CONTROLLABILITY OF PERTURBED LINEAR SYSTEMS*

by

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1. Introduction

The problem of controllability of linear differential systems has recently been considered from a geometrical point of view by several authors [1,2,3,4,5]. The study of the controllability problem leads to the solution of the minimum energy and minimum miss-distance problems in a natural way.

In this paper we study the problem of controllability of abstract linear systems in the presence of additive disturbances. The controllability problem is formulated as the problem of finding an admissible control such that the solution of an operator equation lies in a given target in the presence of the worst disturbance. This study is therefore closely connected to an associated min-max problem. However, the min-max problem does not have a saddle point; hence, conventional game-theoretical techniques cannot be applied. Some related work in this direction has been done by Witsenhausen [6]. Some results on the necessary conditions of optimality for min-max problems have also been reported in the literature [7,8].

The paper may be divided into nine sections.

Section 2 is devoted to the definition of the abstract linear system, and the relevant definitions of controllability are presented in Section 3.

In Section 4 we summarize some mathematical results which are needed in the subsequent development.

In Section 5 we study geometrical properties of various sets and obtain the necessary and sufficient conditions of controllability.

In Section 6 we consider the characterization problem while in Section 7 the associated problem is studied.

Section 8 contains a decomposition scheme to obtain numerical solutions of the expressions arising from the necessary and sufficient conditions.

Finally, in Section 9 we apply the main results to the problems of point and functional controllability for linear systems described by ordinary differential equations.

2. Mathematical Description of the System.

Let X_1 and X_2 be reflexive Banach spaces, and X_3 be a Banach space. X_1 is to be thought of as the control space of the system, X_2 the disturbance space of the system and X_3 the state space of the system. Let $\mathcal{U}$ be a closed, bounded convex subset of X_1, $\mathcal{W}$ a closed bounded convex subset of X_2, $\mathcal{B}$ a closed convex subset of X_3 and let s be a given element in X_3.

Let $L(X_1,X_3)$ be the space of continuous linear maps from X_1 into X_3 and let $L(X_2,X_3)$ be the space of continuous linear maps from X_2 into X_3. Let $S \in L(X_1,X_3)$ and $T \in L(X_2,X_3)$ and consider the abstract linear system defined by the operator equation

$$(L) \qquad x = s + Su + Tw \; .$$

A control $u \in \mathcal{U}$ will be called an admissible control, and a disturbance $w \in \mathcal{W}$ will be called an admissible disturbance. The set $\mathcal{B}$ will be referred to as the target.

3. Definitions of Controllability

Definitions 3.1 The system (L) is said to be controllable under disturbance with respect to $(s,\mathcal{U},\mathcal{W},\mathcal{B})$ if there exists an admissible control $\bar{u}$ such that $s + Su + Tw \in \mathcal{B}$ for all admissible disturbances w in $\mathcal{W}$.

Definition 3.2 If in the above definition $\mathcal{B} = \{x_d\}$, where x_d is a given element in X_3, then (L) is said to be strictly controllable under disturbance with respect to $(s,\mathcal{U},\mathcal{W},x_d)$.

Definition 3.3 If in Definition 3.1 $\mathcal{W} = \{0\}$, then (L) is said to be controllable with respect to $(\mathcal{S}, \mathcal{U}, \mathcal{B})$.

Definition 3.4 If in Definition 3.1 $\mathcal{W} = \{0\}$ and $\mathcal{B} = \{x_d\}$ then (L) is said to be strictly controllable with respect to $(\mathcal{S}, \mathcal{U}, x_d)$.

4. Mathematical Preliminaries

In the following we shall use the theory of paired real locally convex Hausdorff spaces as developed in Kelley and Namioka [9, Chapter 5]. If X is a locally convex space, X^* represents its topological dual. The natural pairing of X and X^* is denoted by $\langle X,X^*\rangle$ and the fixed bi-linear functional by $\langle x,x^*\rangle$. The weak topology for X is denoted by $w(X,X^*)$, and the weak* topology for X^* is denoted by $w(X^*,X)$. Real locally convex Hausdorff topological space will be abbreviated as LCTVS. Strong convergence will be denoted by $\rightarrow$ and weak convergence by $\rightharpoonup$. R denotes the real line and R^+ the positive non-zero reals. The origin in the space X is written 0_X, and all the points different from 0_X, $X \sim 0_X$.

We shall also need various results on lower semi-continuous convex functionals defined on a LCTVS, separation theorems for convex sets, and properties of support functionals of convex sets in LCTVS. These will be summarized below.

Theorem 4.1

Let X be a reflexive Banach space and let $f: X \to R$ be convex and strongly lower semi-continuous. Then f is weakly lower semi-continuous.

Proof: Since we are unable to find any specific reference for this theorem, a proof is given in the Appendix A.

Theorem 4.2 [10, p. 106, Proposition 1]

Let X be a LCTVS, A a non-empty convex, compact subset of X, E the set of extreme points of A, and let $f: A \to R$ be a strongly upper semi-continuous convex function. Then f attains its least upper bound in A on at least one point of E.

The following definitions and results on support functionals are due to Hörmander [11].

Definition 4.3

Let X be a LCTVS and let K be a non-empty, closed, convex subset of X. The support functional $H(x^*)$ of K, $x^* \in X^*$, is defined by

$$H(x^*) = \sup [\langle x, x^* \rangle : x \in K] . \qquad (4.1)$$

$H(x^*)$ is clearly everywhere $> -\infty$. Also, from the definition of the support functional it is clear that the closed convex set K is weakly bounded if and only if $H(x^*)$ is finite for all x^* in X^*.

Theorem 4.4

If the support functionals of two closed convex sets K_1 and K_2 are identically equal, then $K_1 = K_2$.

Theorem 4.5

Let K_1 and K_2 be two closed convex sets, and let $H_1(x^*)$

and $H_2(x^*)$ be their corresponding support functionals.
Then $K_1 \subseteq K_2$ if and only if $H_1(x^*) \leq H_2(x^*)$.

Theorem 4.6

A function $H(x^*)$ defined on X^*, $-\infty < H(x^*) \leq \infty$ is the support functional of a closed, convex set in X if and only if $H(x^*)$ is lower semi-continuous for the $w(X^*,X)$ topology and also convex and positively homogeneous.

Theorem 4.7

The support functional of a closed convex set K is strongly continuous if and only if K is strongly bounded.

Thus the support functional of a closed, bounded, convex set K is everywhere finite.

