FP4 - 3:00

EXACT SOLUTION TO LYAPUNOV'S EQUATION USING ALGEBRAIC METHODS

by

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Abstract

Let A'P + PA = -Q be a Lyapunov equation with A being a stability matrix and both A and Q matrices with rational entries. Multiplying A and Q by a suitable positive integer an equivalent Lyapunov equation $A_1'P + PA_1 = -Q_1$ is obtained, with A_1 and Q_1 having integer entries. Let I(x,y) be the ring of polynomials in x and y over the integers I, and $\mathcal{E}$ be the set of all square matrices with integer entries. The solution P to this equation is given by:

$$P_u = (e_{mn}) = f_{A_1}(q_u(x,y),\ Q_1)$$

$$P = \frac{1}{u^2} \cdot P_u$$

where: $q_u(x,y) \in I(x,y)$ and $u \in I$

$$f_{A_1} : I(x,y) \times \mathcal{E} \to \mathcal{E} \text{ defined as } f_{A_1}(h(x,y),M) = \sum_{j,k} h_{jk}(A_1')^j \cdot M \cdot A_1^k$$

which is a finite sum.

The calculation of u and $q_u(x,y)$ requires finding the characteristic polynomial of A_1, as well as using the Euclidean Algorithm, computations which lead to polynomial coefficient growth. In order to eliminate the space consuming manipulation of large integers in intermediate steps, modular arithmetic is used to obtain the matrix ${}_{p_i}P_u = (e_{mn} \bmod p_i)$ and ${}_{p_i}u = u \bmod p_i$ with p_i a prime, for a sufficient number of primes. The Chinese Remainder Theorem is then applied to obtain the solution P.

The algorithm has been programmed on MACSYMA which is a very suitable computer programming system for all the numerical computations involved.

Numerical results as well as extensions to solving the Algebraic Riccati Equation are presented.

1. Introduction

In the past fifteen years or so there has been impressive progress in the theoretical understanding of the structure, representation and control of linear multivariable systems. In contrast, workers in the field have paid very little attention to the computational aspects of systems problems. This does not mean that algorithms for the solution of systems problems have not been developed. But most of the algorithms that have been proposed have never been seriously studied as far as stability, convergence and similar issues are concerned. Even the LQG problem, bulwark of the so-called "modern control theory", seems to be little understood from the computational point of view.

In this paper we undertake a study of solution methods for Lyapunov's equation

$$PA + A'P = -Q \tag{1.1}$$

using the methods of modern algebra. The emphasis is on the use of finite algebraic procedures which are easily implemented on a digital computer and which lead to an explicit solution to the problem.

It is well known that this is an important equation in the study of stability of linear finite dimensional time-invariant systems. If Q is symmetric and positive definite and if A is a stability matrix (real parts of eigenvalues of A strictly negative) then the unique solution to (1.1) is given by the convergent integral

$$G = \int_0^\infty e^{A't} Q\, e^{At}\, dt \tag{1.2}$$

(cf. BROCKETT).

However, the solution requires the evaluation of an integral over an infinite time interval.

The need for solving this equation also arises when one uses Newton's Method to solve the Algebraic Riccati equation

$$(A-BR^{-1}B'P)'P + P(A-BR^{-1}B'P) = -C'C-PBR^{-1}B'P \tag{1.3}$$

where R is positive definite.

If (A,B) is stabilizable and (A,C) observable, then there exists a unique positive definite solution P to (1.3).

Now let P_0 be a symmetric matrix such that

The research of the first author has been supported by ERDA under Grant ERDA-E(49-18)-2087. The research of the second author has been supported by the Air Force Office of Scientific Research, Air Force Systems Command, under Grant AFOSR-72-2273. The computational work was done using the computer system MACSYMA developed by the Math Lab group at M.I.T. The Math Lab group is supported by the Defense Advanced Research Projects Agency, work order 2095, under Office of Naval Research Contract No. N00014-75-C-0661.

