The Nature of Tournaments

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Abstract

This paper characterizes the optimal way for a principal to structure a rank-order tournament in a moral hazard setting (as in Lazear and Rosen (1981)). We find that it is generally optimal to give rewards to top performers that are smaller in magnitude than corresponding punishments to poor performers. The paper identifies four reasons why the principal might prefer to give larger rewards than punishments: (i) R is small relative to P (where R is risk aversion and P is absolute prudence); (ii) the distribution of shocks to ouput is asymmetric and the asymmetry takes a particular form; (iii) the principal faces a limited liability constraint; and (iv) there is agent heterogeneity of a particular form. An intuition is given as to why these factors affect the optimal prize schedule. Using the theory developed by Green and Stokey (1983), we relate the results about tournaments to the structure of the optimal individual contract. The optimal individual contract typically punishes low output more than it rewards high output. We also give conditions under which the optimal individual contract will be a concave function of output.

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1 Introduction

Lazear and Rosen (1981) argue that rank-order tournaments help to solve a moral hazard problem faced by firms. Lazear-Rosen tournaments have been interpreted as explaining many features of firms, such as within-firm job promotions, wage increases, bonuses, and CEO compensation (Prendergast (1999)).

This paper attempts to characterize the optimal way to structure a tournament that is set up by a principal to deal with a moral hazard problem. These results have considerable practical significance. They allow us to test whether aspects of employee compensation arise because of or in spite of the moral hazard theory of tournaments.

We find that, typically, the optimal prize schedule gives special rewards to a few of the best performers, special punishments to a few of the worst performers, and somewhat smaller rewards/punishments for those whose performance is neither at the top nor bottom of the distribution. Furthermore, the reward for placing ith in the tournament rather than $(i+1)$th is smaller than the punishment for placing $(n-i+1)$th rather than $(n-i)$th (where n is the number of agents in the tournament) when $i \leq \frac{n-1}{2}$. In particular, this means that the punishments for the worst performers are greater in magnitude than the rewards for the best performers.

The particular shape of the optimal prize schedule depends greatly upon the distribution of the shocks to agents' output. We find that a set of weights, $\{\beta_i\}_{i=1}^n$, which can be calculated solely based upon the shock distribution, encapsulates the effect of the shock distribution on the optimal prize schedule. The weight β_i is equal to the marginal change in placing ith in the tournament from a marginal change in effort. In fact, when agents' utility for wealth is logarthmic, the optimal prize schedule is simply an affine transformation of the weight schedule.[1]

While, in general, optimal tournaments punish more than they reward, there are four factors that lead the rewards to be large relative to the punishments. We find that the

[1] When utility for wealth is logarithmic and the shock distribution is symmetric (in the sense that $F(-x) = 1 - F(x)$), we find that the rewards for the best performers are exactly equal to the punishments for the worst performers.

amount of punishment relative to reward depends upon the size of R relative to P, where R is Arrow-Pratt risk aversion and P is the coefficient of absolute prudence. When R is sufficiently low relative to P, it may be optimal for the principal to give larger rewards than punishments.[2] If the principal faces a limited liability constraint (a constraint on how much agents can be punished), this may limit the principal's ability to punish and lead the principal to rely more heavily upon rewards to incentivize agents. The optimal size of rewards relative to punishments also depends upon the distribution of the shocks to agents' output. If the shock distribution is asymmetric ($F(x) \neq 1 - F(x)$) in a manner to be defined below, it may be optimal to give large rewards relative to punishments. Finally, if the agents participating in the tournament are heterogeneous in a manner to be defined below, the principal may wish to give large rewards. Associated with each of these four factors is a distinct intuition that we will attempt to convey below.

While it is generally optimal for the principal to punish more than reward, the principal still gives special rewards to top performers in such cases. In order to examine the importance of rewards relative to punishments as tools to the principal for incentivizing agents, we look at tournaments that give a prize w_1 to the top j performers and a prize w_2 to the bottom $n-j$ performers. We find that, in general, the principal chooses $j \geq \frac{n}{2}$. Furthermore, a winner-take-all tournament ($j = 1$) is usually a less profitable way to structure a tournament than a loser-lose-all tournament ($j = n - 1$). When the shocks to agents' output are uniformly distributed, $j^* = n - 1$. While j^* is not necessarily equal to $n - 1$ when the shocks follow a non-uniform distribution, it is often the case.

These results speak, to some extent, to the importance of punishment as a tool to the principal. Perhaps more importantly, however, we find that the optimal two-prize tournament returns a profit to the principal that closely approximates the profit from the optimal n-prize tournament. The profits to the principal from non-optimal choices of j are generally far from the profits from the optimal n-prize tournament. A consequence of this

[2] The concept of absolute prudence is due to Kimball (1990) who analyzes its role on precautionary saving in a dynamic model. The relationship between risk aversion and absolute prudence has been explored in a variety of settings different to ours (see, for example, Carroll and Kimball (1996) and Caplin and Nalebuff (1991)).

finding is that the optimal loser-lose-all tournament often returns a profit that is close to the profit from the optimal tournament. This cannot be said of the optimal winner-take-all tournament.

Using the theory developed by Green and Stokey (1983), we are able to relate our results about the optimal tournament prize structure to the structure of the optimal individual contract. When optimal tournaments gives larger punishments than rewards, the optimal individual contract also uses more punishment than reward. In the special case where the shocks to output follow a normal distribution, the optimal individual contract is generally concave.

The paper will proceed as follows. Section 2 provides a brief review of the existing literature. Section 3 gives the basic setup of the model and states the problem of the principal designing the tournament. Section 4 establishes the main results of the paper, giving a partial characterization of the optimal prize schedule. Section 5 considers the case in which the principal is limited to awarding only two types of prizes. Section 6 considers the factors that might make it optimal to give larger rewards to winners than punishments to losers. Section 7 considers the implications of our results for the structure of the optimal individual contract. Section 8 contains some concluding remarks.

2 Brief Literature Review

Classic treatments of tournaments are given by Lazear and Rosen (1981), Green and Stokey (1983) and Nalebuff and Stiglitz (1983). The two papers most closely related to this one are Krishna and Morgan (1998) and Moldovanu and Sela (2001). They also focus on optimal prize structures in rank-order tournaments. While the assumptions made in these papers differ from those of Lazear and Rosen (1981), the results are generally seen as applicable to the Lazear-Rosen context. Both of these papers conclude that optimally designed tournaments give a special prize to top performers. In fact, they suggest that roughly the same prize should be given to all but a few of the *best* performers.

In contrast, we find that it is often optimal for the principal to rely more heavily on punishing poor performers than rewarding those who perform well. In fact, we find that the principal would often prefer a loser-lose-all tournament (in which only the worst performer receives a different prize from others) to a winner-take-all tournament (in which only the best performer receives a different prize from others). We see our difference in results as due to important deviations that Krishna and Morgan (1998) and Moldovanu and Sela (2001) have made from the standard moral hazard setting.

Krishna and Morgan (1998) have an ex post participation constraint rather than the standard ex ante participation constraint. This means that the principal must make sure that an agent receives a certain utility ex post rather than in expectation. With an ex ante participation constraint, giving more to agents with high rank in the tournament allows the principal to give *less* to agents with low rank.[3] This is not the case with an ex post participation constraint.

In an elegant paper, Moldovanu and Sela (2001) seek to explain prize structures in tournaments within the framework of private value all-pay auctions. This is formally similar to models analyzed by Weber (1985), Glazer and Hassin (1988), Hillman and Riley (1989), Krishna and Morgan (1997), Clark and Riis (1998) and Barut and Kovenock (1998). Moldovanu and Sela (2001) analyze a model where contestants have different costs of exerting effort, which is private information. The contest designed seeks to maximize the sum of the efforts by determining the allocation of a fixed purse among the contestants. They show that if the contestants have linear or concave cost of effort functions then the optimal prize structure involves allocating the entire prize to the first-place getter. With convex costs, entry fees, or minimum effort requirements, more prizes can be optimal.

With this approach, there is a deterministic relationship between action choice and output. Therefore, agents' attitudes to risk play no role. This contrasts sharply with the stochastic relationship which is present in the Lazear-Rosen framework.

[3]Since an agent has some chance of achieving any rank, giving more for high rank increases an agent's expected utility. Since the PC is binding in equilibrium, this allows the principal to give less for low rank.

3 The Model

In this section, we will give the setup of the problem. The assumptions that we make should be very familiar: they are the same assumptions as those made by Lazear and Rosen (1981), Green and Stokey (1983) and Nalebuff and Stiglitz (1983).

3.1 Statement of the Problem

We will consider a world in which there are n agents available to compete in a rank-order tournament. This tournament is set up by a principal whose goal is to maximize her expected profits. The principal pays a prize w_i to the agent who places ith in the tournament. The profits which accrue to the principal are equal to the sum of the outputs of the participating agents minus the amount she pays out: $\pi = \sum_{i=1}^{n}(q_i - w_i)$. We assume that the principal is risk neutral. At least for now, we will assume that agents are homogeneous in ability. If agent j exerts effort e_j, her output is given by $q_j = e_j + \varepsilon_j + \eta$, where ε_j and η are random variables with mean zero and distributed according to distributions F and G respectively. We assume that the ε_j's are independent of one another and η. We will refer to η as the "common shock" to output and ε_j as the "idiosyncratic shock" to output. Since rank-order tournaments filter out the noise created by common shocks but individual contracts do not, rank-order tournaments are considered most advantageous when common shocks are large.[4]

We will assume that agents have utility that is additively separable in wealth and effort[5]. If agent j places ith in the tournament, her utility is given by: $u(w_i) - c(e_j)$ where $u' \geq 0, u'' \leq 0, c' \geq 0, c'' \geq 0$. Agents have an outside option which guarantees them $\bar{U}$, so unless the expected utility from participation is at least equal to $\bar{U}$, agents will not be willing to participate.

The timing of events is as follows. Time 1: the principal commits to a prize schedule

[4] See Holmström (1982) for a definitive treatment of relative performance evaluation individual contracts. He shows that an appropriately structured individual contract with a relative performance component dominates a rank-order tournament for n finite. Green and Stokey (1983) prove convergence of optimal tournaments to the individual contract second-best as $n \to \infty$ when there are no common shocks.

[5] This implies that preferences for income lotteries are independent of action and that preferences for action lotteries are independent of income (Keeney (1973)). Among other things, this rules out the possibility that stochastic contracts are optimal.

$\{w_i\}_{i=1}^n$. Time 2: agents decide whether or not to participate. Time 3: if everyone has agreed to participate at time 2, individuals choose how much effort to exert. Time 4: output is realized and prizes are awarded according to the prize schedule set at time 1.[6]

3.2 Solving the Model

We will restrict attention to symmetric pure strategy equilibria (as do Green and Stokey (1983) and Krishna and Morgan (1998)). While we cannot rule out the possibility of other equilibria, a unique symmetric pure strategy equilibrium generally exists.[7] In a symmetric equilibrium, every agent will exert effort e^*. Furthermore, every agent has an equal chance of winning any prize. Thus, an agent's expected utility is

$$\frac{1}{n}\sum_i u(w_i) - c(e^*)$$

In order for it to be worthwhile for an agent to participate in the tournament, it is necessary that

$$\frac{1}{n}\sum_i u(w_i) - c(e^*) \geq \bar{U}$$

An agent who exerts effort e while everyone else exerts effort e^* receives expected utility

$$\begin{aligned} U(e,e^*) &= \sum_i \varphi_i(e,e^*)u(w_i) - c(e) \\ \text{where } \varphi_i(e,e^*) &= \Pr(i\text{th place}|e,e^*), \end{aligned}$$

The problem faced by an agent is to choose e to maximize $U(e,e^*)$. The first-order condition for this problem is

$$c'(e) = \sum_i \frac{\partial}{\partial e}\varphi_i(e,e^*)u(w_i)$$

By assumption, the solution to the agent's maximization problem is $e = e^*$. If the

[6]It is sometimes assumed that agents learn the size of the common shock after choosing to participate. This makes a difference when the principal incentivizes agents with individual contracts but makes no difference when the principal incentivizes agents with a tournament, which filters out common shocks.

[7]The appendix to Nalebuff and Stiglitz (1983) contains a detailed discussion of mixed strategy equilibria in tournaments.

first-order condition gives the solution to the agent's maximization problem, it follows that

$$c'(e^*) = \sum_i \beta_i u(w_i)$$
$$\text{where } \beta_i = \left.\frac{\partial}{\partial e}\varphi_i(e, e^*)\right|_{e=e^*}$$

We will often refer to the β_i's as "weights." The β_i's do not depend upon e^* but simply upon the distribution function F. Proposition 1 gives a formula for β_i and some additional properties.