The following theorem is an immediate consequence of the strong separation theorem for convex sets in LCTVS [9, p. 119 Corollary 14.4 and p. 23 Theorem 3.9 and p. 14].

Theorem 4.8

Let X be a Banach space, let A be a weakly compact, convex subset of X and let B be a closed, convex subset of X . Then the following statements are equivalent:

(1) $A \cap B \neq \phi$

(2) $\text{Inf}\{\langle x,x^*\rangle : x \in B\} - \text{Sup}\{\langle x,x^*\rangle : x \in A\} \leq 0$,

$$\forall x^* \in X^* \sim \{0\}$$

(3) $\text{Sup}\{\text{Inf}[\langle x,x^*\rangle : x \in B] - \text{Sup}[\langle x,x^*\rangle : x \in A] :$

$$x^* \in X^* , \|x^*\|_{X^*} = 1\} \leq 0$$

The following definitions and theorem dealing with the Minkowski functional are from Taylor [12, Chapter 3].

Definition 4.9 A set S in a real linear space X is called absorbing if to each $x \in X$ corresponds some $\varepsilon > 0$ such that $\alpha x \in S$ if $0 < |\alpha| \leq \varepsilon$ [12, p. 124].

It is clear at once from the continuity of products and the fact that $0.x = 0$ that each neighborhood of 0 is absorbing.

Definition 4.10 A set S in a real linear space X is called balanced if $\alpha S \subset S$ for all α such that $|\alpha| \leq 1$. [12,p. 123]

Definition 4.11 Let X be a real linear space and K a convex and absorbing subset of X such that $0 \in K$. For each $x \in X$, let A_x be the set of those real α such that $\alpha > 0$ and $x \in \alpha K$. Since K is absorbing A_x is not empty. We then define

$$p(x) = \inf A_x .$$

The functional p is called the Minkowski functional of the set K. [12, p. 134]

Theorem 4.12 Let X be a Banach space. If K is convex, bounded, symmetrical about 0 and has a non-empty interior which contains 0, then the Minkowski functional for K defines a new norm on X. The normed topology resulting from the new norm is the same as the initially given topology.

Proof: Since K has a non-empty interior containing 0, it is absorbing. The proof of the theorem now follows from the properties of K and X [12, p. 132, Theorem 3.4D, p. 135 Theorem 3.41C and p. 136 Theorem 3.41D].

5. Necessary and Sufficient Conditions for Controllability under Disturbance.

In this section we first present a geometrical necessary and sufficient condition for controllability under disturbance. We then consider special restraint and target sets and present an analytical necessary and sufficient condition.

Definition 5.1 For the linear system (L), the unperturbed attainable set A is defined as

$$A = \{ s + \mathcal{S}u : u \in \mathcal{U} \} .$$

Definition 5.2 For the linear system (L), the zone of disturbance D_x at a point $x \in X_3$ is defined as

$$D_x = \{ x + \mathcal{T}w : w \in \mathcal{W} \}.$$

Definition 5.3 For the linear system (L), the reduced target set T_D for the target $\mathcal{B}$ is defined as

$$T_D = \{ x \in X_3 : D_x \subset \mathcal{B} \} .$$

Proposition 5.4 The unperturbed attainable set A and the zone of disturbance D_x are convex and weakly compact.

Proof: The proof of this theorem is an immediate consequence of the linearity and weak continuity of the maps $\mathcal{S}$ and $\mathcal{T}$ and the weak-compactness of the sets $\mathcal{U}$ and $\mathcal{W}$. Q.E.D.

Proposition 5.5 The target set $\mathcal{B}$ and the reduced target set T_D are weakly closed and convex.

Proof: Since $\mathcal{B}$ is convex and closed, it is weakly closed. The convexity of T_D is obvious. We shall show T_D is a strongly closed subset of X_3. Consider a strong Cauchy sequence $\{x_n\}$ in T_D. Since $T_D \subset X$, $x_n \longrightarrow x$, where $x \in X$. For any $w \in \mathcal{W}$, the translated sequence $\{ x_n + \mathcal{T}w \}$ is Cauchy and $x_n + \mathcal{T}w \longrightarrow x + \mathcal{T}w$. However, as points of T_D the x_n's are such that

$x_n + Tw \in \mathcal{B}$, $\forall w \in \mathcal{W}$. But since $\mathcal{B}$ is strongly closed, $x + Tw \in \mathcal{B}$, $\forall w \in \mathcal{W}$, and $D_x \subset \mathcal{B}$ which implies $x \in T_D$. Hence T_D is strongly closed and being convex is thus weakly closed. Q.E.D.

Theorem 5.6 (Geometrical Form of the Necessary and Sufficient Condition for Controllability)

The system (L) is controllable under disturbance with respect to $(s, \mathcal{U}, \mathcal{W}, \mathcal{B})$ if and only if $A \cap T_D \neq \emptyset$.

Proof: (Necessity)

If (L) is controllable under disturbance with respect to $(s, \mathcal{U}, \mathcal{W}, \mathcal{B})$, there exists an admissible $\bar{u}$ such that

$$s + S\bar{u} + Tw \in \mathcal{B}, \ \forall w \in \mathcal{W},$$

and hence $s + S\bar{u} \in T_D$. However, $\bar{u}$ being admissible implies $s + S\bar{u} \in A$, and hence $A \cap T_D \neq \emptyset$.

(Sufficiency) $A \cap T_D \neq \emptyset$ implies that there exists an $x \in X_3$ such that $x \in A$ and $x \in T_D$. Since $x \in A$, there exists a $\bar{u}$ admissible such that $x = s + S\bar{u}$. Likewise $x \in T_D$ implies $x + Tw \in \mathcal{B}$, $\forall w \in \mathcal{W}$ and hence $s + S\bar{u} + Tw \in \mathcal{B}$, $\forall w \in \mathcal{W}$. Q.E.D.

Proposition 5.7 If $\mathcal{W}$ is symmetrical about the origin of X_2 and $\mathcal{B}$ is symmetrical about a given point $x_d \in X_3$, then $T_D \neq \emptyset$ if and only if $x_d \in T_D$.

Proof: Sufficiency is obvious. To prove necessity, let us first define $\mathcal{B}_{x_d} = \{y: y = x - x_d, x \in \mathcal{B}\}$ and $T_D^{x_d} = \{ x : D_x \subset \mathcal{B}_{x_d} \}$. It is clear that $x_d \in T_D$ if and only if $0_{X_3} \in T_D^{x_d}$.