$(A-BR^{-1}B'P_0)$ is a stability matrix, and consider the Lyapunov equation

$$(A-BR^{-1}B'P_0)'P + P(A-BR^{-1}B'P_0) = - C'C - P_0BR^{-1}B'P_0 \qquad (1.4)$$

It is well known that under our hypotheses this equation has a unique positive definite solution P_1. Replace P_0 by P_1 in (1.4) and continue the process. It is known (cf. KLEINMAN [1], [2]) that this is a convergent process. The main computation here is therefore the solution of (1.4).

This paper is based on an important paper by KALMAN. Kalman's concern was the characterization of polynomials whose zeroes lie in certain algebraic domains (and the unification of the ideas of Hermite and Lyapunov). In this paper, we show that the same ideas lead to finite algorithms for the solution of Lyapunov's equation.

This paper is divided into four sections. In section two we present constructive algebraic proofs of two theorems related to a linear matrix equation. This section provides the basis for section 3 where the computational algorithms are presented. In section 4 we present a numerical example.

This is a preliminary report on this work. At the conference we hope to present extensive numerical results.

2. Algebraic Proofs of Two Theorems Related to a Linear Matrix System

In this section we present constructive algebraic proofs for the following two theorems.

Theorem 2.1. Let A be an n×n square matrix over the reals. A is a stability matrix if and only if for any symmetric positive definite matrix Q there exists a unique symmetric positive definite solution P to the matrix equation

$$PA + A'P = -Q. \qquad (2.1)$$

Theorem 2.2. Let A be an n×n square matrix over the reals. If A is a stability matrix and (A,C) is an observable pair then the matrix equation

$$PA + A'P = -C'C \quad (C \text{ is } p\times n) \qquad (2.2)$$

has a unique symmetric positive definite solution P.

Before proceeding with the proofs we shall introduce the algebraic framework in which we work.

Let $\mathcal{R}[x]$ be the ring of polynomials in x over the field $\mathcal{R}$ of real numbers, and $\mathcal{R}[x,y]$ be the ring in x and y over $\mathcal{R}$. If p(x,y) is any element of $\mathcal{R}[x,y]$ we can write it as:

$$\ell'(y)C(p)\ell(x) = p(x,y)$$

where $\ell(z)$ is the column vector $1, z, \ldots z^{n-1}$, n is one plus the largest power of p(x,y) in either x or y and C(p) is an n×n matrix over $\mathcal{R}$. This introduces a bijection between $\mathcal{R}[x,y]$ and $\mathcal{M}$ the set of all square matrices. This (cf. KALMAN) motivates the

Definition 2.1. A polynomial p(x,y) in $\mathcal{R}[x,y]$ is positive if and only if C(p) is i) symmetric and ii) positive definite.

Let Φ denote the ideal $(\phi(x), \phi(y))$ in $\mathcal{R}[x,y]$. Let $[g(x,y)]$ denote the elements of the quotient ring $\mathcal{R}[x,y]/\phi$. We shall denote by pmodΦ the polynomial of minimal degree in the equivalence class [p].

The following two lemmata can be established (cf. KALMAN).

Lemma 2.1 The polynomial p(x,y) in $\mathcal{R}[x,y]$ is positive if and only if there exist polynomials $\pi_1, \pi_2, \ldots \pi_m$ (m = size (C(p)) such that

$$p(x,y) = \sum_{i=1}^{m} \pi_i(x)\pi_i(y)$$

where $\{\pi_i(x)\}$ are a basis for (the vector space over $\mathcal{R}$ of) polynomials of degree less than m.

Lemma 2.2 Let n be the degree of $\phi(x)$. If pmodΦ is positive of degree n-1 in both x and y then $\sigma(x)\sigma(y)p(x,y)\bmod\Phi$ is positive if and only if $\sigma(x)$ and $\phi(x)$ are relatively prime.