Proposition 1 *The following is a formula for β_i as a function of F and the corresponding pdf, f:*

$$\beta_i = \binom{n-1}{i-1} \int_{\mathbb{R}} F(x)^{n-i-1}(1-F(x))^{i-2}\left((n-i)-(n-1)F(x)\right) f(x)^2 dx$$

For all F, $\sum_i \beta_i = 0$, $\beta_1 \geq 0$, and $\beta_n \leq 0$. If F is symmetric ($F(-x) = 1 - F(x)$), $\beta_i = -\beta_{n-i+1}$ for all i.

If F is a uniform distribution on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, $\beta_1 = -\beta_n = \frac{1}{\sigma}$ and $\beta_i = 0$ for $1 < i < n$.

Proposition 1 shows that the weight schedule for the uniform distribution is completely flat in the middle and spikes at the top and bottom. We find that many other distributions have weight schedules that are relatively flat in the middle and spike at the top and bottom. The normal distribution has this pattern. Figure 1 gives a plot of the weights for a normal distribution with standard deviation of 1 and $n = 200$.

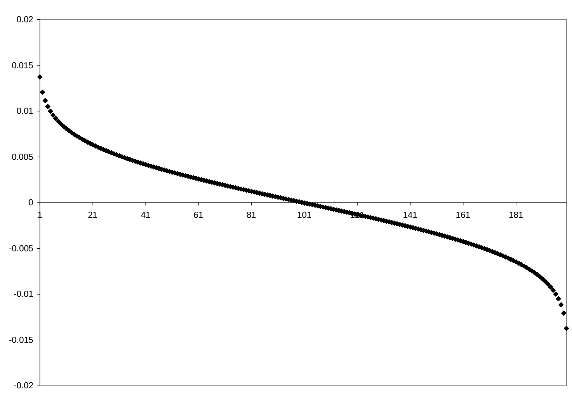

Figure 1: Weights for the Normal

While the weights associated with uniformly-distributed and normally-distributed noise are always decreasing in i, the weights need not be monotonic. When the noise distribution is not single-peaked, non-monotonicities tend to arise. It should be noted that, while the weights can be increasing in i over some range, the weights cannot be increasing over the entire range. As Proposition 1 shows, $\beta_1 > \beta_n$ unless $\beta_1 = \beta_n = 0$. As we will see in the next section, non-monotonicities in the weights lead to non-monotonicities in the optimal prize schedule.

In general, the agents' first-order condition may or may not give the solution to the agents' maximization problem. In order for the first-order condition to give the solution, the second-order must be satisfied. Proposition 2 gives conditions under which the second-order condition will be satisfied at $e = e^*$.

Proposition 2 *Suppose that F is symmetric ($F(-x) = 1 - F(x)$), $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$, and $\sum_{i=1}^{j} \gamma_i \geq 0$ for all $j \leq \frac{n}{2}$, where*

$$\begin{aligned}\gamma_i &= \binom{n-1}{i-1} \int_{\mathbb{R}} (F(x))^{n-i-2} (1-F(x))^{i-3} \begin{bmatrix} (n-i)(n-i-1) \\ -2(n-i)(n-2)F(x) \\ +(n-1)(n-2)F^2(x) \end{bmatrix} f^3(x)dx \\ &\quad + \binom{n-1}{i-1} \int_{\mathbb{R}} F(x)^{n-i-1} (1-F(x))^{i-2} ((n-i) - (n-1)F(x)) f(x) f'(x) dx\end{aligned}$$

Then, the agents' second-order condition is satisfied at $e = e^*$.

The condition on the γ_i's holds when F is a uniform, normal, or Cauchy distribution. In the next section we will give conditions under which the principal will choose a prize schedule for which $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$ when agents act according to the first-order condition.

Proposition 2 gives a necessary but not sufficient condition for the agents to act according to the first-order condition. When F has sufficiently low variance, the prize schedule the principal would choose under the assumption that agents act according to their first-order condition leads the agents to exert zero effort (which violates the agents' first-order condition).[8] In this case, the second-order condition may be satisfied at $e = e^*$ but is not satisfied for all e. This issue stems from the fact that, when F has low variance, it is difficult to tell whether the agent with the lowest rank has slightly low output or very low output. For a given F, rank becomes an increasingly informative signal of output as n increases (this result is due to Green and Stokey (1983)). We believe, therefore, that this particular issue disappears either when n is large or when the variance of F is high. This said, however, we will not offer a sufficient condition in the paper for the agents to act according to the first-order condition.

Now that we have elaborated the agents' problem, we turn to the principal's problem. We have assumed that the principal is risk neutral. This implies that the principal's objective is to maximize expected profits

$$E(\pi) = \sum_j e_j - \sum_i w_i = n\left(e^* - \frac{1}{n}\sum_i w_i\right).$$

If the agents' first-order condition is equivalent to the agents' incentive compatibility constraint, the problem of the principal can be stated as follows

$$\max_{w_i}\left(e^* - \frac{1}{n}\sum_i w_i\right)$$

subject to

$$\frac{1}{n}\sum_i u(w_i) - c(e^*) \geq \bar{U} \tag{IR}$$

$$c'(e^*) = \sum_i \beta_i u(w_i) \tag{IC}$$

Substituting $(c')^{-1}\left(\sum_i \beta_i u(w_i)\right)$ for e^*, and $u^{-1}(u_i)$ for w_i, we can rewrite the principal's

[8] For a detailed discussion of this issue, see Nalebuff and Stiglitz (1983), Theorem 4.

problem as:

$$\max_{u_i}\left((c')^{-1}\left(\sum_i \beta_i u(w_i)\right)-\frac{1}{n}\sum_i u^{-1}(u_i)\right)$$

subject to

$$\bar{U}-\frac{1}{n}\sum_i u_i+(c')^{-1}\left(\sum_i \beta_i u(w_i)\right)\leq 0$$

The Lagrangian associated with this maximization problem is:

$$\mathcal{L}=\left((c')^{-1}\left(\sum_i \beta_i u_i\right)-\frac{1}{n}\sum_i u^{-1}(u_i)\right)-\lambda\left(\bar{U}-\frac{1}{n}\sum_i u_i+c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)\right)$$

Just as the agents' first-order condition does not necessarily solve the agents' maximization problem, the first-order conditions of the Lagrangian may not solve the principal's maximization problem. The following Lemma gives a condition under which the principal will act according to the first-order conditions of the Lagrangian.

Lemma 1 *Let* $s(u_1,...,u_n)=(c')^{-1}\left(\sum_i \beta_i u_i\right)-\frac{1}{n}\sum_i u^{-1}(u_i)$ *and* $l(u_1,...,u_n)=\bar{U}-\frac{1}{n}\sum_i u_i+c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$*. If* $c'''\geq 0$ *and* $\frac{c''}{c'}\geq\frac{c'''}{c''}$*, then* s *is concave and* l *is convex. Therefore, if* $c'''\leq 0$*,* $\frac{c''}{c'}\geq\frac{c'''}{c''}$*, and* $(u_1,...,u_n,\lambda)$ *satisfies the Kuhn-Tucker conditions of* $\mathcal{L}$*,* $(u_1,...,u_n)$ *solves the principal's problem.*

These conditions on the cost of effort function are somewhat restrictive, but they do hold for all functions of the form $c(e)=de^{\alpha}$ for which $\alpha\geq 2$.

4 The Optimal Prize Schedule

We will begin by giving a partial characterization of the principal's optimal prize schedule. It will be shown that, when u has the property that $R\geq\frac{P}{2}$ (R is risk aversion and P is absolute prudence), the noise distribution is symmetric, and the weights are monotonic, the rewards given at the top of the prize schedule are smaller than the punishments given at the bottom of the prize schedule. When $R \leq \frac{P}{2}$, the noise distribution is symmetric, and the weights are monotonic, the rewards given at the top of the prize schedule are *larger* than the punishments given at the bottom. $R \geq \frac{P}{2}$ for all CARA utility fuctions and CRRA utility functions with $\theta \geq 1$.

For many common noise distributions, the prize schedule is relatively flat except at the top and at the bottom. In particular, when the noise distribution is uniformly distributed, we find that there is a special prize for first place, a special prize (or punishment) for last place, and a single prize for everyone else. We also show that, when agents have CRRA utility with a coefficient of relative risk aversion equal to one, the principal's optimal prize schedule is an affine transformation of the weights, β_i.

The first-order conditions of the Lagrangian lead to the following proposition, which tells us a great deal about the optimal prize schedule.

Proposition 3 *Suppose $w^* = (w_1^*, ...w_n^*)$ is the optimal prize schedule and let $v_i = u'(w_i^*)$. If the agents act according to their first-order condition, $c''' \geq 0$, and $\frac{c''}{c'} \geq \frac{c'''}{c''}$, then*
$\frac{\frac{1}{v_i} - \frac{1}{v_{i+k}}}{\frac{1}{v_j} - \frac{1}{v_{j+l}}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$ *for all $i, j, k,$ and l.*

It should be noted that the equations of the proposition plus the individual rationality constraint captures almost everything about the optimal tournament. They are just one equation short of a complete characterization.

Consider the implications of Proposition 3 when F is a uniform distribution. Because the weights are constant for the uniform distribution for $1 < i < n$, it follows that the prize schedule is flat except at the top and the bottom. Therefore, it is optimal for the principal to give one prize for the best performance, one prize (or punishment) for the worst performance, and a prize for everyone else.

Corollary 1 *If F is uniformly distributed, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition:*

$$w_i^* = w_j^*,\ 1 < i, j < n$$

As mentioned above, many distributions (such as the normal) have weight schedules that are relatively flat for $1 < i < n$ and spike at the top and bottom. The optimal prize schedules associated with these distributions will be relatively flat for $1 < i < n$, although they will not be perfectly flat.

Proposition 3 allows us to bound the slope of the prize schedule. Proposition 4, which is the main result of this section, summarizes what we can conclude about the slope.

Proposition 4 *Suppose $i, j \geq 0$ and $\min(i, i+k) \geq \max(j, j+l)$ (k and l can be positive or negative). Suppose further that $\beta_i - \beta_{i+k} \geq 0$ and $\beta_j - \beta_{j+l} \geq 0$. Let $R = -\frac{u''}{u'}$ denote the Arrow-Pratt measure of risk aversion. Let $P = -\frac{u'''}{u''}$ denote the coefficient of absolute prudence. Suppose $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and the agents act according to their first-order condition.*

(i) If $R \geq \frac{P}{2}$:

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w^*_i - w^*_{i+k}}{w^*_j - w^*_{j+l}} \leq \left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(ii) If $R \leq \frac{P}{2}$:

$$\left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w^*_i - w^*_{i+k}}{w^*_j - w^*_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(iii) Let $u^*_i = u(w^*_i)$. If $R \geq \frac{P}{3}$:

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \leq \left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

(iv) If $R \leq \frac{P}{3}$:

$$\left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \leq \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

Notice that $R \geq \frac{P}{2}$ is a stronger condition than $R \geq \frac{P}{3}$. We will explain why the size of R relative to P is important in Section 5 in a case where the intuition is easy to see. For most utility functions of interest, $R \geq \frac{P}{2}$. $R \geq \frac{P}{2}$ for all CARA utility functions and CRRA utility functions with $\theta \geq 1$. $R \geq \frac{P}{3}$ for all CARA utility functions and CRRA utility functions with $\theta \geq \frac{1}{2}$. $R \leq \frac{P}{3}$ for CRRA utility functions with $\theta \leq \frac{1}{2}$, and $R \leq \frac{P}{2}$ for CRRA utility functions with $\theta \leq 1$. This makes $\theta = 1$ and $\theta = \frac{1}{2}$ interesting cases since $R = \frac{P}{2}$ for $\theta = 1$ and $R = \frac{P}{3}$ for $\theta = \frac{1}{2}$.

In the previous section, Proposition 2 gave a condition on the prize schedule under which the agents' second-order condition will hold for certain F at $e = e^*$. The following corollary to Proposition 4 gives us conditions under which the principal will choose a prize schedule that meets the condition of Proposition 2.

Corollary 2 *If F is symmetric, $\{\beta_i\}$ is decreasing in i, $R \geq \frac{P}{3}$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition:*

$$u(w_i^*) - u(w_j^*) \leq u(w_{n-j+1}^*) - u(w_{n-i+1}^*) \text{ for all } i \leq j \leq \frac{n+1}{2}$$

Therefore, when the principal assumes that agents act according to the first-order condition, F is symmetric, $\{\beta_i\}$ is decreasing in i, $\sum_{i=1}^{j} \gamma_i \geq 0$ for $j \leq \frac{n}{2}$, $R \geq \frac{P}{3}$, $c''' \geq 0$, and $\frac{c''}{c'} \geq \frac{c'''}{c''}$, the principal will choose a prize schedule that satisfies the agent's second-order condition at $e = e^*$.

When F is symmetric, Proposition 4 allows us to compare the size of punishments inflicted at the bottom of the prize schedule ($w_i^* - w_{i+1}^*$, $i \geq \frac{n+1}{2}$) to corresponding rewards given at the top of the prize schedule ($w_{n-i}^* - w_{n-i+1}^*$). We find that when R is large relative to P, the punishments ($w_i^* - w_{i+1}^*$) will be larger than the rewards ($w_{n-i}^* - w_{n-i+1}^*$). When R is small relative to P, the punishments ($w_i^* - w_{i+1}^*$) will be smaller than the rewards ($w_{n-i}^* - w_{n-i+1}^*$).