Let $x \in T_D^{x_d}$. Then $D_x \subset \mathcal{B}_{x_d}$. Let $H_1(x^*)$ and $H_2(x^*)$ be the support functionals of D_x and $\mathcal{B}_{x_d}$. Then from Theorem 4.5 $D_x \subset \mathcal{B}_{x_d}$ if and only if $H_1(x^*) \leq H_2(x^*), \forall x^* \in X_3^*$. But

$$H_1(x^*) = \sup [\langle z,x^* \rangle : z \in D_x] = \langle x,x^* \rangle + \sup [\langle w,T^* x^* \rangle : w \in W],$$

and $H_2(x^*) = \sup [\langle y,x^* \rangle : y \in B_{x_d}]$. Hence $D_x \subset B_{x_d}$ if and only if

$$h(x,x^*) = \langle x,x^* \rangle + \sup [\langle w,T^* x^* \rangle : w \in W] - \sup [\langle y,x^* \rangle : y \in B_{x_d}] \le 0,$$

$\forall x^* \in X_3^*$. Now $h(-x,x^*) = h(x,-x^*)$, since W and B_{x_d} are symmetrical about 0_{X_2} and 0_{X_3} respectively. Hence

$$\sup [h(-x,x^*) : \|x^*\|_{X_3^*} = 1] = \sup [h(x,x^*) : \|x^*\|_{X_3^*} = 1] \le 0$$

which implies that $-x \in T_D^{x_d}$. But $T_D^{x_d}$ is convex. Hence $0_{X_3} \in T_D^{x_d}$. Q.E.D.

To obtain necessary and sufficient conditions in analytical form, we shall specialize the sets U, W and B to

$$U = \{u \in X_1 : ||u||_{X_1} \le \rho \ , \ 0 \le \rho < \infty\}$$
$$W = \{w \in X_2 : ||w||_{X_2} \le \beta \ , \ 0 \le \beta < \infty\}$$
$$B = \{x \in X_3 : \|x-x_d\|_{X_3} \le \varepsilon \ , \ 0 \le \varepsilon < \infty \ , \ x_d \in X_3 \text{ given}\} .$$

<u>Proposition 5.8</u> $T_D \neq \emptyset$ if and only if $\sup \{ \beta\|T^* x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1 \} \le \varepsilon$, where $T^* : X_2^* \to X_3^*$ is the adjoint operator of T.

<u>Proof</u>: This follows easily from Proposition 5.7. Q.E.D.

<u>Theorem 5.9</u> <u>(Necessary and Sufficient Condition for Controllability, in Analytical Form).</u>

The system (L) is controllable under disturbance with respect to (s,U,W,B) if and only if

(i) $\sup \{ \beta||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1 \} \le \varepsilon$ (5.1)

(ii) $\langle s-x_d,x^* \rangle - \rho||S^* x^*||_{X_1^*} - \sup \{\langle x,x^* \rangle : x \in T_D^{x_d}\} \le 0, \forall x^* \in X_3^* \sim 0_{X_3^*}$, (5.2)

where $S^* : X_1^* \to X_3^*$ is the adjoint operator of S and $T_D^{x_d}$ is the translate of T_D :

$$T_D^{x_d} = \{ x \in X_3 : \sup [\langle x,x^* \rangle + \beta||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1] \le \varepsilon \}.$$

Condition (ii) of the Theorem is equivalent to

(iii) $\sup\{<s - x_d, x^*> - \rho || S^* x^* ||_{X_1^*} - \sup [<x,x^*> : x \in T_D^{x_d}] : ||x^*||_{X_3^*} = 1\} \leq 0$ ...(5.3)

Proof: From Theorem 5.6, (L) is controllable under disturbance with respect to $(s, U, W.B)$ if and only if $A \cap T_D \neq \emptyset$. From Theorem 4.8, given the non-empty sets A and T_D, $A \cap T_D \neq \emptyset$ if and only if

$$\inf\{<x,x^*> : x \in A\} - \sup\{<x,x^*> : x \in T_D\} \leq 0, \quad \forall x^* \in X_3^* \sim 0_{X_3^*} \quad (5.4)$$

Since $s \in A$, $A \neq \emptyset$, and from condition (i) of the theorem $T_D \neq \emptyset$, and hence (5.4) makes sense. All that remains is to compute explicitly the inf and sup operations. Now,

$$\inf\{<x,x^*> : x \in A\} = \inf\{<s + Su, x^*> : u \in U\} = <s, x^*> - \rho ||S^* x^*||_{X_1^*} .$$

To compute $\sup\{<x,x^*> : x \in T_D\}$, let us first define

$$T_D^{x_d} = \{x \in X_3 : D_x \subset B\}, \quad \text{where } B = \{x \in X_3 : ||x||_{X_3} \leq \varepsilon\} .$$

Hence $\sup\{<x,x^*> : x \in T_D\} = \sup\{<x + x_d, x^*> : x \in T_D^{x_d}\}$

$$= <x_d, x^*> + \sup\{<x,x^*> : x \in T_D^{x_d}\} .$$

Corollary 5.10 In Condition (ii) of Theorem 5.9, the set $T_D^{x_d}$ may be replaced by $\partial T_D^{x_d}$, where $\partial T_D^{x_d}$ is the boundary of $T_D^{x_d}$ in the norm topology of X_3. Moreover,

$$\partial T_D^{x_d} = \{x \in X_3 : \sup[<x,x^*> + \beta ||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1] = \varepsilon\} .$$

Proof: The proof of the above Corollary can be found in the Appendix B.

Corollary 5.11 Let the assumptions of Theorem 5.9 be true. Further, let X_3 be a reflexive Banach space. Then in Condition (ii) of Theorem 5.9, $T_D^{x_d}$ can be replaced by E, where E is the set of extreme points of $T_D^{x_d}$.

Proof: The proof of the corollary follows from the weak compactness of $T_D^{x_d}$ and from Theorems 4.6 and 4.2. A characterization of the extreme points of T_D is not available although recent work of Choquet [13] may be useful.

Corollary 5.12 If $\mathcal{W} = \{0\}$, then (L) is controllable with respect to $(\delta, \mathcal{U}, \mathcal{B})$ if and only if

$$\sup\{\langle \delta - x_d, x^* \rangle - \rho \, \|S^* x^*\|_{X_1^*} : \|x^*\|_{X_3^*} = 1\} \le \varepsilon, \forall\, x^* \in X_3^* .$$

The following proposition verifies the intuitive perceived fact that strict controllability cannot be achieved in the perturbed case.