Let $f_A: \mathcal{R}[x,y] \times \mathcal{M} \to \mathcal{M}$ be the action (cf. KALMAN) defined in the following manner

$$f_A(h(x,y),M) = \sum_{j,k} h_{jk}(A')^j \cdot M \cdot (A)^k \qquad (2.3)$$

We list here explicitly some properties of this map:

i) $f_A(u,M) = uM$ (u a unit in $\mathcal{R}[x,y]$)

ii) $f_A(g(x,y) + h(x,y), M) = f_A(g(x,y), M) + f_A(h(x,y), M)$

iii) $f_A(g(x,y)\cdot h(x,y),M) = f_A(g(x,y), f_A(h(s,y),M)) = f_A(h(x,y), f_A(g(x,y),M))$

iv) Let $g(x) = \det(Ix-A)$ and $H=(g(x),g(y))$ $f_A(h(x,y), M) = f_A(g\bmod H, M)$.

Suppose that A is an n×n stability matrix with characteristic polynomial $\phi_2(x) = \det(Ix-A)$. Define $\phi_1(x)$ and $P_\phi(x,y)$ in the following manner.

$$\phi_1(x) = \phi_2(-x) \qquad (2.4)$$

$$P_\phi(x,y) = \frac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x + y} \qquad (2.5)$$

It can be shown that $P_\phi(x,y)$ is an element of $\mathcal{R}[x,y]$ of degree n-1 in both x and y, and that it is positive. Since $\phi_1(x)$, $\phi_2(x)$ are relatively prime there exist polynomials T_u, λ_u such that

$$T_u(x)\phi_1(x) + \lambda_u(x)\phi_2(x) = u \quad (\text{u a unit in } \mathcal{R}[x,y]) \qquad (2.6)$$

This implies (Lemma 2.2) that

$$q_u(x,y) = T_u(x)T_u(y)P_\phi(x,y)\bmod\Phi \qquad (2.7)$$

is positive.

Using the above we have:

$$\begin{aligned}
&(x+y)\cdot T_u(x)T_u(y)P_\phi(x,y)\\
&\quad = T_u(x)T_u(y)[\phi_2(x)\phi_2(x)-\phi_1(x)\phi_1(y)]\\
&\quad = +T_u(x)T_u(y)\phi_2(x)\phi_2(y)\\
&\qquad -T_u(x)T_u(y)\phi_1(x)\phi_1(y)\\
&\quad = (+T_u(x)T_u(y) - \lambda_u(x)\lambda_u(y))\phi_2(x)\phi_2(y)\\
&\qquad +u\lambda_u(x)\phi_2(x) + u\lambda_u(y)\phi_2(y) - u^2
\end{aligned}$$

which implies that:

$$((x+y)\cdot T_u(x)T_u(y)P_\phi(x,y))\bmod\Phi = -u^2 \qquad (2.8)$$

$$f_A((x+y)\cdot T_u(x)T_u(y)P_\phi(x,y),G) = -u^2\cdot G \qquad (2.9)$$

using property iv) of the action for any n×n matrix G.

We have now developed all the necessary structure to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Suppose that A is a stability matrix. We claim that

$$P = \frac{1}{u^2} f_A(q_u(x,y),\ Q)$$

is the unique solution of PA + A'P = -Q. Using the above mentioned properties of the action:

$$\begin{aligned}
&\frac{1}{u^2}\left[f_A(q_u(x,y),\ Q)\cdot A + A'\cdot f_A(q_u(x,y),Q)\right]\\
&= \frac{1}{u^2}\Big[f_A(x,f_A(q_u(x,y),Q))\\
&\qquad + f_A(y,\ f_A(q_u(x,y),Q))\Big]\\
&= \frac{1}{u^2}\left[f_A((x+y),\ f_A(q_u(x,y),\ Q))\right]\\
&= \frac{1}{u^2}\left[f_A((x+y)q_u(x,y),Q)\right]\\
&= \frac{1}{u^2}\left[f_A\left(((x+y)q_u(x,y)\bmod\Phi),\ Q\right)\right]\\
&= \frac{1}{u^2}\left[-u^2\cdot Q\right] = -Q
\end{aligned}$$

Uniqueness follows by observing that the linear operator $L:R^{n^2} \to R^{n^2}$ defined by

$$L(P) = PA + A'P$$

is onto since no restriction was placed on Q. This implies that L is one-one.

