively studied in the literature on partial al equations.

eover in this situation we capture the fact that the reachable space of ite dimensional systems will usually be only dense in the state space.

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