Proposition 5.13 For the non-trivial ($\beta \neq 0$, $T \neq 0$) system (L) it is impossible to achieve strict controllability.

Proof: Suppose strict controllability is possible, that is $\varepsilon = 0$. Then from Proposition 5.8 $T_D \neq \phi$ if and only if

$$\sup\{\beta \, \|T^* x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\} \le 0$$

which implies

$$\|T^* x^*\|_{X_2^*} = 0 , \quad \forall\, x^* \in X_3^* \text{ such that } \|x^*\|_{X_3^*} = 1 .$$

This is possible if and only if T is the identically zero operator (which is a trivial form of (L)). Therefore, if T is not the zero operator,

$$\sup\{\beta \|T^* x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\} > 0 ,$$

which implies that the reduced target set is empty, making controllability impossible. Q.E.D.

6. The Characterization Problem

If the system (L) is controllable under disturbance, then it is useful to characterize the minimum values of ρ and ε and the maximum value of β for which the system is controllable. This is done in the following theorems.

Theorem 6.1 (Minimum Norm Control)

For given β and ε, assume that (L) is controllable under disturbance for some ρ, $0 \leq \rho < \infty$. Then there exists a unique minimum bound ρ^* for which the system (L) is controllable under disturbance. Furthermore, ρ^* is given by

(1) $\rho^* = 0$ if $\sup\{\langle s - x_d, x^*\rangle - \sup[\langle x, x^*\rangle : x \in T_D^{x_d}] : \|x^*\|_{X_3^*} = 1\} \leq 0$

and

(2) if $\sup\{\langle s - x_d, x^*\rangle - \sup[\langle x, x^*\rangle : x \in T_D^{x_d}] : \|x^*\|_{X_3^*} = 1\} > 0$

$\rho^* = \bar{\rho}$ is the unique solution of

$$\sup\{\langle s - x_d, x^*\rangle - \bar{\rho}\, \|S^* x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in T_D^{x_d}] : \|x^*\|_{X_3^*} = 1\} = 0$$

Consider the function

$$f : (R^+ \cup \{0\}) \times X_3^* \to R : (\rho, x^*) \mapsto \langle s - x_d, x^*\rangle - \rho\, \|S^* x^*\|_{X_1^*} - \sup[\langle x, x^*\rangle : x \in E]$$

and the function

$$g : R^+ \cup \{0\} \to R : \rho \mapsto \sup \{f(\rho, x^*) : ||x^*||_{X_3^*} = 1\}$$

We show that g is monotonically decreasing, continuous, convex function of ρ. For $\rho_2 \geq \rho_1 \geq 0$,

$$f(\rho_2, x^*) \leq f(\rho_1, x^*) \ , \forall x^* \in X_3^* \ ,$$

and hence $g(\rho_2) \leq g(\rho_1)$ showing that g is monotonically decreasing. For $\rho_2 \neq \rho_1$ and $\lambda \in [0,1]$,

$$f(\lambda\rho_1 + (1-\lambda)\rho_2, x^*) = \lambda f(\rho_1, x^*) + (1-\lambda) f(\rho_2, x^*) \ , \forall x^* \in X_3^*,$$

which implies that g is convex. For $\rho_2 \geq \rho_1 \geq 0$,

$$|g(\rho_2) - g(\rho_1)| = g(\rho_1) - g(\rho_2) \leq \sup [(\rho_2 - \rho_1) \ ||S^* x^*||_{X_1^*} : ||x^*|| = 1]$$
$$\leq |\rho_2 - \rho_1| \ .K \ ,$$

where $\sup [\ ||S^* x^*||_{X_1^*} : ||x^*|| = 1\ ] = K < \infty$, since S^* is continuous. This shows that g is a continuous function of ρ.

Since (L) is controllable under disturbance for $\tilde{\rho} \geq 0$, we have $g(\tilde{\rho}) \leq 0$. There are two cases to consider:

(i) $g(0) \leq 0$. Then the unique minimum bound $\rho^* = 0$.

(ii) $g(0) > 0$. Then by virtue of the properties of the function $g(\rho)$, ρ^* is given by the unique solution in R^+ of $g(\rho) = 0$. Q.E.D.

Theorem 6.2 (Maximum Norm Disturbance)

Given the bounds ε and ρ, assume that the system (L) ($T \neq 0$) is controllable under disturbance for $\beta = 0$. Then there exists a unique maximum bound β^* such that (L) is controllable under disturbance if and only if $\beta \leq \beta^*$. Moreover, defining $\bar{\beta}$ by

$$\bar{\beta} = \frac{\varepsilon}{\sup [\ ||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\ ]}$$

β^* is given by

(i) $\beta^* = \bar{\beta}$ if $f(\bar{\beta}) \leq 0$,

(ii) $\beta^* = \hat{\beta}$ if $f(\bar{\beta}) \leq 0$, where $\hat{\beta}$ is the unique solution of $f(\beta) = 0$ in $[0,\bar{\beta}]$. $f(\beta)$ is defined by

$$f(\beta) = \sup\{<\delta - x_d, x^*> - \rho||S^*x^*||_{X_1^*} - \sup[<x,x^*> : x \in T_D^{x_d}(\beta)] : ||x^*||_{X_3^*} = 1\} .$$

Proof: A necessary condition for controllability of (L) is $T_D \neq \emptyset$ which implies $\beta \leq \bar{\beta}$. We shall show that f is monotonically increasing, convex and continuous on $[0,\bar{\beta}]$. Hence there are two cases:

(i) $f(\bar{\beta}) \leq 0$. In that case (L) is controllable and hence $\beta^* = \bar{\beta}$.

(ii) $f(\bar{\beta}) > 0$. Since $f(0) \leq 0$ by hypothesis and $f(\bar{\beta}) > 0$, $f(\beta)$ has a unique solution $\hat{\beta}$ on $[0,\bar{\beta}]$ and $\beta^* = \hat{\beta}$.

Now if $0 \leq \beta_1 \leq \beta_2 \leq \bar{\beta}$,

$$\beta_1 \sup[||T^*x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1] \leq \beta_2 \sup[||T^*x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1] ,$$

and therefore

$$T_D(\beta_1) \supset T_D(\beta_2) ,$$

and hence using Theorem 4.5 $f(\beta_1) \leq f(\beta_2)$, which shows that f is monotonically increasing.