We now show that P is positive definite.

Since $q_u(x,y)$ is positive this implies that (Lemma 2.2)

$$q_u(x,y) = \sum_{i=1}^{u} \pi_i(x)\pi_i(y)$$

where $\{\pi_i\}$ are a basis.

Therefore

$$\begin{aligned}
P &= \frac{1}{u^2} f_A(q_u(x,y),\ Q)\\
&= \frac{1}{u^2} f_A\left(\sum_{i=1}^{n} \pi_i(x)\pi_i(y),\ Q\right)\\
&= \frac{1}{u^2} \sum_{i=1}^{n} \pi_i(A')\cdot Q\cdot\pi_i(A).
\end{aligned}$$

P is symmetric and since $Q \geq 0$, it must at least be positive semidefinite. Suppose that there exists an n-vector $z\neq0$ such that $z'Pz = 0$.

This implies that $\pi_i(A)\cdot z = 0$ for all $1 \leq i \leq n$. The polynomials $\pi_1,\pi_2,\ldots\pi_n$ form a basis for the vector space of polynomials of degree less than n. Therefore there exist constant $k_1,k_2\ldots k_n$ such that

$$\sum_{i=1}^{u} k_i\pi_i(x) = 1$$

$$\Longrightarrow f_A\left(\sum_{i=1}^{u} k_i\pi_i(x),\ I\right) = I \quad \text{(I n×n identity)}$$

$$\Longrightarrow \sum_{i=1}^{u} k_i\pi_i(A) = I$$

$$\Longrightarrow \sum_{i=1}^{u} k_i\pi_i(A)\cdot z = I\cdot z$$

Since $\pi_i(A)\cdot z = 0$ for all i the l.h.s. of the above equality is zero. This is a contradiction since for $z\neq0$, $Iz\neq0$.

Therefore P is positive definite.

Suppose now that for each symmetric positive definite matrix Q there exists a symmetric positive definite solution P of (2.1).

Let z be some eigenvector corresponding to the eigenvalue λ.

$$-\bar{z}'\cdot Qz < 0 \qquad (\bar{z} \text{ denotes complex conjugate})$$

$$\Longrightarrow \bar{z}'(PA+A'P)z < 0$$

$$\Longrightarrow \bar{z}'P\lambda z + \bar{\lambda}\,\bar{z}'Pz < 0$$

$$\Longrightarrow (\lambda + \bar{\lambda})\ \bar{z}'Pz < 0$$

Since P > 0 this implies that $\lambda + \bar{\lambda} < 0$, i.e. that $\text{Re}(\lambda) < 0$. Therefore A is stable.

Proof of Theorem 2.2. Suppose that A is a stability matrix. This implies that

$$q_u(x,y) = T_u(x)T_u(y)P_\phi(x,y)\bmod\Phi$$

is positive. By lemma 2.2 $q_u(x,y)$ can be written as:

$$q_u(x,y) = \sum_{i=1}^{n} \pi_i(x)\pi_i(y)$$

where $\{\pi_i\}$ is a basis. Following the proof of Theorem 2.1 the solution P of (2.2) exists and can be written as:

$$P = \frac{1}{u^2} f_A\,(q_u(x,y),\ C'C)$$

$$= \frac{1}{u^2} \sum_{i=1}^{n} \pi_i(A')\cdot C'C\ \pi_i(A)$$

Since $C'C \geq 0$ we have

$$z'\cdot\pi_i(A')\cdot C'\cdot C\cdot\pi_i(A)\cdot z = ||C\pi_i(A)z|| \geq 0$$

for all $1 \leq i \leq n$. This means that $P \geq 0$. Suppose now that there exists $z \neq 0$ such that $z'\cdot P\cdot z = 0$. This implies that we must have