Proposition 5 *Let $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ and $q_i = \frac{u_i^* - u_{i+1}^*}{u_{n-i}^* - u_{n-i+1}^*}$. Suppose F is symmetric, $\{\beta_i\}$ is decreasing in i, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and agents act according to their first-order condition.*

(i) If $R \geq \frac{P}{2}$:

$$\begin{aligned} r_i &\geq 1 \text{ for } i \geq \frac{n+1}{2} \\ r_{i+1} &\geq r_i \text{ for all } i \end{aligned}$$

(ii) If $R \leq \frac{P}{2}$:

$$\begin{aligned} r_i &\leq 1 \text{ for } i \geq \frac{n+1}{2} \\ r_{i+1} &\leq r_i \text{ for all } i \end{aligned}$$

(iii) If $R \geq \frac{P}{3}$:

$$\begin{aligned} q_i &\geq 1 \text{ for } i \geq \frac{n+1}{2} \\ q_{i+1} &\geq q_i \text{ for all } i \end{aligned}$$

(iv) If $R \leq \frac{P}{3}$:

$$\begin{aligned} q_i &\leq 1 \text{ for } i \geq \frac{n+1}{2} \\ q_{i+1} &\leq q_i \text{ for all } i \end{aligned}$$

As an illustration of this result, consider the optimal prize schedule when $n = 200$, F is a normal distribution with standard deviation 1, $c(e) = \frac{e^2}{2}$, and the utility function is CRRA with $\theta = 2$. For this case, $R \geq \frac{P}{2}$. Figure 2 shows the prize schedule in money (as opposed to utils). We observe that the punishments at the bottom of the prize schedule ($w_i^* - w_{i+1}^*$, $i \geq \frac{n+1}{2}$) are greater than the corresponding rewards at the top ($w_{n-i}^* - w_{n-i+1}^*$). Figure 3 gives the ratios, r_i.

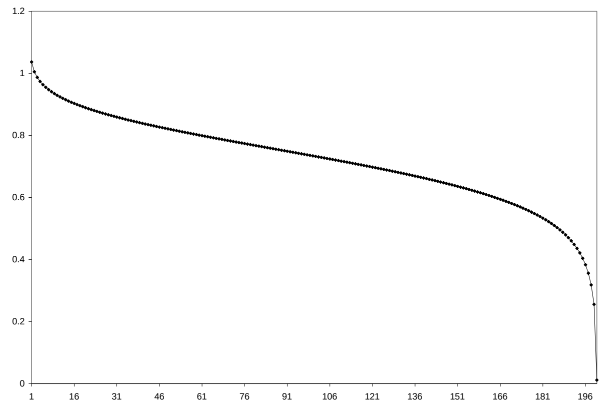

Figure 2: Optimal Prize Schedule

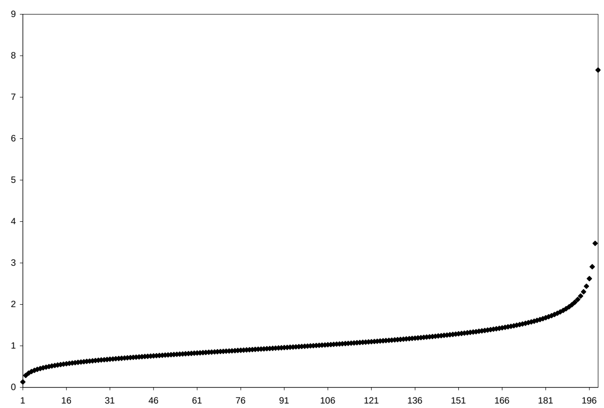

Figure 3: r_i

As mentioned, if the utility function is CRRA with $\theta = 1$, $R = \frac{P}{2}$. If the utility function is CRRA with $\theta = \frac{1}{2}$, $R = \frac{P}{3}$. Therefore, these are interesting special cases. This yields the following corollary.

Corollary 3 *If $u(w) = \log(w)$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, and the agents act according to their first-order condition, then*

$$\frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

for all i, j, k, and l. Furthermore, the vector $w^* = (w_1^*, ..., w_n^*)$ is an affine transformation of the vector $\beta = (\beta_1, ..., \beta_n)$. If F is also symmetric,

$$\frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*} = 1$$

for all i.

If $u(w) = w^{1/2}$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *and the agents act according to their first-order condition, then*

$$\frac{u_i^* - u_{i+k}^*}{u_j^* - u_{j+l}^*} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

for all i, j, k, *and* l. *Furthermore, the vector* $u^* = (u_1^*, ..., u_n^*)$ *is an affine transformation of the vector* $\beta = (\beta_1, ..., \beta_n)$. *If* F *is also symmetric,*

$$\frac{u_i^* - u_{i+1}^*}{u_{n-i}^* - u_{n-i+1}^*} = 1$$

for all i.

In this sense, optimal prize schedules tend to look similar to affine transformations of the weight schedules. When R is large relative to P, the optimal prize schedule differs from an affine transformation of the weight schedule in that the prizes at the top are revised in the direction of the median prize while the prizes at the bottom are revised in the opposite direction from the median prize. We see this in comparing the prize schedule in Figure 2 to the corresponding weights shown in Figure 1. When R is small relative to P, the optimal prize schedule differs from an affine transformation of the weight schedule in that the slope of the prize schedule at the top is revised *upward* relative to the slope of the prize schedule at the bottom.

5 Two-Prize Tournaments

In the previous section, we found that when R is large relative to P, the principal relies more heavily on punishment than on reward. To examine how important punishments are relative to rewards, we will consider what happens when the principal is limited to using just two prizes. That is, suppose she can only give a prize w_1 to the top j performers and a prize w_2 to the bottom $n-j$ performers. When the principal is restricted in this way, where would she like to set j? One possibility would be to set $j = \frac{n}{2}$, so that the top half earns one prize and the bottom half earns another. Another possibility would be to set $j = 1$, which gives a special prize to the best performer. The opposite would be to set $j = n - 1$, so that there is a special punishment in store for the worst performer.

We will find that, when $R \geq \frac{P}{3}$ and F is symmetric, it is always optimal to set $j \geq \frac{n}{2}$ and it is often optimal for the principal to set $j = n - 1$, giving a special punishment to the worst performer. This is somewhat indicative of the importance of punishments to the principal relative to the importance or rewards.

In order to further examine the importance of punishment relative to reward, we will compare the profits to the principal from the optimal n-prize tournament and two-prize tournaments for different choices of j. We find that, when R is large relative to P and F is symmetric, the principal's profits from the optimal $j = n - 1$ tournament are frequently close to her profits from the optimal n prize tournament while the profits from a $j = 1$ tournament are usually far from optimal. These results contrast with the notion that it is optimal or nearly optimal for the principal to implement a winner-take-all tournament (see Moldovanu and Sela (2001), and Krishna and Morgan (1998)).

Definition 1 *We will call a tournament a "j tournament" when the principal pays a prize* w_1 *to the top* j *performers and a prize* w_2 *to the bottom* $n-j$ *performers. Let* $u_1 = u(w_1)$ *and* $u_2 = u(w_2)$*. We will call a tournament a "winner-prize tournament" if* $j \leq \frac{n}{2}$ *and a "strict winner-prize tournament" if* $j = 1$*. We will call a tournament a "loser-prize tournament" is* $j \geq \frac{n}{2}$ *and a "strict loser-prize tournament" if* $j = n - 1$*.*

We will consider when the principal prefers to implement a loser-prize tournament rather than a winner-prize tournament. To answer this question, we will compare a j tournament and an $n - j$ tournament that induce the same level of effort and both meet the individual rationality constraint. It will be shown that, when R is large relative to P, F is symmetric, and $j \leq \frac{n}{2}$, the payment made to agents by the principal is greater when she uses the j tournament. When R is small relative to P, F is symmetric, and $j \leq \frac{n}{2}$, the payment made to agents by the principal is smaller when she uses the j tournament.

First, we must know when a j tournament and an $n - j$ tournament induce the same effort. The following lemma provides the answer.

Lemma 2 *If F is symmetric and agents act according to the first-order condition, a j tournament and an $n-j$ tournament for which $u_1 - u_2$ is the same induce the same level of effort. This level of effort is given by*

$$c'(e) = \left(\sum_{i=1}^{j} \beta_i \right) (u_1 - u_2)$$

Using this lemma, we will now establish the main result of this section.

Proposition 6 *Suppose the principal is restricted to use a j tournament (but has a choice over w_1 and w_2) and that the principal is restricted to implementing a tournament that induces effort level e. Let π_j denote the expected profits from the optimal choice of w_1 and w_2. Suppose further that F is symmetric and agents act according to the first-order condition.*

(i) If $R \geq \frac{P}{3}$,

$$\pi_j \leq \pi_{n-j} \text{ for } j \leq \frac{n}{2}$$

(ii) If $R \leq \frac{P}{3}$,

$$\pi_j \geq \pi_{n-j} \text{ for } j \leq \frac{n}{2}$$

The following is an immediate corollary.

Corollary 4 *Suppose the principal is restricted to implementing a j tournament, but can choose whatever j she likes. Suppose F is symmetric and agents act according to the first-order condition. If u satisfies $R \geq \frac{P}{3}$, then the optimal j tournament is a loser-prize tournament (a tournament with $j \geq \frac{n}{2}$). If u satisfies $R \leq \frac{P}{3}$, then the optimal j tournament is a winner-prize tournament (a tournament with $j \leq \frac{n}{2}$).*

Let us consider the intuition behind Proposition 6. Suppose the j tournament gives payments u_1 and u_2 in utils while the $n-j$ tournament gives payments $\tilde{u}_1$ and $\tilde{u}_2$ in utils. Because the individual rationality constraint will bind for both tournaments and because they induce the same effort level, we can conclude that: $\tilde{u}_2 \leq u_2 \leq \tilde{u}_1 \leq u_1$ and that $j(u_1 - \tilde{u}_2') = (n-j)(\tilde{u}_1' - u_2)$.

The principal pays the dollar equivalent of $\tilde{u}_1$ rather than the dollar equivalent of u_2 to $n-j$ agents when using the $n-j$ tournament. This is a cost to using the $n-j$ tournament. But, the principal also pays the money equivalent of $\tilde{u}_2$ rather than the money equivalent of u_1 to j agents. This is a benefit to using the $n-j$ tournament.

The cost will be small if the utility function is relatively vertical between u_2 and $\tilde{u}_1$. The benefit will be large if the utility function is relatively flat between $\tilde{u}_2$ and u_2 and between $\tilde{u}_1$ and u_1. Of course, we assume that $u'' \leq 0$. Therefore, the utility function must be flatter between u_2 and $\tilde{u}_1$ than between $\tilde{u}_2$ and u_2. What this means is that for the benefit to be large relative to the cost, the utility function must get flatter and get flatter at an increasing rate. Furthermore, the more quickly the utility function flattens, the more important it is that the utility function is flattening at an increasing rate (the flattening from bottom to middle favors the j tournament, which needs to be made up for by extra flattening from middle to top in order for the $n-j$ tournament to dominate).

R gives a measure of the flattening of the utility function and P gives a measure of whether the flattening takes place at an increasing or decreasing rate. This explains why R large relative to P leads the $n-j$ tournament to be favored. As we showed in Proposition 5, when the principal uses n prizes and R is large relative to P, it is also optimal to punish more than reward. The reason for this is exactly the same.

So far, we have given conditions under which the optimal two-prize tournament is a loser-prize tournament. We can go further and make comparisons between loser-prize tournaments when we assume that the idiosyncratic noise distribution is uniform.

Proposition 7 *Suppose the principal is restricted to use a j tournament, that F is a symmetric uniform distribution, and that agents act according to the first-order condition. If u satisfies $R \geq \frac{P}{3}$, the optimal j tournament is the strict loser-prize tournament. If u satisfies $R \leq \frac{P}{3}$, the optimal j tournament is the strict winner-prize tournament.*

When the noise distribution is not uniform, the optimal j depends upon the utility function as well as the distributional weights. However, as mentioned above, many distributions (including the normal distribution) have weight schedules that are similar to the uniform distribution: they are relatively flat for $1 < i < n$ and spike at the top and bottom. The strict loser-prize tournament tends to be optimal when $R > \frac{P}{3}$ and the noise distribution has weights that look similar to those of a uniform distribution. In the numerical examples that we have considered, we have generally found $j = n - 1$ to be the optimal two-prize tournament when F is normal and $R > \frac{P}{3}$.

5.1 Numerical Examples

Our results above give no sense of how much the choice of j matters to the principal's profits. In a case where $j = n - 1$ is optimal, we would like to know how much worse off the principal would be if she chose $j = 1$ instead. We have looked at numerical examples in order to get a sense of the magnitude of the loss.