Since $\lambda T_D(\beta_1) + (1 - \lambda)T_D(\beta_2) \subset T_D(\lambda\beta_1 + (1 - \lambda)\beta_2)$, $\forall \lambda \in [0,1]$, it follows from Theorems 4.5 and 4.6 that

$$\lambda \sup[<x,x^*> : x \in T_D(\beta_1)] + (1 - \lambda)\sup[<x,x^*> : x \in T_D(\beta_2)]$$
$$\leq \sup[<x,x^*> : x \in T_D(\lambda\beta_1 + (1 - \lambda)\beta_2)] , \forall x^* \in X_3^*.$$

Therefore, $f(\lambda\beta_1 + (1 - \lambda)\beta_2) \leq \lambda f(\beta_1) + (1 - \lambda)f(\beta_2)$, which shows that f is convex on $[0,\bar{\beta}]$.

The convexity of f implies its continuity on $(0,\bar{\beta}]$. The continuity of f at 0 may be proved in a manner analogous to Theorem 6.1. Q.E.D.

Theorem 6.3 (Minimum Miss-distance)

Given the bounds ρ and β, there exists a unique minimum bound ε^* such that (L) is controllable if and only if $\varepsilon \geq \varepsilon^*$. Moreover defining $\bar{\varepsilon}$ by

$$\bar{\varepsilon} = \sup\{\beta ||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\} ,$$

ε^* is given by

(i) $\varepsilon^* = \bar{\varepsilon}$ if $f(\bar{\varepsilon}) \leq 0$

(ii) $\varepsilon^* = \hat{\varepsilon}$ if $f(\varepsilon) > 0$, where $\hat{\varepsilon}$ is the unique solution of $f(\varepsilon) = 0$ in $[\bar{\varepsilon}, \infty)$. $f(\varepsilon)$ is defined by

$$f(\varepsilon) = \sup\{<\delta - x_d, x^*> - \rho ||S^* x^*||_{X_1^*} - \sup[<x,x^*> : x \in T_D^{x_d}(\varepsilon)]: ||x^*||_{X_3^*} = 1\}.$$

Proof: We first show that for some $\varepsilon_c \geq 0$, the system (L) is controllable. Let α and ε_c be defined as

$$\alpha = \sup\{<\delta - x_d, x^*> - \rho ||S^* x^*||_{X_1^*} : ||x^*||_{X_3^*} = 1\}$$

$$\varepsilon_c = |\alpha| + \bar{\varepsilon} .$$

Clearly $B(|\alpha|) = \{x \in X_3 : ||x||_{X_3} \leq |\alpha|\} \subset T_D^{x_d}$, and for any $x^* \in X_3^*$, the norm of which is unity

$$\alpha = \sup\{<x,x^*> : x \in B(|\alpha|)\} \leq \sup\{<x,x^*> : x \in T_D^{x_d}\},$$

which implies that (L) is controllable for $\varepsilon = \varepsilon_c$ since $f(\varepsilon_c) \leq \alpha - |\alpha| \leq 0$.

A necessary condition for controllability of (L) is $T_D \neq \emptyset$, which implies $\varepsilon \geq \bar{\varepsilon}$. We shall show that f is monotonically decreasing, convex on $(\bar{\varepsilon}, \infty)$ and continuous on $(\bar{\varepsilon}, \varepsilon_c]$. Hence there are two cases:

(i) $f(\bar{\varepsilon}) \leq 0$. In that case (L) is controllable and hence $\varepsilon^* = \bar{\varepsilon}$.

(ii) $f(\bar{\varepsilon}) > 0$. Since $f(\varepsilon_c) \leq 0$ and $f(\bar{\varepsilon}) > 0$, $f(\varepsilon) = 0$ has a unique solution on $(\bar{\varepsilon}, \varepsilon_c]$, and $\varepsilon^* = \hat{\varepsilon}$.

The properties of the function f may be proved in a manner analogous to Theorems 6.1 and 6.2. Q.E.D.

7. A Min-max Problem

It is not too surprising to find that there is a relation between the bound $\tilde{\varepsilon}$ on the minimal target and the following expression:

$$\min \{\max [||s + Su + Tw - x_d||_{X_3} : w \in W]: u \in U\}. \quad (7.1)$$

This min-max expression naturally arises when one desires to use the best control to get as close as possible to the center of the target in the presence of the worst disturbance; it is basically a game problem with no saddle point.

Theorem 7.1 Let ε^* be the minimal bound on the target set B as defined in Theorem 6.3 corresponding to the given sets of admissible controls and disturbances, U and W; then :

(i) $\inf \{\sup [||s + Su + Tw - x_d||_{X_3} : w \in W]: u \in U\} = \varepsilon^*$

(ii) if X_3 is a finite dimensional Banach space, there exists u^* U and w^* W such that

$$||s + Su^* + Tw^* - x_d||_{X_3} = \inf\{\sup[\ ||s + Su + Tw - x_d||_{X_3} : w \in W]: u \in U\}.$$

Proof: (i) Let $\varepsilon_o = \inf\{\sup [||s + Su + Tw - x_d||_{X_3} : w \in W]: u \in U\}$, then $\varepsilon_o \leq \varepsilon^*$. The proof is by contradiction. If $\varepsilon_o < \varepsilon^*$, $\forall \ \eta > 0$, there exists $\bar{u} \in U$ such that

$$\sup\{|| s + S\bar{u} + Tw - x_d|| _{X_3} : w \in W\} < \varepsilon_o + \eta ;$$

in particular for $\eta = \varepsilon^* - \varepsilon_o$, the above expression contradicts the minimality of ε^*.

(ii) It is readily seen from part (i) and the definition of ε^* that there exists $u^* \in U$ such that

$$\inf\{\sup[||s + Su + Tw - x_d||_{X_3} : w \in W]: u \in U\} = \sup\{||s + Su^* + Tw - x_d||_{X_3} : u \in U\} .$$

Consider now the function

$$n : X_2 \to R : w \mapsto ||s + Su^* + Tw - x_d||_{X_3}$$

Let $T(w) = \{Tw : w \in W\}$. The set $T(w)$ is compact.

$$\sup[||s + Su^* + Tw - x_d||_{X_3} : w \in W] = \sup[||s + Su^* + x - x_d : x \in T(w)]$$

The function n is continuous with respect to x and the set $T(w)$ is compact. Hence there exists a $w^* \in W$ at which n assumes its maximum value.

The previous results concerning the existence of an admissible control and disturbance satisfying the min-max problem naturally call for a geometrical interpretation.