$$||C\pi_i(A)\cdot z|| = 0 \qquad \text{for } 1 \leq i \leq n$$

$$\Longrightarrow C\pi_i(A)\cdot z = 0 \qquad \text{for } 1 \leq i \leq n.$$

Since $\{\pi_i\}$ are a basis there exists an $n\times n$ matrix K such that:

$$K\cdot\begin{bmatrix}\pi_1(x)\\ \pi_2(x)\\ \vdots\\ \pi_n(x)\end{bmatrix} = \begin{bmatrix}1\\ x\\ \\ x^{n-1}\end{bmatrix}$$

The above represents n equations of the form

$$k_{i1}\pi_1(x) + k_{i2}\pi_2(x)\ldots + k_{in}\pi_n(x) = x^{i-1},$$
$$1 \leq i \leq n$$

with $(k_{i1}, \ldots k_{in})$ being the i^{th} row of K.

$$f_A(k_{i1}\pi_1(x) + \ldots. + k_{in}\pi_n(x),\ C) = C\cdot A^{i-1}$$
$$1 \leq i \leq n$$

$$\Longrightarrow \sum_{j=1}^{n} k_{ij}C\cdot\pi_j(A) = C\cdot A^{i-1} \qquad 1 \leq i \leq n$$

Define the operator $H:R^n \to R^{n\cdot p}$ by:

$$H(w) = \begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}\cdot w$$

Since (A,C) is an observable pair the null space of H is {0}.

Since $C\cdot\pi_i(A)z = 0 \quad 1 \leq i \leq n$ this implies that

$$\sum_{j=1}^{n} k_{ij}C\cdot\pi_j(A)z = 0 \qquad \text{for all } 1 \leq i \leq n$$

$$\Longrightarrow H(z) = 0.$$

This is a contradiction since $z \neq 0$ and the null space of H is {0}.

3. Computational Algorithm

The proof of Theorem 2.1 is constructive and purely algebraic. It therefore gives great insight into how a computational algorithm should be constructed, for obtaining the solution P of an equation of the form

$$PA + A'P = -Q \qquad (2.10)$$

where A is a stability matrix. The algorithm so constructed, basically involves obtaining $\phi_2(x)$ the characteristic polynomial of A. Using the Extended Euclidean algorithm a polynomial T_u as in (2.6) can be obtained. With these polynomials $P_\phi(x,y)$, $q_u(x,y)$ and the solution P are formed.

By restricting the field of interest $\mathcal{R}$, to that of the rational numbers F, the procedure for obtaining the exact solution of (2.10) is fully implementable, using the remarkable facilities provided by the computer programming system MACSYMA available at M.I.T.

Three algorithms are presented here, the Rational, Integer and Modular, which are based on the constructive proof of Theorem 1.

MACSYMA (Project MAC's SYmbolic MAnipulation System) is a large computer programming system used for performing symbolic as well as numerical mathematical computations. This would easily allow us to make parametric studies.

We now describe the algorithms.

The Rational Algorithm

This algorithm is a mere implementation of the steps outlined in the proof of Theorem 2.1.

R_1) Obtain ϕ_2, the characteristic polynomial of A.

R_2) Set $P_\phi(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x + y}$

R_3) Using the Extended Euclidean Algorithm obtain T_u and u.

R_4) Set $q_u(x,y) = T_u(x)T_u(y)P_\phi(x,y)\bmod\Phi$

R_5) Form $P_u = f_A(q_u(x,y),\ Q)$

R_6) Set $P = \frac{1}{u^2}\cdot P_u$

The Integer Algorithm

Multiplying A and Q by a suitable positive integer an equivalent Lyapunov equation

$$A_1'P = PA_1 = -Q_1 \qquad (3.1)$$

is obtained with A_1, Q_1 having integer entries. Suppose that $\phi_2^1(x)$ is the characteristic polynomial of A_1 in x. It is clear that $\phi_2^1(x)$ has integer coefficients and therefore it is an element of Z[x,y] (the ring of polynomials in x and y with integer coefficients).