The numerical examples we have considered suggest that the profits from the optimal j tournament are generally close to the profits from the optimal n prize tournament. The induced effort level is also similar. However, we find that the choice of j matters a great deal. When j is not chosen optimally, the principal's profit may be quite far from the profit from the optimal j tournament and the profit from the optimal n-prize tournament.

Since $j = n - 1$ is often the optimal j when $R > \frac{P}{3}$, we find that there are many cases where the optimal $j = n-1$ tournament closely approximates the optimal n prize tournament while the optimal $j = 1$ tournament returns a profit that is markedly worse. Therefore, in many cases, punishing the worst performer is the most important incentive the principal has at her disposal.

Table 1 gives the results of a particular numerical example in order to give a sense of our findings. Table 1 considers the profits from tournaments in which F is a normal distribution with standard deviation of 1, $u(w) = 1 - \frac{1}{w}$ (CRRA with $\theta = 2$), $c(e) = \frac{e^2}{2}$, and $\bar{U} = -2$. For different values of n, it compares the profits from the optimal n prize tournament, the optimal $j = n - 1$ tournament, the optimal $j = \frac{n}{2}$ tournament, and the optimal $j = 1$

tournament. This is a case where $R > \frac{P}{3}$ and F is symmetric, so Proposition 6 implies that the optimal j tournament is a loser-prize tournament. In fact, we find that in all of the cases considered in Table 1, the strict loser-prize tournament ($j = n - 1$) is the optimal two-prize tournament.

Table 1: Prize Structures

n	$\pi_{first-best}$	$\pi_{optimal}$	$\pi_{j=n-1}$	$\pi_{j=1}$	$\pi_{j=n/2}$
2	1.076	0.288	0.288	0.288	0.288
4	1.076	0.514	0.492	0.152	0.377
6	1.076	0.582	0.552	0.061	0.409
8	1.076	0.617	0.583	-0.000	0.426
10	1.076	0.639	0.603	-0.044	0.436
20	1.076	0.689	0.652	-0.155	0.456
50	1.076	0.733	0.699	-0.245	0.469
100	1.076	0.757	0.728	-0.283	0.473
200	1.076	0.777	0.752	-0.306	0.475

We see in Table 1 that the principal's profit from the $j = n - 1$ tournament is similar to the principal's profit from the optimal n prize tournament ($\pi_{optimal}$). The profits from the $j = 1$ and $j = \frac{n}{2}$ tournaments are considerably lower. We also see that the profit from the optimal n prize tournament is increasing in n and appears to be converging. This is an illustration of the finding of Green and Stokey (1983) that the profits from the optimal rank-order tournament converge, as n increases, to the individual contract second-best level of profits (in the case where there is no common shock to output).

6 When is it optimal to give winners large rewards?

Our analysis above suggests that, in many instances, the principal will choose a tournament that gives larger punishments to losers than rewards to winners (in the sense of $w_i^* - w_{i+1}^* \geq w_{n-i}^* - w_{n-i+1}^*$ for $i \geq \frac{n+1}{2}$). In this section, we will explore the circumstances in which it is optimal for the principal to give large rewards relative to punishments.

We see four reasons why a principal might want to give larger rewards than punishments. The first, which we have already considered, is that R may be low relative to P. Proposition 5 says that, if $R \leq \frac{P}{2}$, F is symmetric, and the weights are decreasing in i, $w_i^* - w_{i+1}^* \leq w_{n-i}^* - w_{n-i+1}^*$ for $i \geq \frac{n+1}{2}$. The other reasons to reward winners are: the presence of a limited liability constraint, a noise distribution which is asymmetric in a particular way, and agent heterogeneity.

Having already addressed the first reason, we will consider the other three in turn. We will attempt to give an intuition as to why each of these factors plays a role in determining the size of rewards relative to punishments.

6.1 Limited Liability

Suppose the principal faces a limited liability constraint (a limit on how much the agents can be punished). We might write this constraint, for example, as: $w_i \geq \bar{w}$ for all i. A limited liability constraint may lead the principal to implement a tournament in which winners are rewarded more than losers are punished. The reason is that such a constraint makes it comparatively more difficult to incentivize agents through punishment.

Consider an example. Let us suppose the principal is trying to induce an effort level e and is choosing between a $j = 1$ tournament and a $j = n - 1$ tournament. Suppose, in the absence of a limited liability constraint, it is optimal to choose prizes w_1 and w_2 in the $j = 1$ tournament and prizes $\tilde{w}_1$ and $\tilde{w}_2$ in the $j = n - 1$ tournament. Suppose the profit from these tournaments are π and $\tilde{\pi}$ respectively. If F is symmetric and $R \geq \frac{P}{3}$, Proposition 6 implies that $\tilde{\pi} > \pi$.

We know that $\tilde{w}_2 < w_2 < \tilde{w}_1 < w_1$. Now suppose that there is a limited liability constraint such that $\tilde{w}_2 < \bar{w} < w_2 < \tilde{w}_1 < w_1$. The optimal $j = 1$ tournament is the same with or without the limited liability constraint, so $\pi_L = \pi$. The limited liability constraint does have an effect, though, on the optimal $j = n - 1$ tournament. The amount given to the loser must be increased to meet the limited liability constraint. To induce effort level e, the prize for everyone else must be increased as well. These changes are costly. So, $\tilde{\pi}_L < \tilde{\pi}$.

When F is symmetric and $R \geq \frac{P}{3}$, $\tilde{\pi} > \pi$. However, it is possible that $\pi_L > \tilde{\pi}_L$. This gives a sense of why a limited liability constraint might matter.

A limited liability constraint ensures agents a certain amount ex post. In this sense, it is similar to the ex post participation constraint in Krishna and Morgan (1998). If the limited liability constraint makes the ex-ante participation constraint non-binding, this is equivalent to the Krishna and Morgan case. They find that, in this case, it is generally optimal for the principal to implement a winner-take-all tournament. This is consistent with our observation that limited liability constraints tend to increase rewards relative to punishments.

Our results from Section 4 of the paper can easily be extended to cases where the principal faces a limited liability constraint. The following Lemma is a generalization of Proposition 3.

Lemma 3 *Suppose that the principal faces a limited liability constraint of the form:* $w_i \geq \bar{w}$ *for all* i. *Suppose* $w^* = (w_1^*, ...w_n^*)$ *is the optimal prize schedule and let* $v_i = u'(w_i^*)$. *If the agents act according to their first-order condition,* $c''' \geq 0$, *and* $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *then* $\frac{\frac{1}{v_i} - \frac{1}{v_{i+k}}}{\frac{1}{v_j} - \frac{1}{v_{j+l}}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$ *for all* $i, j, k,$ *and* l *whenever* $w_i, w_j, w_k, w_l > \bar{w}$.

Lemma 3 shows that the results of Section 3 hold whenever the limited liability constraint is nonbinding. In particular, Proposition 5 can be extended to the case where the principal faces a limited liability constraint.

Corollary 5 *Let* $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ *and* $q_i = \frac{u_i^* - u_{i+1}^*}{u_{n-i}^* - u_{n-i+1}^*}$. *Suppose* F *is symmetric,* $\{\beta_i\}$ *is decreasing in* i, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, *and agents act according to their first-order condition. Suppose that the principal faces a limited liability constraint of the form:* $w_i \geq \bar{w}$ *for all* i. *Then the optimal prize schedule,* $\{w_i^*\}$ *is decreasing in* i. *Let* j *be such that* $w_j^* > \bar{w}$ *and either* $w_{j+1}^* = \bar{w}$ *or* $j = n$.

(i) If $R \geq \frac{P}{2}$:

$$\begin{aligned} r_i &\geq 1 \textit{ for } j > i \geq \frac{n+1}{2} \\ r_i &= \infty \textit{ for } i \leq n-j+1 \\ r_{i+1} &\geq r_i \textit{ for } j > i > n-j+1 \end{aligned}$$

(ii) If $R \leq \frac{P}{2}$:

$$\begin{aligned} r_i &\leq 1 \textit{ for } j > i \geq \frac{n+1}{2} \\ r_i &= \infty \textit{ for } i \leq n-j+1 \\ r_{i+1} &\leq r_i \textit{ for } j > i > n-j+1 \end{aligned}$$

(iii) If $R \geq \frac{P}{3}$:

$$\begin{aligned} q_i &\geq 1 \textit{ for } j > i \geq \frac{n+1}{2} \\ q_i &= \infty \textit{ for } i \leq n-j+1 \\ q_{i+1} &\geq q_i \textit{ for } j > i > n-j+1 \end{aligned}$$

(iv) If $R \leq \frac{P}{3}$:

$$\begin{aligned} q_i &\leq 1 \textit{ for } j > i \geq \frac{n+1}{2} \\ q_i &= \infty \textit{ for } i \leq n-j+1 \\ q_{i+1} &\leq q_i \textit{ for } j > i > n-j+1 \end{aligned}$$

This corollary gives us a good picture of when punishments will be large relative to rewards and when rewards will be large relative to punishments.

It should be noted that, in the absence of a limited liability constraint, in cases where $R \geq \frac{P}{2}$ and F is symmetric, losers are not necessarily boiled in oil (for an example, see Figure 2). Since the punishments for losers are typically not exorbitant, it is possible to imagine cases in which a limited liability constraint might be non-binding.

6.2 Asymmetric Noise Distribution

Corollary 3 shows that, when u is logarthmic, the optimal prize schedule is an affine tranformation of the weights, β_i. When F is symmetric, $\beta_i = -\beta_{n-i+1}$. This means that, when F is symmetric, the optimal prize schedule rewards winners and punishes losers equally.

But, when F is asymmetric, β_i may be larger or smaller than $-\beta_{n-i+1}$. There are F for which the weight schedule–and hence the prize schedule–is steep for low i and flat for high i. The prize schedule in this case clearly rewards winners more than it punishes losers.

Why does a weight schedule that is flat at the bottom lead to a prize schedule that is flat at the bottom? Suppose, for the sake of argument, that $\beta_{n-1} = \beta_n$. What this says is that a marginal change in agent effort does not affect the probability of placing $(n-1)$th relative to nth. Therefore, placing nth rather than $(n-1)$th is a matter of luck rather than effort. In punishing agents for placing nth rather than $(n-1)$th, the principal gives a reward for luck without giving a reward for effort. Since agents are risk averse, it is costly to the principal to reward luck. Therefore, it does not make sense for the principal to reward agents for placing nth rather than $(n-1)$th. So, $w^*_{n-1} = w^*_n$. If, in contrast, $\beta_{n-1} > \beta_n$, punishing nth place relative to $(n-1)$th place rewards effort as well as luck. So, it makes sense for the principal to punish nth place in this case.

It should be noted that there are asymmetric F that produce weight schedules that are steeper for high i than for low i. Such F lead to prize schedules that reward winners less than they punish losers. Therefore, asymmetry of the noise distribution can lead to more or less reward for winners depending upon the particular type of asymmetry.

6.3 Heterogeneity

Agent heterogeneity can have an effect on the optimal size of rewards relative to punishments. Whether heterogeneity increases rewards relative to punishments, decreases rewards relative to punishments, or is neutral depends, however, on the exact type of heterogeneity that exists.

First, there is a question of how an agent's type, θ, affects her output. We will consider two cases. We will say that there is *additive heterogeneity* if agent i's output is given by $q_i = e_i + \theta_i + \varepsilon_i + \eta$, where θ_i is agent i's type, ε_i is idiosyncratic noise, and η is a common shock to output. We will say that there is *multiplicative heterogeneity* if agent i's output is given by $q_i = \theta_i e_i + \varepsilon_i + \eta$. Multiplicative heterogeneity is perhaps a more relevant type of heterogeneity to consider, but we will examine both cases.

A second question is whether the principal ever learns the agents' types, and if so, whether the principal can contract upon type. If the principal becomes aware of the agents' types and can contract upon it, then the principal can handicap the tournament. Handicapping effectively restores agent homogeneity and eliminates any effects of heterogeneity on the optimal prize schedule. For this reason, heterogeneity only affects the optimal prize schedule when the principal does not observe agents' types or cannot contract upon it. We will assume in this section that the principal does not observe agents' types. It is potentially interesting to assume that the principal observes but cannot contract upon type, but we will not analyze this case in this section.

The third important question is when agents become aware of their types. There are three cases to consider. Case 1: agents learn their types after deciding whether to participate in the tournament and after choosing an effort level. Case 2: agents learn their types before choosing an effort level but after deciding whether to participate. Case 3: agents learn their types before deciding whether to participate. We will consider cases 1 and 2 in some detail and briefly discuss case 3.

In case 1, since agents only learn their types after choosing their effort levels, agents have the same individual rationality and incentive compatibility constraints. When there is additive heterogeneity in case 1, θ_i is just additional idiosyncratic noise. Hence, this case is identical to the homogeneous case we have already considered. Therefore, we only need to consider multiplicative heterogeneity in case 1. The following proposition gives an analysis.