<u>Theorem 7.2</u>. Let X_3 be a finite dimensional Banach space and let $\bar{u}$, $\bar{w}$ be such that $||s + S\bar{u} + T\bar{w} - x_d||_{X_3} = \min\{\max[||s + Su + Tw - x_d||_{X_3} : w \in W] : u \in U\} = \varepsilon^*$, then $s + S\bar{u} + T\bar{w} \in \partial B(\varepsilon^*) \cap \partial D_{s+S\bar{u}}$; in particular $T\bar{w} \in \partial D_0$.

<u>Proof</u>: Clearly $||s + S\bar{u} + T\bar{w} - x_d||_{X_3} = \varepsilon^*$ if and only if $s + S\bar{u} + T\bar{w} \in \partial B(\varepsilon^*)$ since $B(\varepsilon^*)$ is the closed ball in X_3 with centre x_d and radius ε^*. The remaining part of the proof is by contradiction; if $s + S\bar{u} + T\bar{w} \notin \partial D_{s + S\bar{u}}$, then $s + S\bar{u} + T\bar{w} \in \text{int } D_{s + S\bar{u}}$. Since $D_{s+S\bar{u}} \subset B(\varepsilon^*)$, $s + S\bar{u} + T\bar{w} \in \text{int } B(\varepsilon^*)$, which is in contradiction with the first result.

Q.E.D.

<u>Corollary 7.3</u>. In the unperturbed case ($W = \{0\}$), if $\bar{u}$ is such that

$$||s + S\bar{u} - x_d||_{X_3} = \min\{||s + Su - x_d||_{X_3} : u \in U\} = \varepsilon^*,$$

then $s + S\bar{u} \in \partial B(\varepsilon^*)$.

8. A Decomposition Scheme and the Numerical Problem.

The relatively simple structure of the analytical expressions derived in the previous sections is sufficient motivation to look for a general iterative scheme involving successive approximations in order to solve the controllability problem. We shall restrict ourselves to the case where the target, control and disturbance sets are balls in their respective Banach spaces:

$$\mathcal{B} = \{x \in X_3 : ||x - x_d||_{X_3} \leq \varepsilon \} \tag{8.1}$$

$$\mathcal{U} = \{x \in X_1 : ||x||_{X_1} \leq \rho \} \tag{8.2}$$

$$\mathcal{W} = \{x \in X_2 : ||x||_{X_2} \leq \beta \}. \tag{8.3}$$

In the above case the necessary and sufficient conditions for controllability are described in Theorem 5.9.

The decomposition will aim at separating the effect of the perturbations from the effect of the controls by judiciously constructing a new norm in the state space X_3. In this we remain faithful to our previous techniques which have involved the definition of the reduced target set T_D.

Proposition 8.1 For $\mathcal{B}$, $\mathcal{U}$ and $\mathcal{W}$ as described in(8.1), (8.2) and (8.3), $T_D^{x_d}$ has a non-empty interior which contains 0 if and only if

$$\sup \{\beta||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\} < \varepsilon . \tag{8.4}$$

Proof: Clearly ε cannot be zero, and the proof is obvious from Proposition 5.8 and the fact that 0 has an open neighborhood. Q.E.D.

Theorem 8.2 Given $\mathcal{B}$, $\mathcal{U}$ and $\mathcal{W}$ as described in (8.1), (8.2) and (8.3) such that

$$\sup\{\beta||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\} < \varepsilon \quad ,$$

and the new norm for X_3 defined by the Minkowski functional p on $T_D^{x_d}$,

$$\{|x|\}_{X_3} = \varepsilon . p(x), \qquad (8.5)$$

the necessary and sufficient condition for controllability reduces to

$$\sup\{<\delta - x_d, x^*> - \rho ||S^* x^*||_{X_1^*} : \{|x^*|\}_{X_3^*} = 1\} \leq \varepsilon, \qquad (8.6)$$

where

$$\{|x^*|\}_{X_3^*} = \sup\{<x,x^*> : \{|x|\}_{X_3} \leq 1\} = \frac{1}{\varepsilon} \sup\{<x,x^*>: x \in T_D^{x_d}\} \qquad (8.7)$$

is the natural norm in X_3^* arising from the new norm in X_3.

Furthermore, the norm $\{|x|\}$ for x in X_3 can be computed as follows:

$$\{|x|\} = \begin{cases} 0 \ , \text{ if } x = 0 \\ \varepsilon/c \ , \text{ if } x \neq 0 \end{cases} , \qquad (8.8)$$

where c is the unique non-negative solution of the equation

$$\sup\{c<x,x^*> + \beta ||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\} = \varepsilon; \qquad (8.9)$$

the left-hand side of expression (8.9) is a non-negative, monotonically increasing, convex and continuous function for all non-negative real c.

<u>Corollary 8.3</u> In the unperturbed case ($\beta = 0$, $W = \{0\}$), if $\varepsilon > 0$, $\{|x|\}_{X_3} = ||x||_{X_3}$, and the necessary and sufficient condition for controllability is

$$\sup\{<\delta - x_d, x^*> - \rho ||S^* x^*||_{X_1^*} : ||x^*|| = 1\} \leq \varepsilon \ .$$

<u>Proof of Theorem 8.2</u>: The expressions (8.5), (8.6) and (8.7) clearly follow from the definition of a new norm in X_3. The equivalence of the definitions of $\{|x|\}$ in the expressions (8.5) and (8.8) is the only point that requires a proof. Let f be defined as

$$f: R^+ \cup \{0\} \to R^+ \cup \{0\} \ ,$$

$$c \longmapsto \sup\{c<x,x^*> + \beta ||T^* x^*||_{X_2^*} : ||x^*||_{X_3^*} = 1\} \ .$$

It is readily seen that f is well-defined, monotone increasing, convex and continuous on $R^+ \cup \{0\}$. If x is not 0, then the solution $\bar{c}$ of the expression (8.9) is finite and non-zero, so $\left(|x|\right)$ is well defined. The Minkowski functional p(x) (Definition 4.11) is zero if x is 0; if x is not 0, then

$$p(x) = \inf\{ a \mid \sup[\langle \tfrac{1}{a}x, x^* \rangle + \beta \| T^* x^* \|_{X_2^*} : \| x^* \|_{X_3^*} = 1] \leq \varepsilon \}.$$

The properties of the function f show that there exists a unique $\bar{c}$ in R^+ such that $f(\bar{c}) = \varepsilon$. Since f is monotone increasing,

$$\tfrac{1}{a}x \in T_D^{x_d} \iff 0 < \tfrac{1}{a} \leq \bar{c} ;$$

however, $0 < \bar{c}$ by expression (8.4), and hence $p(x) = \frac{1}{\bar{c}}$. Q.E.D.