Let

$$\phi_1'(x) = \phi_2'(-x)$$

$$P_\phi'(x,y) = \frac{\phi_2'(x)\phi_2'(y)-\phi_1'(x)\phi_1'(y)}{x + y} \qquad (3.2)$$

It can be shown that $P'(x,y) \in Z[x,y]$, as well as the fact that there exist polynomials $T_u'(x)$ $\lambda_u'(x)$ elements of Z[x,y] and u' an integer such that

$$T_u'(x)\phi_1'(x) + \lambda_u'(x)\phi_2'(x) = u' \qquad (3.3)$$

Since the leading coefficient of ϕ_2' is unity, division is possible and with Φ' the ideal $(\phi_2'(x), \phi_2'(y))$ in Z[x,y] we have

$$q_u'(x,y) = T_u'(x)T_u'(y)P_\phi'(x,y) \bmod \Phi' \qquad (3.4)$$

being an element of Z[x,y]. Consequently $P_u^* = f_{A_1}(q_u'(x,y), Q_1)$ has integer entries with the solution now being expressed as:

$$P = \frac{1}{(u')^2} \cdot P_u^*. \qquad (3.5)$$

In (3.3) it is required that polynomials $T_u'(x)$, $\lambda_u'(x)$ and integer u' be found such that (3.3) is satisfied. Existence can be shown in this manner.

Let $\phi_2'(x) = \det(Ix-A_1) = a_0x^n + a_1x^{n-1} + \ldots$ $..+a_n$. Define S to be the n-dimensional matrix

$$S = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ \vdots & & & \vdots & & & & \vdots \\ a_{2n-1} & a_{2n-2} & \ldots & & & & & a_n \end{bmatrix}$$

where $a_k = 0$ for $k > n$ and $a_0 = 1$. Since $\phi_2'(x)$ is a stability polynomial S is positive definite (cf. BROCKETT). Since $\det S > 0$ it can be shown that for each allowable integer value of u' there exist unique polynomials $T_u'(x)$, $\lambda_u'(x)$ of degree less than n such that

$$T_u'(x)\phi_1'(x) + \lambda_u'(x)\phi_2'(x) = u'$$

If $T_u'(x) = d_1x^{n-1} + d_2x^{n-2} + \ldots + d_n$ then

$$d_i = \frac{M_{ni} \cdot u'}{2\det S} \qquad 1 \leq i \leq n$$

where $M_{ni} = \det S_{ni}$, with S_{ni} the matrix obtained from S by deleting the nth row and ith column.

By letting $u' = k \cdot (2\det S)$, k an integer greater than zero we have $u' \in Z$ and $T_u' \in Z[x,y]$.