Proposition 8 *Consider a tournament with n heterogeneous agents. Agents learn their* *ability after signing a contract and after choosing an effort level. The output of an individual i exerting effort level e_i is $q_i = \theta_i e_i + \varepsilon_i + \eta$ where ε_i is an idiosyncratic shock (distributed according to F), η is a common shock (distributed according to G), and θ_i is the agent's ability level ($\theta_i - E(\theta_i)$ distributed according to H). We assume that the ε_i's are independent of one another, the θ_i's, and η. We further assume that the θ_i's are independent of one another and η. For all individuals participating in the tournament, utility is given by $u(w) - c(e)$ where w is the prize received, e is the effort exerted, and $u' \geq 0, u'' \leq 0, c' \leq 0, c'' \geq 0$. All agents have the same outside option, which gives them utility $\bar{U}$. If we restrict attention to symmetric equilibria, when the principal offers the agents a tournament awarding prize w_i for ith place, the first-order condition for the agents' problem can be written as*

$$c'(e) = \sum_{i=1}^{n} \beta_i u(w_i)$$

where

$$\sum_{i=1}^{n} \beta_i = 0$$

And, if F and H are both symmetric, then

$$\beta_i = -\beta_{n-i+1}$$

The effect of the distribution of $\theta - E(\theta)$ on the optimal prize schedule is therefore analogous to the effect of the idiosyncratic noise distribution on the optimal prize schedule. When $\theta - E(\theta)$ is distributed according to H and H is symmetric, heterogeneity will not lead to large rewards relative to punishments. However, a particular type of asymmetry of H can lead to large rewards relative to punishments.

We turn now to case 2. In case 2, since agents learn their types after deciding whether to participate but before choosing their effort levels, agents will have the same individual rationality constraint but different incentive compatibility constraints. Agents of different types will therefore choose to exert different levels of effort.

Let us begin by considering additive heterogeneity of agents. A high θ agent needs to receive a considerably worse draw from the idiosyncratic noise distribution or exert considerably less effort in order to place below a low θ agent in the tournament. Therefore, high θ agents have a low probability of placing in the bottom of the tournament. As a result, rewards for placing well in the tournament incentivize high θ agents more than punishments for placing poorly in the tournament. In contrast, low θ agents are given a greater incentive to exert effort by punishments than by rewards.

If the principal were to switch from a tournament structure in which rewards are large relative to punishments to a tournament structure in which punishments are large relative to rewards, the principal would find a decline in the effort level of high θ agents but an increase in the effort level of low θ agents.

Whether this is advantageous to the principal depends in part upon the distribution of θ. This is analogous to the effect of the distribution of H in case 1. For example, imagine a setting in which one agent always receives a θ far below any other but no agent ever receives a θ far above any other. A strict loser-prize tournament (which punishes more than it rewards) will induce virtually no effort in this setting since the agent who receives the lowest θ is all-but-certain to rank at the bottom. In contrast, a strict winner-prize tournament (which rewards more than it punishes) will induce a group of agents with high θ's to compete for first place in the tournament. A strict winner-prize tournament clearly dominates a strict-loser prize tournament in this case. If, instead, one agent always receives a θ far above any other but no agent ever receives a θ far below any other, a strict loser-prize tournament will be preferred to a strict winner-prize tournament.

In order to examine a separate issue, let us assume that the ability distribution is symmetric. It turns out that, in this context, switching from a tournament that rewards more than punishes to a tournament that punishes more than rewards leads to an increase in effort among low θ agents that is generally *greater* than the decrease in effort among high θ agents. In this setting, heterogeneity increases the desire to give punishments that are large relative to rewards.

We will illustrate this result under somewhat restrictive assumptions, but we believe that the result applies widely. We will show that a principal can induce more effort with a strict-loser prize tournament in which $w_1 - w_2 = \Delta$ than with a strict-winner prize tournament in which $w_1 - w_2 = \Delta$. Recall that Lemma 2 shows that, in the homogeneous case, two-prize tournaments for which $w_1 - w_2 = \Delta$ induce the *same* effort level.

We begin by giving a careful definition of the moral hazard setting.

Definition 2 *A tournament is called a θ-Additive tournament when: there are n individuals (where n is even), $\frac{n}{2}$ of whom are low ability and $\frac{n}{2}$ of whom are high ability. Agents learn their ability after signing a contract but before choosing an effort level. The output of a low ability individual exerting effort e_i is $q_i = e_i + \varepsilon_i + \eta$ where ε_i is an idiosyncratic shock and η is a common shock. The output of a high ability individual exerting effort e_j is $q_j = e_j + \theta + \varepsilon_j + \eta$ where ε_j is an idiosyncratic shock and η is a common shock, and $\theta \geq 0$. All of the idiosyncratic shocks are independent and are distributed according to F. For all individuals participating in the tournament, utility is given by $u(w) - c(e)$ where w is the prize received, e is the effort exerted, $u' \geq 0$, $u'' \leq 0$, $c' \leq 0$, and $c'' \geq 0$. All individuals have the same outside option, which gives them utility $\bar{U}$.*

Proposition 9 gives the result.

Proposition 9 *Consider a θ-Additive tournament with $c(e) = \frac{e^2}{2}$. If F is symmetric and the agents act according to their first-order conditions, the strict loser-prize tournament which pays out prizes w_1 and w_2 induces a higher level of effort than the strict winner-prize tournament which pays out prizes w_1 and w_2.*

Unlike the homogeneous case, which induces the same level of effort when the prizes are the same in the loser- and winner-prize tournaments (Lemma 2), the heterogeneous case results in greater effort in the loser-prize case. It follows from the same logic as that in the proof of Proposition 6, that the strict loser-prize tournament dominates the strict winner-prize tournament when $R \geq \frac{P}{3}$. Furthermore, the strict loser-prize tournament may dominate even when $R \leq \frac{P}{3}$.

Corollary 6 *Consider a θ-Additive tournament with $c(e) = \frac{e^2}{2}$. Suppose that F is symmetric and the agents act according to their first-order conditions, and that $R \geq \frac{P}{3}$ for u. Then,* *the principal's profits are greater when she implements a strict loser-prize tournament than when she implements a strict winner-prize tournament. Furthermore, the strict winner-prize tournament does not necessarily dominate when $R \leq \frac{P}{3}$.*

We conclude from these results that additive heterogeneity in case 2 increases punishment relative to reward when the ability distribution is symmetric.

Multiplicative heterogeneity in case 2 has a different impact on the optimal tournament. Multiplicative heterogeneity introduces a new effect. Because, high θ agents are more productive than low θ agents, the principal cares more about inducing effort among high θ agents than low θ agents.

As with additive heterogeneity, high θ agents have a low probability of placing at the bottom of the tournament and low θ agents have a low probability of placing at the top. Therefore, high θ agents are given a greater incentive to exert effort by rewards than by punishments and low θ agents are given a greater incentive to exert effort by punishments than by rewards.

Unlike the additive heterogeneity case, the effort of high θ types is more valuable to the principle than the effort of low θ types. This gives the principal a strong reason to rely more upon rewarding winners than punishing losers. We will give an illustration of this effect.

As in all of the cases we have considered, the ability distribution can have an impact on the optimal tournament. We will therefore assume a symmetric ability distribution in order to illustrate the effect.

Definition 3 *A tournament is called a θ-Multiplicative tournament when: there are n individuals (where n is even), $\frac{n}{2}$ of whom are low ability and $\frac{n}{2}$ of whom are high ability. Agents learn their abilities after signing a contract but before choosing an effort level. The output of a low ability individual exerting effort e_i is $q_i = e_i + \varepsilon_i + \eta$ where ε_i is an idiosyncratic shock and η is a common shock. The output of a high ability individual exerting effort e_j is $q_j = \theta e_j + \varepsilon_j + \eta$ where ε_j is an idiosyncratic shock and η is a common shock, and $\theta \geq 1$. All of the idiosyncratic shocks are independent and are distributed according to F. For all individuals participating in the tournament, utility is given by $u(w) - c(e)$ where w is the* *prize received, e is the effort exerted, $u' \geq 0$, $u'' \leq 0$, $c' \leq 0$, and $c'' \geq 0$. All individuals have the same outside option, which gives them utility $\bar{U}$.*

A strict winner-prize tournament is effective at incentivizing high ability types while a strict loser-prize tournament is effective at incentivizing low ability types. We know that when there is no heterogeneity ($\theta = 1$) and $R \geq \frac{P}{3}$, the principal prefers the strict loser-prize tournament to the strict winner-prize tournament. But, as θ increases, the *importance* of incentivizing high ability types relative to low ability types increases and the strict winner-prize tournament does a better and better job of incentivizing high ability agents relative to the strict loser-prize tournament. This means that when θ is very large, it is better for the principal to implement a strict winner-prize tournament even if $R \geq \frac{P}{3}$. Proposition 10 illustrates this result.

Proposition 10 *Consider a θ-Multiplicative tournament with $c(e) = de^{\alpha}$, $d > 0$ and $\alpha > 1$. If F is symmetric and the agents act according to their first-order conditions, then for sufficiently large θ, the principal will prefer a strict winner-prize tournament to a strict loser-prize tournament.*

We conclude, therefore, that multiplicative heterogeneity can lead it to be optimal for the principal to give rewards that are large relative to punishments.

Finally, we turn to case 3, in which agents learn their types before deciding whether to participate in the tournament. In case 3, the tournament that the principal offers affects which agents will choose to participate in the tournament. Therefore, the principal also faces a screening problem.

The major way in which the screening problem affects the structure of the optimal tournament is that it affects the distribution of types participating in the tournament. Screening is likely to result in a distribution of types that is asymmetric. The type of screening that the principal can engage in screens out agents with $\theta < \bar{\theta}$. Therefore, screening may cut off the left-hand tail of the ability distribution.

Agents are therefore likely to be more clumped at low θ than at high θ. If this is the case, punishment will tend to give more incentivization to low θ agents than reward will give to high θ agents. Therefore, we believe that screening on the part of the principal is likely to lead to larger punishments relative to rewards.

7 Individual Contracts

In this section, we will relate our results about the shape of the optimal tournament prize schedule to the shape of the optimal individual contract.

Let us return to the moral hazard setting from Section 2, in which agents are homogeneous. Let us suppose that there is no common shock to output ($\eta = 0$). Under these assumptions, Green and Stokey (1983) have shown that incentivizing agents with the optimal individual contract, which gives agents $w^*(q)$, yields a profit that is greater than the profit from the optimal n-person tournament.

The reason for this result is that an agent's rank is a noisy signal of an agent's output. Therefore, the tournament gives agents a more variable payoff for a given effort level than the individual contract. Since agents are risk averse, the principal needs to compensate agents for this additional risk, which is costly.

Green and Stokey observe, however, that as n increases, an agent's rank gives an increasingly informative signal of an agent's output. Rank becomes a less and less noisy signal of output. As a result, they find that the profit from the tournament approaches the profit from the individual contract as n approaches infinity. We will use this result to relate the structure of the optimal tournament to the structure of the optimal individual contract. We will show that the optimal individual contract, like the optimal tournament, generally gives smaller rewards than punishments. Furthermore, we will be able to give conditions under which the optimal individual contract is concave or convex.

Let us construct an individual contract, $w_n(q)$, that is continuous and for which $w_n(E(q^n_{(i)})) = w^n_i$ where $E(q^n_{(i)})$ is the expected output of the agent who ranks ith in an n-agent tournament and w^n_i is the optimal prize to give for ith place in the n-agent tournament. We can define

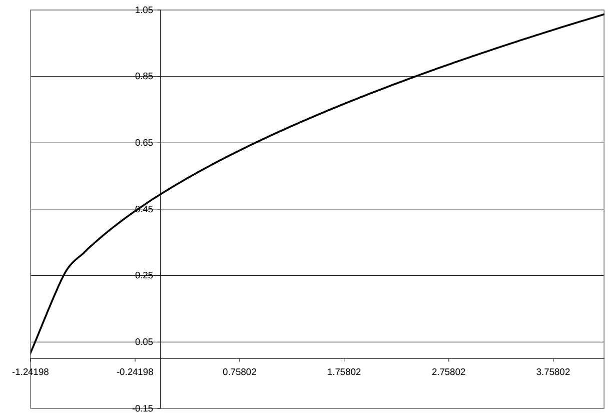

Figure 1: Figure 4: $w_{200}(q)$

$w_n(q)$ as follows:

$$w_n(q) = \begin{cases} \left(\frac{E(\varepsilon^n_{(i)})-(q-e^*_n)}{E(\varepsilon^n_{(i)})-E(\varepsilon^n_{(i+1)})}\right) w^n_{i+1} + \left(\frac{(q-e^*)-E(\varepsilon^n_{(i+1)})}{E(\varepsilon^n_{(i)})-E(\varepsilon^n_{(i+1)})}\right) w^n_i, & q-e^*_n \in [E(\varepsilon^n_{(i+1)}), E(\varepsilon^n_{(i)})] \\ w^n_1 + \left(\frac{w^n_1-w^n_2}{E(\varepsilon^n_{(1)})-E(\varepsilon^n_{(2)})}\right)\left((q-e^*_n)-E(\varepsilon^n_{(1)})\right), & q-e^*_n \geq E(\varepsilon^n_{(1)}) \\ w^n_n - \left(\frac{w^n_{n-1}-w^n_n}{E(\varepsilon^n_{(n-1)})-E(\varepsilon^n_{(n)})}\right)\left(E(\varepsilon^n_{(n)})-(q-e^*_n)\right), & q-e^*_n \leq E(\varepsilon^n_{(n)}) \end{cases}$$

where e^*_n is the effort induced by the optimal n-agent tournament and $\varepsilon^n_{(i)}$ is the ith order statistic of the idiosyncratic shocks.