It is interesting to note that the decomposition is achieved by distorting the original unit ball in X_3. In doing so the effects of the controls and the disturbances are separated from each other, reducing the numerical problem to the determination of the supremum of each of the following expressions:

1. $\langle \delta - x_d, x^* \rangle - \rho \| S^* x^* \|_{X_1^*}$, for $\left(|x^*|\right)_{X_3^*} = 1$,
2. $\langle cx, x^* \rangle + \beta \| T^* x^* \|_{X_2^*}$, for $\| x^* \|_{X_3^*} = 1$,
3. $\langle x, x^* \rangle$, for $\left(| x |\right)_{X_3} = 1$.

The above expressions are continuous and positively homogeneous functions which are either convex, concave or linear on a unit ball in a Banach space.

9. Applications to Differential Equations.

In this section we shall show how the theory developed in the previous sections may be applied to differential systems.

We consider two cases:

(a) Controllability under disturbance of linear differential systems in a Banach space

(b) Functional Controllability under disturbance of linear systems described by ordinary differential equations.

<u>Controllability of Linear Differential Systems in a Banach Space</u>

Let U , W and X be real Banach spaces. Let $t_1 > 0$ and $1 < p < \infty$. We define $L^p_{t_1}(U)$ as the Banach space of all U-valued strongly measurable functions defined on $[o,t_1]$ such that

$$\int_0^{t_1} ||u(t)||^p_U \, dt < \infty .$$

The Banach space $L^p_{t_1}(U)$ is normed by

$$||u||_p = (\int_0^{t_1} || u(t)||^p_U \, dt)^{\frac{1}{p}}$$

We shall assume that the space $L^p_{t_1}(U)$ is reflexive. This will be the case if U is reflexive and separable or uniformly convex. The dual space $[L^p_{t_1}(U)]^*$ is isometrically isomorphic to $L^q_{t_1}(U^*)$, $\frac{1}{p} + \frac{1}{q} = 1$.

In an analogous manner, define $L^{p'}_{t_1}(W)$, $1 < p' < \infty$, as the Banach space of all W-valued strongly measurable functions with norm

$$||w||_{p'} = (\int_0^{t_1} ||w(t)||^{p'}_W dt)^{\frac{1}{p'}}$$

and assume that $L^{p'}_{t_1}(W)$ is reflexive.

Consider the linear differential system

$$(L_D) \qquad \dot{x}(t) = Ax(t) + Bu(t) + Cw(t) \tag{9.1}$$

with initial condition $x(0) \in X$, where A is a linear closed operator with domain D(A) which is the infinitesimal generator of a strongly continuous semi-group T(t), $t \geq 0$, of linear bounded operators, B is a linear bounded operator mapping U into D(A) and C is a linear bounded operator mapping W into D(A). We shall say that x(.) is a solution of (9.1) with initial condition $x(0) \in X$ if x(.) satisfies the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-\tau)Bu(\tau)d\tau + \int_0^t T(t-\tau)Cw(\tau)d\tau \tag{9.2}$$

where the integrals in the right hand side of (9.2) are in the sense of Bochner [15, Sec. 3.7, p. 78].

For given $t_1 \geq 0$, define linear bounded transformations

$$R_T: L^p_{t_1}(U) \to X : u \mapsto \int_0^{t_1} T(t_1 - \tau)Bu(\tau)d\tau$$

$$S_T: L^{p'}_{t_1}(W) \to X : w \mapsto \int_0^{t_1} T(t_1 - \tau)Cw(\tau)d\tau \ ,$$

and define $T(t_1)x(0) = x_o$. Then (9.2) can be written as

$$x(t_1) = x_o + R_{t_1}u + S_{t_1}w \tag{9.3}$$

Let $\Omega_u \subset L^p_{t_1}(U)$, $\Omega_w \subset L^{p'}_{t_1}(W)$ and $K \subset X$ be defined by

$$\Omega_u = \{ u : ||u||_p \leq \rho \}$$

$$\Omega_w = \{ w : ||w||_{p'} \leq \beta\}$$

and $$K = \{ x : ||x - x_d||_X \leq \varepsilon\},$$

where x_d is a given element in X.

Definition 9.1

(L_D) is controllable under disturbance with respect to $(x_o, \Omega_u, \Omega_w, K, t_1)$ if there exists a $\bar{u} \in \Omega_u$ such that $x_o + R_{t_1}\bar{u} + S_{t_1}w \in K$ for all $w \in \Omega_w$.

With this specification of the problem and the above definition of controllability the theory developed in the previous sections is applicable.

Functional Controllability

Consider the linear differential system

$$(L_o) \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)w(t), \qquad (9.4)$$

where $x(t) \in x = R^n$, $u(t) \in U = R^m$, $w(t) \in W = R^p$ and A(t), B(t), C(t) are matrices of appropriate order which are bounded measurable on the given compact interval $[0,t_1]$. Let u(t) and w(t) be Lebesgue-measurable functions and define $L^p_{t_1}(U)$, $L^{p'}_{t_1}(W)$, $1 < p,p' < \infty$, and the sets Ω_u, Ω_w in a manner analogous to the previous case, and let $K \subset L^2_{t_1}(x)$ be defined by

$$K = \{ x : ||x - x_d||_2 \leq \varepsilon \}$$

where x_d is a given element in $L^2_{t_1}(X)$. For given $x(o) \in X$, $u \in L^p_{t_1}(U)$ and $w \in L^{p'}_{t_1}(W)$, and interpreting Eq. (9.4) in the Caratheodory sense, there exists an absolutely continuous function $x(\cdot;u,w)$ defined on the compact interval $[0,t_1]$ which satisfies Eq. (9.4) almost everywhere. Since $x(\cdot;u,w)$ is absolutely continuous on the compact interval $[0,t_1]$, $x(\cdot;u,w) \in L^2_{t_1}(X)$.

The solution of (9.4) is given by

$$x(t) = \Phi(t,o)\, x(o) + \int_o^t \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau + \int_o^t \Phi(t,\tau)\, C(\tau)\, w(\tau)\, d\tau \qquad (9.5)$$

where Φ is the fundamental matrix of solutions. Let

$$R : L^p_{t_1}(U) \to L^2_{t_1}(X)$$

be the linear bounded transformation defined by

$$(Ru)(t) = \int_o^t \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau \qquad 0 < t \leq t_1 ,$$

and let

$$S : L^{p'}_{t_1}(W) \to L^2_{t_1}(X)$$

be the linear bounded transformation defined by

$$(Sw)(t) = \int_o^t \Phi(t,\tau)\, C(\tau)\, w(\tau)\, d\tau \qquad 0 < t \leq t_1 ,$$

and let $\Phi(t,o)\, x(o) = x_o$.