The algorithm proceeds as follows.

```
I_1)  Obtain $A_1$, $Q_1$.
I_2)  Find $\phi_2'$ the characteristic polynomial of $A_1$.
I_3)  Set $P_\phi'(x,y) = \frac{\phi_2'(x)\phi_2'(y)-\phi_1'(x)\phi_1'(y)}{x + y}$
I_4)  Find $T_u'$ and u'.
I_5)  Set $q_u'(x,y) = T_u'(x)T_u'(y)P_\phi'(x,y) \bmod \Phi'$
I_6)  $P_u^* = f_{A_1}(q_u'(x,y), Q_1)$
I_7)  Set $P = \frac{1}{(u')^2} \cdot P_u^*$
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Doing all calculations in integer arithmetic may save time since greatest common divisor computations will not be performed in intermediate steps.

The Modular Algorithm

The Integer algorithm paves the way for a modular approach to the solution.

Suppose p is a prime that does not divide 2 detS. If $A_1 = (a_{ij})$ and $Q_1 = (q_{ij})$ let ${}_pA = (a_{ij} \bmod p)$, ${}_pQ = (q_{ij} \bmod p)$, both ${}_pA$ and ${}_pQ$ being matrices with elements in Z_p, the field of integers modulo p. Let $Z_p[x,y]$ be the ring of polynomials in x and y over Z_p.

Let

$${}_p\phi_2(x) = \det(Ix - {}_pA), \quad {}_p\phi_2(x) \in Z_p[x,y]$$

and

$${}_p\phi_1(x) = {}_p\phi_2(-x)$$

It can easily be shown that:

$${}_p\phi_2(x) = \phi_2'(x) \bmod p$$
$${}_p\phi_1(x) = \phi_1'(x) \bmod p$$

(the notation $\phi_2'(x) \bmod p$ means reduce each coefficient of $\phi_2'(x)$ modulo p, considering the derived polynomial as an element of $Z_p[x,y]$). Let

$${}_pP_\phi(x,y) = \frac{{}_p\phi_2(x)\,{}_p\phi_2(y) - {}_p\phi_1(x)\,{}_p\phi_1(y)}{x + y}$$

(x+y being thought of as an element of $Z_p[x,y]$ ${}_pP_\phi(x,y)$ is an element of $Z_p[x,y]$.)

It follows that there exist polynomials ${}_pT_u(x)$, ${}_p\lambda_u(x)$ and ${}_pu \in Z_p$ such that

$${}_pT_u(x)\,{}_p\phi_1(x) + {}_p\lambda_u(x)\,{}_p\phi_2(x) = {}_pu$$

where: ${}_pT_u(x) = T_u'(x) \bmod p$

${}_p\lambda_u(x) = \lambda_u'(x) \bmod p$

${}_pu = u' \bmod p$

Let ${}_p\phi$ be the ideal $({}_p\phi_2(x), {}_p\phi_2(y))$ in $Z_p[x,y]$ and

$$
\begin{aligned}
{}_p q_u(x,y) &= {}_p T_u(x)\,{}_p T_u(y)\,{}_p P_\phi(x,y) \bmod \Phi \\
&= e_{00} + e_{10}y + e_{01}x + e_{11}xy + \ldots \\
&\quad \ldots + e_{(n-1)(n-1)}x^{n-1}\,y^{n-1}
\end{aligned}
$$

we have that

$$ {}_p q_u(x,y) = q'_u(x,y) \bmod p. $$

Let

$$ {}_p P_u = \sum_{j,k} e_{k,j}\cdot({}_p A')^{k}\cdot{}_p Q\cdot({}_p A)^{j} $$

with all operations done modulo p.
If $P^*_u = (g_{ij})$ in (3.5), then

$$ {}_p P_u = (g_{ij} \bmod p). $$

Now if ${}_p P_u$ ${}_p u$ are obtained for a sufficient number of primes, the Chinese Remainder Theorem can be used to find P^*_u and u' making it possible to obtain the solution

$$ P = \frac{1}{(u')^2}\cdot P^*_u $$

Since considerable coefficient growth takes place in intermediate computations of the Integer algorithm it may be advantageous to implement the modular algorithm.

M_1) Obtain ${}_p A$, ${}_p Q$.

M_2) Let ${}_p\phi_2 = \det(Ix - {}_p A)$

M_3) Set $\displaystyle {}_p P_\phi(x,y) = \frac{{}_p\phi_2(x)\,{}_p\phi_2(y) - {}_p\phi_1(x)\,{}_p\phi_1(y)}{x+y}$

M_4) Obtain ${}_p T_u$ ${}_p u$.

M_5) Set ${}_p q_u(x,y) = {}_p T_u(x)\,{}_p T_u(y)\,{}_p P_\phi(x,y) \bmod {}_p\Phi$

M_6) Obtain ${}_p P_u$.

M_7) Repeat steps 1-6 for a sufficient number of primes and by use of the Chinese Remainder Theorem find P^*_u, u'.

M_8) Set $\displaystyle P = \frac{1}{(u')^2}\cdot P^*_u$

6. Numerical Results

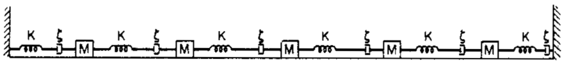
All three algorithms have been programmed on MACSYMA. The example shown here corresponds to the evaluation of