Figure 4 shows $w_{200}(q)$ in the case where F is a normal distribution with standard deviation of 1, $u(w) = 1 - \frac{1}{w}$ (CRRA with $\theta = 2$), $c(e) = \frac{e^2}{2}$, and $\bar{U} = -2$. Figure 5 gives the slope. Notice that the shape is not identical to the shape of the optimal tournament prize schedule. The optimal tournament prize schedule has a slope that is increasing at low i (equivalent to high q). In contrast, $w_{200}(q)$ has a slope that is decreasing at high q (equivalent to low i). The reason for this result is that $E(\varepsilon^n_{(\lceil n/2 \rceil - 1)}) - E(\varepsilon^n_{(\lceil n/2 \rceil)}) \leq .... \leq E(\varepsilon^n_{(2)}) - E(\varepsilon^n_{(3)}) \leq E(\varepsilon^n_{(1)}) - E(\varepsilon^n_{(2)})$ for F normal.

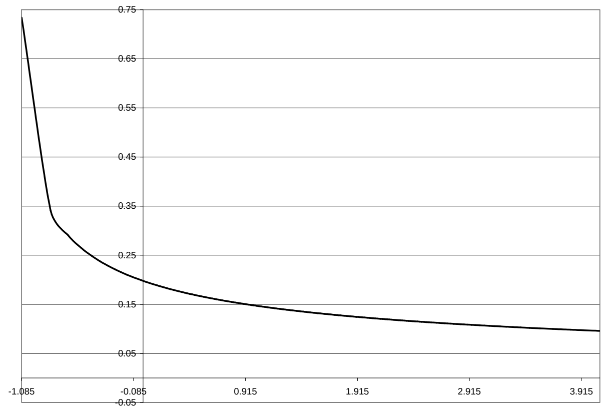

Figure 5: Slope of $w_{200}(q)$

In fact, it can be shown that, when F is a normal distribution, $w_n(q)$ is concave if $R \geq \frac{P}{2}$ and $w_n(q)$ is convex if $\frac{P}{2} \geq R \geq \frac{P}{3}$. This result relies upon the fact that $E(\varepsilon^n_{(i)}) = n\sigma^2\beta^n_i$ for the normal distribution.

In general, we cannot determine whether $w_n(q)$ will be concave or convex, but we can determine whether the $w_n(q)$ gives larger rewards or punishments. When F is symmetric and the optimal n-prize tournament gives larger punishments than rewards:

$$w_n(e^*_n + (\theta_1 + \theta_2)) - w_n(e^*_n + \theta_1) \leq w_n(e^*_n - \theta_1) - w_n(e^*_n - (\theta_1 + \theta_2))$$

for $\theta_1, \theta_2 \geq 0$. Similarly, if F is symmetric and the optimal n-prize tournament gives larger rewards than punishments:

$$w_n(e^*_n + (\theta_1 + \theta_2)) - w_n(e^*_n + \theta_1) \geq w_n(e^*_n - \theta_1) - w_n(e^*_n - (\theta_1 + \theta_2))$$

for $\theta_1, \theta_2 \geq 0$.

Green and Stokey (1983) implies that $w_n(q)$ converges pointwise to $w^*(q)$ as n approaches infinity. As a result, we can apply our findings about the shape of $w_n(q)$ to $w^*(q)$. This gives us the following proposition.

Proposition 11 *Suppose that the optimal individual contract for the principal is $w^*(q)$, $\eta = 0$, $c''' \geq 0$, $\frac{c''}{c'} \geq \frac{c'''}{c''}$, F is symmetric, $\{\beta^n_i\}^n_{i=1}$ is decreasing in i for all n, $\sum_{i=1}^{j} \gamma^n_i \geq 0$ for* *all $j \leq \frac{n}{2}$ and all n, and that the agents' second-order conditions are satisfied for sufficiently large n. Then $w^*(q)$ is increasing in q. Suppose $\theta_1, \theta_2 \geq 0$ and e^* is the effort induced by $w^*(q)$.*

If $R \geq \frac{P}{2}$,

$$w^*(e^* + \theta_1 + \theta_2) - w^*(e^* + \theta_1) \leq w^*(e^* - \theta_1) - w^*(e^* - \theta_1 - \theta_2)$$

If $\frac{P}{2} \geq R \geq \frac{P}{3}$,

$$w^*(e^* + \theta_1 + \theta_2) - w^*(e^* + \theta_1) \geq w^*(e^* - \theta_1) - w^*(e^* - \theta_1 - \theta_2)$$

If $R \geq \frac{P}{3}$,

$$u(w^*(e^* + \theta_1 + \theta_2)) - u(w^*(e^* + \theta_1)) \leq u(w^*(e^* - \theta_1)) - u(w^*(e^* - \theta_1 - \theta_2))$$

Suppose, additionally, that F is a normal distribution (which implies that the conditions on β_i^n and γ_i^n hold). If $R \geq \frac{P}{3}$, $u(w^*(q))$ is concave. If $R \geq \frac{P}{2}$, $w^*(q)$ is concave. If $\frac{P}{2} \geq R \geq \frac{P}{3}$, $w^*(q)$ is convex.

In the standard principal-agent model, one can obtain an implicit expression for the slope of optimal incentive scheme. For instance, Holmström (1979) shows that where the first-order approach is valid (and the principal is risk-neutral) the optimal individual contract has the following form

$$\frac{1}{u'(w(q))} = \mu - \lambda \frac{f'(q - e^*)}{f(q - e^*)}$$

where λ and μ are Lagrange multipliers on the participation and incentive compatibility constraints respectively. We find that the results of Proposition 11 can also be derived using this framework. While Holmström's approach is well known, these results do not appear to be.

8 Conclusion

This paper gives a framework and an intuition for thinking about how prizes should be structured in rank-order tournaments created to deal with moral hazard. These results allow us to test whether aspects of employee compensation are explained by the moral hazard theory of tournaments or arise for other reasons. Within-firm job promotions, wage increases, bonuses, and CEO compensation have often been interpreted as prizes for top performers in Lazear-Rosen rank-order tournaments. Our results, for example, cast some doubt on the idea that tournaments that reward winners without punishing losers exist *purely* to solve a moral hazard problem.

The paper identifies four key factors that influence the size of rewards for top performers relative to punishments for poor performers: (i) the size of R relative to P, (ii) the shape of the idiosyncratic noise distribution, (iii) limited liability constraints, and (iv) agent heterogeneity. When R is sufficiently large relative to P, the noise distribution is symmetric, agents are homogeneous, and the principal does not face a limited liability constraint, it is generally optimal to give larger punishments than rewards.

The paper also attempts to determine how important punishment is as a tool to the principal relative to reward. We find that a loser-lose-all tournament often returns a profit that closely approximates the profit of the optimal tournament. In contrast, the winner-take-all tournament usually returns a profit that is far from the profit of the optimal tournament.

We are able to apply our results about optimal tournaments to analyze the structure of the optimal individual contract. We find that when R is sufficiently large relative to P and the noise distribution is symmetric, the optimal individual contract generally gives larger punishments for low output than rewards for high output. In the special case where F is a normal distribution, the optimal individual contract is concave when R is sufficiently large relative to P.

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9 Appendix

Proof of Proposition 1.

$$\begin{aligned}\varphi_i(e,e^*) &= \Pr(i\text{th place}|e,e^*)\\ &= \int_{\mathbb{R}} \binom{n-1}{i-1} \left(F(e-e^*+x)\right)^{n-i} \left(1-F(e-e^*+x)\right)^{i-1} f(x)dx\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial e}\varphi_i(e,e^*) &= \int_{\mathbb{R}} \binom{n-1}{i-1} \left[\begin{array}{c}(n-i)\left(F(e-e^*+x)\right)^{n-i-1}\left(1-F(e-e^*+x)\right)^{i-1}\\ -(i-1)\left(F(e-e^*+x)\right)^{n-i}\left(1-F(e-e^*+x)\right)^{i-2}\end{array}\right] f(x)f(e-e^*+x)dx\\ &= \int_{\mathbb{R}} \binom{n-1}{i-1} \left(F(e-e^*+x)\right)^{n-i-1}\left(1-F(e-e^*+x)\right)^{i-2}\\ &\quad \cdot \left[\begin{array}{c}(n-i)\left(1-F(e-e^*+x)\right)\\ -(i-1)\left(F(e-e^*+x)\right)\end{array}\right] f(x)f(e-e^*+x)dx\\ &= \int_{\mathbb{R}} \binom{n-1}{i-1} \left(F(e-e^*+x)\right)^{n-i-1}\left(1-F(e-e^*+x)\right)^{i-2}\\ &\quad \cdot \left[(n-i)-(n-1)\left(F(e-e^*+x)\right)\right] f(x)f(e-e^*+x)dx\end{aligned}$$

$$\beta_i = \frac{\partial}{\partial e}\varphi_i(e,e^*)\bigg|_{e=e^*} = \binom{n-1}{i-1}\int_{\mathbb{R}} F(x)^{n-i-1}(1-F(x))^{i-2}\left((n-i)-(n-1)F(x)\right)f(x)^2dx$$

Since $\sum_{i=1}^n \varphi_i(e,e^*)=1$, $\sum_{i=1}^n \frac{\partial}{\partial e}\varphi_i(e,e^*)=0$. Hence, $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \frac{\partial}{\partial e}\varphi_i(e,e^*)\big|_{e=e^*}=0$.

$$\begin{aligned}\beta_1 &= (n-1)\int_{\mathbb{R}} F(x)^{n-2}f(x)^2dx \geq 0\\ \beta_n &= -(n-1)\int_{\mathbb{R}} (1-F(x))^{n-2}f(x)^2dx \leq 0\end{aligned}$$

If F is symmetric, $F(-x)=1-F(x)$. Differentiating both sides, it follows that $f(-x)=f(x)$.

$$\beta_i = \binom{n-1}{i-1}\int_{\mathbb{R}} F(x)^{n-i-1}(1-F(x))^{i-2}\left((n-i)-(n-1)F(x)\right)f(x)^2dx$$

Therefore,

$$\beta_{n-i+1} = \binom{n-1}{n-i}\int_{\mathbb{R}} F(x)^{i-2}(1-F(x))^{n-i-1}\left((i-1)-(n-1)F(x)\right)f(x)^2dx$$

Since, $\binom{n-1}{n-i} = \binom{n-1}{i-1}$,

$$\begin{aligned}
\beta_{n-i+1} &= \binom{n-1}{i-1} \int\limits_{\mathbb{R}} F(x)^{i-2}(1-F(x))^{n-i-1}\left((i-1)-(n-1)F(x)\right) f(x)^2 dx \\
&= \binom{n-1}{i-1} \int\limits_{\infty}^{-\infty} F(-x)^{i-2}(1-F(-x))^{n-i-1}\left((i-1)-(n-1)F(-x)\right) f(-x)^2 d(-x) \\
&= \binom{n-1}{i-1} \int\limits_{\mathbb{R}} (1-F(x))^{i-2}F(x)^{n-i-1}\left((i-1)-(n-1)(1-F(x))\right) f(x)^2 dx \\
&= -\binom{n-1}{i-1} \int\limits_{\mathbb{R}} (1-F(x))^{i-2}F(x)^{n-i-1}\left((n-i)-(n-1)F(x)\right) f(x)^2 dx \\
&= -\beta_i
\end{aligned}$$

Hence, $\beta_{n-i+1} = -\beta_i$ for F symmetric. If F is symmetric, $F(-x) = 1 - F(x)$. Differentiating both sides, it follows that $f(-x) = f(x)$. Suppose F is uniform on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$. $F(x) = \frac{x+\frac{\sigma}{2}}{\sigma}$ on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$. $f(x) = \frac{1}{\sigma}$ on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$ and $f(x) = 0$ on $[-\frac{\sigma}{2}, \frac{\sigma}{2}]^C$.

$$\begin{aligned}
\beta_i &= \binom{n-1}{i-1} \int\limits_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} F(x)^{n-i-1}(1-F(x))^{i-2}\left((n-i)-(n-1)F(x)\right) f(x)^2 dx \\
&= \binom{n-1}{i-1} \int\limits_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \left(\frac{x+\frac{\sigma}{2}}{\sigma}\right)^{n-i-1} \left(\frac{-x+\frac{\sigma}{2}}{\sigma}\right)^{i-2} \left((n-i)-(n-1)\left(\frac{x+\frac{\sigma}{2}}{\sigma}\right)\right) \frac{1}{\sigma^2} dx \\
&= \binom{n-1}{i-1} \int\limits_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \left(\frac{x+\frac{\sigma}{2}}{\sigma}\right)^{n-i-1} \left(\frac{-x+\frac{\sigma}{2}}{\sigma}\right)^{i-2} \left((\frac{n-2i+1}{2}) - \left(\frac{n-1}{\sigma}\right)x\right) \frac{1}{\sigma^2} dx \\
&= \binom{n-1}{i-1} \left(\frac{x+\frac{\sigma}{2}}{\sigma}\right)^{n-i} \left(\frac{-x+\frac{\sigma}{2}}{\sigma}\right)^{i-1} \frac{1}{\sigma}\Bigg|_{x=-\frac{\sigma}{2}}^{\frac{\sigma}{2}}
\end{aligned}$$

From this, we see that

$$\begin{aligned}
\beta_1 &= -\beta_n = \frac{1}{\sigma} \\
\beta_i &= 0, \quad 1 < i < n
\end{aligned}$$

■

Proof of Proposition 2. Suppose that F is symmetric and $u(w_i) - u(w_j) \leq u(w_{n-j+1}) - u(w_{n-i+1})$ for all $i \leq j \leq \frac{n+1}{2}$.