Then Eq. (9.5) can be written as

$$x = x_o + Ru + Sw \quad , \qquad (9.6)$$

where

$$x = x(\cdot;u,w) \quad .$$

Definition 9.2

(L_o) is functionally controllable under disturbance with respect to $(x_o, \Omega_u, \Omega_w, K, t_1)$ if there exists a $\bar{u} \in \Omega_u$ such that $x(\cdot;\bar{u},w) \in K$, for all $w \in \Omega_w$.

The problem of functional controllability under disturbance can now be treated using the results of previous sections.

APPENDIX A

Definition A.1

The functional f is called weakly lower semi-continuous at the point x_o if, for any $\{x_n\}$ which converges weakly to x_o,

$$f(x_o) \leq \underline{\lim}_{n\to\infty} f(x_n)$$

([16] p. 73, Definition 8.1).

Theorem A.2

Let X be a reflexive Banach space, and let $f : X \to R$ be convex and strongly lower semi-continuous. Then f is weakly lower semi-continuous.

Proof: The proof is by contradiction. Let us assume that the functional f is not weakly semi-continuous. Hence for some x_o in X Definition A.1 is not satisfied; that is there exists a weakly convergent sequence, $\{x_n\}$,

$$\{x_n\} \rightharpoonup x_o$$

such that

$$f(x_o) > \underline{\lim}_{n\to\infty} f(x_n) .$$

This implies that for some positive real ε, we can find a subsequence, also denoted by $\{x_n\}$, for which

$$f(x_n) \leq f(x_o) - \varepsilon \quad , \quad \text{for all } n . \tag{A.1}$$

Now since X is a reflexive Banach space, for every weakly convergent sequence and any positive real δ, we can find elements x_i, $i = 1,\ldots,N(\delta)$

such that

$$||x_o - \sum_{i=1}^{N} \alpha_i^N x_i||_X < \delta \text{ , where } \alpha_i^N \geq 0 \text{ and } \sum_{i=1}^{N} \alpha_i^N = 1$$

([17] p. 422, Corollary 14).

Since f is strongly lower semi-continuous, we can choose δ sufficiently small such that

$$f(\sum_{i=1}^{N} \alpha_i^N x_i) > f(x_o) - \frac{\varepsilon}{2} \quad . \tag{A.2}$$

However, the convexity of the functional f and Eq. (A.1) and (A.2) lead to a contradiction :

$$f(x_o) - \frac{\varepsilon}{2} < f(\sum_{i=1}^{N} \alpha_i^N x_i) \leq \sum_{i=1}^{N} \alpha_i^N f(x_i) \leq \sum_{i=1}^{N} \alpha_i^N (f(x_o) - \varepsilon) = f(x_o) - \varepsilon$$

and

$$\frac{\varepsilon}{2} > \varepsilon > 0 \quad .$$

Q.E.D.

APPENDIX B

We may characterize the boundary (with respect to the norm topology) of T_D as follows:

Theorem B.1

If the set T_D is not empty, its boundary ∂T_D (in the norm topology) is the set of points x in X_3 such that

$$\sup\{\langle x-x_d, x_3^* \rangle + \beta \| T^* x_3^* \|_{X_2^*} : \|x_3^*\|_{X_3^*} = 1\} = \varepsilon \ .$$

Proof: Let the function f on X_3 into R be defined as

$$x \mapsto \sup\{\langle x-x_d, x_3^* \rangle + \beta \| T^* x_3^* \|_{X_2^*} : \|x_3^*\|_{X_3^*} = 1\} \ .$$

From Theorem 4.5,

$$T_D = \{x \in X_3 : f(x) \leq \varepsilon\} \ .$$

(a) Let x_o be a point of T_D such that $f(x_o) = \varepsilon$. We assume x_o to be an interior point of T_D (in the strong topology) and obtain a contradiction. There exists an open ball $B_\delta(x_o)$ at x_o entirely contained in T_D, since x_o is an interior point of T_D. However,

$$\sup\{f(x) : x \in B_\delta(x_o)\} = \varepsilon + \delta \quad ,$$

as can easily be computed by changing the order of the two suprema ([14], p. 352, Proposition 9). This clearly implies that there exists some x in $B_\delta(x_o) \subset T_D$ for which $f(x) > \varepsilon$ in contradiction with the definition of T_D.

(b) Let x_o be a boundary point of T_D (in the strong topology). We assume that

$$\alpha = f(x_o) < \varepsilon$$

and show that x_o cannot be a boundary point of T_D. Let $\delta = (\varepsilon - \alpha)/2$ and $N_\delta(x_o)$ be an open ball of radius δ at x_o. For all y in $N_\delta(x_o)$

$$f(y) \le \alpha + \|y - x_o\|_{X_3} \le \alpha + \frac{\varepsilon - \alpha}{2} < \varepsilon \; ;$$

x_o is an interior point of T_D, since there is an open neighborhood of x_o entirely contained in T_D.

Q.E.D.

<u>Theorem B.2</u>

If $T_D^{x_d}$ is not empty,

$$\sup\{<x,x^*> \; : \; x \in T_D^{x_d}\} = \sup\{<x,x^*> \; : \; x \in \partial T_D^{x_d}\}$$

<u>Proof</u>: By Theorem 4.5, for any $x \in T_D^{x_d}$,

$$\alpha = \sup\{<x,x^*> + \beta \, \|T^* x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\} \le \varepsilon \quad .$$

But by Theorem 8.2, there exists $c \ge 1$ such that

$$\sup\{<cx,x^*> + \beta \, \|T^* x^*\|_{X_2^*} : \|x^*\|_{X_3^*} = 1\} = \varepsilon \; ,$$

and by Theorem B.1, $cx \in \partial T_D^{x_d}$. We then have for any x in $T_D^{x_d}$

$$|<x,x^*>| \le |<cx,x^*>|$$

and

$$\sup\{|\langle x,x^*\rangle| : x \in T_D^{x_d}\} \leq \sup\{|\langle x,x^*\rangle| : x \in \partial T_D^{x_d}\} \quad .$$

The theorem follows from the linearity of the functional $\langle x,x^*\rangle$ and from the fact that $T_D^{x_d}$ is symmetrical about the origin 0_{X_3} .

Q.E.D.

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