$$ G = \int_0^\infty \underline{x}'(t)\cdot Q\cdot\underline{x}(t)\,dt $$

where $\underline{x}(t)$ is a solution of

$$ \dot{\underline{x}}(t) = A\,\underline{x}(t) \qquad x(0) = C \qquad (*) $$

The system modeled by (*) is given below:



The A matrix of the system is given by:

$$
\frac{1}{M}\cdot\begin{bmatrix}
0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2K & -2\zeta & K & \zeta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 \\
K & \zeta & -2K & -2 & K & \zeta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\
0 & 0 & K & \zeta & -2K & -2\zeta & K & \zeta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
0 & 0 & 0 & 0 & K & \zeta & -2K & -2\zeta & K & \zeta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M \\
0 & 0 & 0 & 0 & 0 & 0 & K & \zeta & -2K & -2\zeta
\end{bmatrix}
$$

The example shown is run with K = 10,000, M=1, ζ=1.

The Q matrix is given by:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The solution to the equation PA + A'P = -Q is:

$$
\begin{bmatrix}
5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{12} & 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 & \frac{1}{6} & 0 & \frac{1}{12} \\
0 & 0 & 5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 5000 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{3}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 5000 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5000 & 0 \\
0 & \frac{1}{12} & 0 & \frac{1}{6} & 0 & \frac{1}{4} & 0 & \frac{1}{3} & 0 & \frac{5}{12}
\end{bmatrix}
$$

By appropriately choosing the values K,M,ζ the system can be made ill-conditioned.

Preliminary tests on the algorithms show that the Integer algorithm performs the fastest, but that it requires much more storage than the Modular algorithm.

Since the solution P is exact it is quite possible that its elements which are rational numbers may have large numerators and denominators. But in a physical situation only a limited number of digit accuracy is required. Since the exact solution is available it is presently under investigation whether a scheme can be devised that would guarantee an arbitrary digit accuracy of an approximate solution.

7. Generalizations

Using the same algebraic framework the solution of the matrix equation

$$PA + BP = C$$

can also be obtained in the case when A and B are two arbitrary $n \times n$ stability matrices. In this case let Ψ be the ideal $(\phi_2(x), \psi_2(y))$ in $\mathcal{R}[x,y]$ where $\phi_2(x)$, $\psi_2(y)$, $\phi_1(y)$, $\psi_1(y)$ are defined as:

$$\phi_2(x) = \det(Ix - A)$$

$$\psi_2(y) = \det(Iy - B)$$

$$\phi_1(x) = \phi_2(-x)$$

$$\psi_1(y) = \psi_2(-y)$$

where

$$P_{\phi,\psi}(x,y) = \frac{\phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x)}{x + y}$$

It can be shown that there exist polynomials $\lambda_u(x)$, $\mu_u(x)$, $\lambda'_u(x)$, $\mu'_u(x)$ such that:

$$\lambda_u(x)\psi_1(x) + \mu_u(x)\phi_2(x) = u \qquad u \text{ a unit in } \mathcal{R}[x,y]$$

$$\lambda'_u(x)\psi_2(x) + \mu'_u(x)\phi_1(x) = u$$

Let

$$q'_u(x,y) = -\lambda_u(x) \cdot \mu'_u(y) P_{\phi\psi}(x,y) \bmod \Psi$$

Define a new action $f_{AB}: \mathcal{R}[x,y] \times \mathcal{M} \to \mathcal{M}$ by

$$f_{AB}(g(x,y), M) = \sum_{jk} g_{jk}(B)^j \cdot M \cdot (A)^k$$

$$\text{where} \qquad g = \sum_{jk} g_{jk}\, x^k y^j$$

The solution P of equation $PA + BP = C$ is then given by:

$$P = \frac{1}{u^2} f_{AB}(q'_u(x,y), C).$$

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