The second-order condition of the agent's problem is:

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial e^2}\varphi_i(e, e^*)u(w_i) - c''(e) \leq 0$$

Since $c'' \leq 0$ by assumption, the second-order condition will hold at $e = e^*$ if:

$$\begin{aligned} \sum_{i=1}^{n} \gamma_i u(w_i) &\leq 0 \\ \text{where } \gamma_i &= \left. \frac{\partial^2}{\partial e^2} \varphi_i(e, e^*) \right|_{e=e^*} \end{aligned}$$

In the proof of Proposition 1, a formula was given for $\frac{\partial}{\partial e}\varphi_i(e, e^*)$. Differentiating this expression with respect to e, we find that:

$$\begin{aligned} \gamma_i &= \left. \frac{\partial^2}{\partial e^2} \varphi_i(e, e^*) \right|_{e=e^*} \\ &= \binom{n-1}{i-1} \int_{\mathbb{R}} (F(x))^{n-i-2} (1-F(x))^{i-3} \left[\begin{array}{c} (n-i)(n-i-1) \\ -2(n-i)(n-2)F(x) + (n-1)(n-2)F^2(x) \end{array} \right] f^3(x)dx \\ &\quad + \binom{n-1}{i-1} \int_{\mathbb{R}} F(x)^{n-i-1} (1-F(x))^{i-2} \left[(n-i) - (n-1)F(x)\right] f(x)f'(x)dx \end{aligned}$$

Since $\sum_{i=1}^{n} \varphi_i(e, e^*) = 1$, $\sum_{i=1}^{n} \frac{\partial^2}{\partial e^2}\varphi_i(e, e^*)\Big|_{e=e^*} = 0$ and $\sum_{i=1}^{n} \gamma_i = 0$.
Since F is symmetric:

$$\begin{aligned} \gamma_{n-i+1} &= \binom{n-1}{n-i} \int_{\mathbb{R}} (F(x))^{i-3} (1-F(x))^{n-i-2} \left[\begin{array}{c} (i-1)(i-2) \\ -2(i-1)(n-2)F(x) + (n-1)(n-2)F^2(x) \end{array} \right] f^3(x)dx \\ &\quad + \binom{n-1}{i-1} \int_{\mathbb{R}} F(x)^{i-2} (1-F(x))^{n-i-1} \left[\begin{array}{c} (i-1) \\ -(n-1)F(x) \end{array} \right] f(x)f'(x)dx \\ &= \binom{n-1}{i-1} \int_{\mathbb{R}} (F(-x))^{i-3} (1-F(-x))^{n-i-2} \left[\begin{array}{c} (i-1)(i-2) - 2(i-1)(n-2)F(-x) \\ +(n-1)(n-2)F^2(-x) \end{array} \right] f^3(-x)dx \\ &\quad - \binom{n-1}{i-1} \int_{\mathbb{R}} F(-x)^{i-2} (1-F(-x))^{n-i-1} \left[(n-i) - (n-1)(1-F(-x))\right] f(-x)f'(-x)dx \\ &= \binom{n-1}{i-1} \int_{\mathbb{R}} (F(x))^{n-i-2} (1-F(x))^{i-3} \left[\begin{array}{c} (i-1)(i-2) - 2(i-1)(n-2)(1-F(x)) \\ +(n-1)(n-2)(1-F(x))^2 \end{array} \right] f^3(x)dx \\ &\quad + \binom{n-1}{i-1} \int_{\mathbb{R}} F(-x)^{i-2} (1-F(-x))^{n-i-1} \left[(n-i) - (n-1)(1-F(-x))\right] f(x)f'(x)dx \\ &= \binom{n-1}{i-1} \int_{\mathbb{R}} (F(x))^{n-i-2} (1-F(x))^{i-3} \left[\begin{array}{c} (n-i)(n-i-1) - 2(n-i)(n-2)F(x) \\ +(n-1)(n-2)F^2(x) \end{array} \right] f^3(x)dx \\ &\quad + \binom{n-1}{i-1} \int_{\mathbb{R}} F(x)^{n-i-1} (1-F(x))^{i-2} \left[(n-i) - (n-1)F(x)\right] f(x)f'(x)dx \\ &= \gamma_i \end{aligned}$$

Let us define γ_i' as follows. $\gamma_i' = \gamma_i$ for $i \neq \frac{n+1}{2}$. If $i = \frac{n+1}{2}$, $\gamma_i' = \frac{1}{2}\gamma_i$

Notice that, since $\gamma_i = \gamma_{n-i+1}$,

$$\begin{aligned}\sum_{i=1}^{\lceil n/2 \rceil} \gamma'_i &= \frac{1}{2}\sum_{i=1}^{n} \gamma_i \\ &= 0\end{aligned}$$

Let $P = \{i \leq \lceil n/2 \rceil \,|\gamma'_i \geq 0\}$ and $N = \{i \leq \lceil n/2 \rceil \,|\gamma'_i < 0\}$. By assumption $\sum_{i=1}^{j} \gamma_i \geq 0$ for all $j \leq \frac{n}{2}$. Furthermore, $\sum_{i=1}^{\lceil n/2 \rceil} \gamma'_i = 0$. As a result, for $i \in P$ it is possible to write γ'_i as:

$$\gamma'_i = -\sum_{k \in N} \delta_{ik}\gamma'_k$$

where $\delta_{ik} \geq 0$ for $i > k$, $\delta_{ik} = 0$ for $k < i$, and

$$\sum_{i \in P} \delta_{ik} = 1 \quad \text{for all } k \in N$$

$$\begin{aligned}\sum_{i=1}^{n} \gamma_i u(w_i) &= \sum_{i=1}^{\lceil n/2 \rceil} \gamma'_i(u(w_i) + u(w_{n-i+1})) \\ &= \sum_{i \in P} \gamma'_i(u(w_i) + u(w_{n-i+1})) + \sum_{k \in N} \gamma'_k(u(w_k) + u(w_{n-k+1})) \\ &= -\sum_{i \in P}\sum_{k \in N} \delta_{ik}\gamma'_k(u(w_i) + u(w_{n-i+1})) + \sum_{i \in N} \gamma'_i(u(w_i) + u(w_{n-i+1})) \\ &= \sum_{k \in N}(-\gamma'_k)(\sum_{i \in P} \delta_{ik}(u(w_i) + u(w_{n-i+1})) - (u(w_k) + u(w_{n-k+1}))\end{aligned}$$

For $i \leq k \leq \frac{n+1}{2}$,

$$\begin{aligned}u(w_i) - u(w_k) &\leq u(w_{n-k+1}) - u(w_{n-i+1}) \\ u(w_i) + u(w_{n-i+1}) &\leq u(w_k) + u(w_{n-k+1})\end{aligned}$$

Since $i \leq k \leq \frac{n+1}{2}$ when $\delta_{ik} > 0$, it follows that

$$\sum_{k \in N}(-\gamma_k)(\sum_{i \in P} \delta_{ik}(u(w_i) + u(w_{n-i+1})) - (u(w_k) + u(w_{n-k+1})) \leq 0$$

Therefore, $\sum_{i=1}^{n} \gamma_i u(w_i) \leq 0$. Hence, the second-order condition holds at $e = e^*$. ■

Proof of Lemma 1. $s(u_1, ..., u_n) = (c')^{-1}\left(\sum_i \beta_i u_i\right) - \frac{1}{n}\sum_i u^{-1}(u_i)$

Since $u'' \leq 0$ and $u' \geq 0$, $(u^{-1})''(x) = \frac{-u''(u^{-1}(x))}{(u'(u^{-1}(x)))^3} \geq 0$. Therefore, $-u^{-1}$ is concave. Hence, $-\frac{1}{n}\sum_i u^{-1}(u_i)$. Since $\sum_i \beta_i u_i$ is linear in u_i, $(c')^{-1}\left(\sum_i \beta_i u_i\right)$ is concave if $(c')^{-1}$ is concave. For simplicity of notation, let $z_1(x) = (c')^{-1}(x)$. $z_1''(x) = \frac{-c'''((c')^{-1}(x)))}{(c''((c')^{-1}(x)))^3}$. Therefore, $(c')^{-1}$ is concave if and only if $\frac{c'''}{c''} \geq 0$. Since $c'', c''' \geq 0$, $(c')^{-1}$ is indeed concave. Since $(c')^{-1}\left(\sum_i \beta_i u_i\right)$ and $-\frac{1}{n}\sum_i u^{-1}(u_i)$ are both concave, s is a concave function.

$q(u_1, ..., u_n) = -\frac{1}{n}\sum_i u_i + c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right) + \bar{U}$

$-\frac{1}{n}\sum_i u_i + \bar{U}$ is linear, and therefore both concave and convex. $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ will be convex if $c\left((c')^{-1}(x)\right)$ is convex since $\sum_i \beta_i u_i$ is linear. For simplicity of notation, let $z_2(x) = c\left((c')^{-1}(x)\right)$. $z_2''(x) = \frac{\left(c''((c')^{-1}(x))\right)^2 - \left(c'((c')^{-1}(x))\right)\left(c'''((c')^{-1}(x))\right)}{(c''((c')^{-1}(x)))^3}$. Since $-\frac{c'''}{c''} \geq -\frac{c''}{c'}$, it follows that

$z_2'' \geq 0$. Hence, $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ is convex. Since $-\frac{1}{n}\sum_i u_i + \bar{U}$ and $c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)$ are both convex, q is convex. ■

Proof of Proposition 3.

$$\mathcal{L} = \left((c')^{-1}\left(\sum_i \beta_i u_i\right) - \frac{1}{n}\sum_i u^{-1}(u_i)\right) - \lambda\left(\bar{U} - \frac{1}{n}\sum_i u_i + c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)\right)$$

Let $h(x) = (c')^{-1}(x)$, $v(x) = u'(x)$, and $v_i = u'(w_i) = u'(u^{-1}(u_i))$.
The first order condition for u_i is as follows:

$$\begin{aligned}
\left(\beta_i h'\left(\sum_i \beta_i u_i\right) - \frac{1}{nu'(u^{-1}(u_i))}\right) + \lambda\left(\frac{1}{n} - c'\left(h\left(\sum_i \beta_i u_i\right)\right)h'\left(\sum_i \beta_i u_i\right)\right) &= 0 \\
\beta_i n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right) + \lambda &= \frac{1}{v_i}
\end{aligned}$$

It follows that, for any i and k,

$$\frac{1}{v_i} - \frac{1}{v_{i+k}} = (\beta_i - \beta_{i+k}) n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right)$$

Similarly, for any j and l,

$$\frac{1}{v_j} - \frac{1}{v_{j+l}} = \frac{1}{v_i} - \frac{1}{v_{i+k}} = (\beta_j - \beta_{j+l}) n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right)$$

Therefore,

$$\frac{\frac{1}{v_i} - \frac{1}{v_{i+k}}}{\frac{1}{v_j} - \frac{1}{v_{j+l}}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

■

Proof of Corollary 1. Assume that agents act according to the first-order condition, F is uniform, and $1 < i \leq j < n$. It follows from Proposition 3 that:

$$\frac{\frac{1}{v_i} - \frac{1}{v_j}}{\frac{1}{v_1} - \frac{1}{v_n}} = \frac{\beta_i - \beta_j}{\beta_1 - \beta_n}$$

Since F is uniform, $\beta_i = \beta_j = 0$. Hence, $\frac{1}{v_i} = \frac{1}{v_j}$, which implies that $w_i^* = w_j^*$. ■

Proof of Proposition 4. Let $r(w) = \frac{1}{v(w)} = \frac{1}{u'(w)}$.

$$\begin{aligned}
r'(w) &= \frac{-u''}{(u')^2} \geq 0 \\
r''(w) &= \frac{2(u'')^2 - u'u'''}{(u')^3} \\
&= \left(\frac{-u''}{(u')^2}\right)\left(2\left(\frac{-u''}{u'}\right) - \left(-\frac{u'''}{u''}\right)\right) \\
&= \left(\frac{-u''}{(u')^2}\right)(2R - P)
\end{aligned}$$

Where $R = -\frac{u''}{u'}$ is the Arrow-Pratt measure of risk aversion and $P = -\frac{u'''}{u''}$ is the coefficient of absolute prudence. Since $u' \geq 0$ and $u'' \leq 0$ by assumption, $\frac{-u''}{(u')^2} \geq 0$. Therefore, $R \geq \frac{P}{2}$ implies $r'' \geq 0$ and $R \leq \frac{P}{2}$ implies $r'' \leq 0$. Let us now consider two cases.

Case 1: $R \geq \frac{P}{2}$

Since $\beta_i \geq \beta_{i+k}$ and r is increasing, it follows that $w_i^* \geq w_{i+k}^*$. Because $r'' \geq 0$ (since $R \geq \frac{P}{2}$), it follows that:

$$r'(w_{i+k}^*)(w_i^* - w_{i+k}^*) \leq r(w_i^*) - r(w_{i+k}^*) \leq r'(w_i^*)(w_i^* - w_{i+k}^*)$$

Similarly, since $\beta_j \geq \beta_{j+l}$, $w_j^* \geq w_{j+l}^*$ and:

$$r'(w_{j+l}^*)(w_j^* - w_{j+l}^*) \leq r(w_j^*) - r(w_{j+l}^*) \leq r'(w_j^*)(w_j^* - w_{j+l}^*)$$

Hence,

$$\left(\frac{r'(w_{i+k}^*)}{r'(w_j^*)}\right) \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)} \leq \left(\frac{r'(w_i^*)}{r'(w_{j+l}^*)}\right) \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*}$$

And,

$$\left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right) \frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right) \frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)}$$

By Proposition 3, $\frac{r(w_i^*) - r(w_{i+k}^*)}{r(w_j^*) - r(w_{j+l}^*)} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$. Therefore,

$$\left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right) \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right) \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$r'' \geq 0$ and $\min(i, i+k) \geq \max(j, j+l)$ implies that:

$$1 \leq \frac{r'(w_{j+l}^*)}{r'(w_i^*)} \leq \frac{r'(w_j^*)}{r'(w_{i+k}^*)}$$

And,

$$\frac{r'(w_{j+l}^*)}{r'(w_i^*)} = \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right) \left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2$$

So,

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right) \left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{w_i^* - w_{i+k}^*}{w_j^* - w_{j+l}^*} \leq \left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right) \left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^2 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

Case 2: $R \leq \frac{P}{2}$

Since $w_i^* \geq w_{i+k}^*$ and $r'' \leq 0$ (since $R \leq \frac{P}{2}$), it follows that:

$$r'(w_i^*)(w_i^* - w_{i+k}^*) \leq r(w_i^*) - r(w_{i+k}^*) \leq r'(w_{i+k}^*)(w_i^* - w_{i+k}^*)$$

Similarly,

$$r'(w_j^*)(w_j^* - w_{j+l}^*) \leq r(w_j^*) - r(w_{j+l}^*) \leq r'(w_{j+l}^*)(w_j^* - w_{j+l}^*)$$

Hence,

$$\left(\frac{r'(w_i^*)}{r'(w_{j+l}^*)}\right)\frac{w_i^*-w_{i+k}^*}{w_j^*-w_{j+l}^*} \leq \frac{r(w_i^*)-r(w_{i+k}^*)}{r(w_j^*)-r(w_{j+l}^*)} \leq \left(\frac{r'(w_{i+k}^*)}{r'(w_j^*)}\right)\frac{w_i^*-w_{i+k}^*}{w_j^*-w_{j+l}^*}$$

And,

$$\left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right)\frac{r(w_i^*)-r(w_{i+k}^*)}{r(w_j^*)-r(w_{j+l}^*)} \leq \frac{w_i^*-w_{i+k}^*}{w_j^*-w_{j+l}^*} \leq \left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right)\frac{r(w_i^*)-r(w_{i+k}^*)}{r(w_j^*)-r(w_{j+l}^*)}$$

By Proposition 3, $\frac{r(w_i^*)-r(w_{i+k}^*)}{r(w_j^*)-r(w_{j+l}^*)} = \frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$. Therefore,

$$\left(\frac{r'(w_j^*)}{r'(w_{i+k}^*)}\right)\frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}} \leq \frac{w_i^*-w_{i+k}^*}{w_j^*-w_{j+l}^*} \leq \left(\frac{r'(w_{j+l}^*)}{r'(w_i^*)}\right)\frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$$

$r'' \leq 0$ and $\min(i, i+k) \geq \max(j, j+l)$ implies that:

$$\frac{r'(w_j^*)}{r'(w_{i+k}^*)} \leq \frac{r'(w_{j+l}^*)}{r'(w_i^*)} \leq 1$$

And,

$$\frac{r'(w_j^*)}{r'(w_{i+k}^*)} = \left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^2$$

So,

$$\left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^2\frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}} \leq \frac{w_i^*-w_{i+k}^*}{w_j^*-w_{j+l}^*} \leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^2\frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}} \leq \frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$$

Let $z(x) = \frac{1}{u'(u^{-1}(x))}$.

$$\begin{aligned}
z'(x) &= \frac{-u''(u^{-1}(x))}{(u'(u^{-1}(x)))^3} \geq 0 \\
z''(y) &= \frac{3(u'')^2 - u'u'''}{(u')^5} \\
&= \left(\frac{-u''}{(u')^4}\right)\left(3\left(\frac{-u''}{u'}\right) - \left(\frac{-u'''}{u''}\right)\right) \\
&= \left(\frac{-u''}{(u')^4}\right)(3R - P)
\end{aligned}$$

Since $u' \geq 0$ and $u'' \leq 0$ by assumption, $\frac{-u''}{(u')^4} \geq 0$. Therefore, $R \geq \frac{P}{3}$ implies $z'' \geq 0$ and $R \leq \frac{P}{3}$ implies $z'' \leq 0$. Following the same procedure as above with r, we consider two cases.

Case 1: $R \geq \frac{P}{3}$

Since $R \geq \frac{P}{3}$, it follows that $z'' \geq 0$. Hence,

$$z'(u_{i+k}^*)(u_i^* - u_{i+k}^*) \leq z(u_i^*) - z(u_{i+k}^*) \leq z'(u_i^*)(u_i^* - u_{i+k}^*)$$

Similarly,

$$z'(u_{j+l}^*)(u_j^* - u_{j+l}^*) \leq z(u_j^*) - z(u_{j+l}^*) \leq z'(u_j^*)(u_j^* - u_{j+l}^*)$$

Hence,

$$\left(\frac{z'(u^*_{i+k})}{z'(u^*_j)}\right)\frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})} \le \left(\frac{z'(u^*_i)}{z'(u^*_{j+l})}\right)\frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}}$$

And,

$$\left(\frac{z'(u^*_{j+l})}{z'(u^*_i)}\right)\frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})} \le \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \left(\frac{z'(u^*_j)}{z'(u^*_{i+k})}\right)\frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})}$$

By Proposition 3 and the definition of z, it follows that $\frac{z(u^*_i)-z(u^*_{i+k})}{z(u^*_j)-z(u^*_{j+l})} = \frac{\frac{1}{v_i}-\frac{1}{v_{i+k}}}{\frac{1}{v_j}-\frac{1}{v_{j+l}}} = \frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$. So,

$$\left(\frac{z'(u^*_{j+l})}{z'(u^*_i)}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \le \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \left(\frac{z'(u^*_j)}{z'(u^*_{i+k})}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$z'' \ge 0$ and $\min(i, i+k) \ge \max(j, j+l)$ implies that:

$$1 \le \frac{z'(u^*_{j+l})}{z'(u^*_i)} \le \frac{z'(u^*_j)}{z'(u^*_{i+k})}$$

And,

$$\frac{z'(u^*_{j+l})}{z'(u^*_i)} = \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3$$

So,

$$\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \le \left(\frac{u''(w^*_{j+l})}{u''(w^*_i)}\right)\left(\frac{u'(w^*_i)}{u'(w^*_{j+l})}\right)^3\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \le \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \left(\frac{u''(w^*_j)}{u''(w^*_{i+k})}\right)\left(\frac{u'(w^*_{i+k})}{u'(w^*_j)}\right)^3\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

Case 2: $R \le \frac{P}{3}$

Since $R \le \frac{P}{3}$, it follows that $z'' \le 0$. Hence,

$$z'(u^*_i)(u^*_i - u^*_{i+k}) \le z(u^*_i) - z(u^*_{i+k}) \le z'(u^*_{i+k})(u^*_i - u^*_{i+k})$$

Similarly,

$$z'(u^*_j)(u^*_j - u^*_{j+l}) \le z(u^*_j) - z(u^*_{j+l}) \le z'(u^*_{j+l})(u^*_j - u^*_{j+l})$$

Hence,

$$\left(\frac{z'(u^*_i)}{z'(u^*_{j+l})}\right)\frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})} \le \left(\frac{z'(u^*_{i+k})}{z'(u^*_j)}\right)\frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}}$$

And,

$$\left(\frac{z'(u^*_j)}{z'(u^*_{i+k})}\right)\frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})} \le \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \left(\frac{z'(u^*_{j+l})}{z'(u^*_i)}\right)\frac{z(u^*_i) - z(u^*_{i+k})}{z(u^*_j) - z(u^*_{j+l})}$$

By Proposition 3 and the definition of z, it follows that $\frac{z(u^*_i)-z(u^*_{i+k})}{z(u^*_j)-z(u^*_{j+l})} = \frac{\frac{1}{v_i}-\frac{1}{v_{i+k}}}{\frac{1}{v_j}-\frac{1}{v_{j+l}}} = \frac{\beta_i-\beta_{i+k}}{\beta_j-\beta_{j+l}}$. So,

$$\left(\frac{z'(u^*_j)}{z'(u^*_{i+k})}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \le \frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} \le \left(\frac{z'(u^*_{j+l})}{z'(u^*_i)}\right)\frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$z'' \leq 0$ and $\min(i, i+k) \geq \max(j, j+l)$ implies that:

$$\frac{z'(u_j^*)}{z'(u_{i+k}^*)} \leq \frac{z'(u_{j+l}^*)}{z'(u_i^*)} \leq 1$$

And,

$$\frac{z'(u_j^*)}{z'(u_{i+k}^*)} = \left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^3$$

So,

$$\left(\frac{u''(w_j^*)}{u''(w_{i+k}^*)}\right)\left(\frac{u'(w_{i+k}^*)}{u'(w_j^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{u_i^* - u_{i+k}^*}{u_j^* - u_{j+l}^*} \leq \left(\frac{u''(w_{j+l}^*)}{u''(w_i^*)}\right)\left(\frac{u'(w_i^*)}{u'(w_{j+l}^*)}\right)^3 \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}} \leq \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

■

Proof of Corollary 2. Suppose $R \geq \frac{P}{3}$, F is symmetric, $\{\beta_i\}$ is decreasing in i, agents act according to the first-order condition, and $i \leq j \leq \frac{n+1}{2}$. It follows from Proposition 4 that:

$$\frac{u_{n-j+1}^* - u_{n-i+1}^*}{u_i^* - u_j^*} \geq \frac{\beta_i - \beta_j}{\beta_{n-j+1} - \beta_{n-i+1}}$$

Since F is symmetric, $\beta_i = -\beta_{n-i+1}$ and $-\beta_j = \beta_{n-j+1}$. Hence, $\frac{\beta_i - \beta_j}{\beta_{n-j+1} - \beta_{n-i+1}} = 1$. It therefore follows that:

$$\begin{aligned} \frac{u_{n-j+1}^* - u_{n-i+1}^*}{u_i^* - u_j^*} &\geq 1 \\ u_{n-j+1}^* - u_{n-i+1}^* &\geq u_i^* - u_j^* \\ u(w_{n-j+1}^*) - u(w_{n-i+1}^*) &\leq u(w_i^*) - u(w_j^*) \end{aligned}$$

■

Proof of Proposition 5. Let $r_i = \frac{w_i^* - w_{i+1}^*}{w_{n-i}^* - w_{n-i+1}^*}$ and $q_i = \frac{u_i^* - u_{i+1}^*}{u_{n-i}^* - u_{n-i+1}^*}$ and assume that $i \geq \frac{n+1}{2}$.

It follows from Proposition 4 that if $R \geq \frac{P}{2}$,

$$\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \leq \left(\frac{r'(w_{n-i+1}^*)}{r'(w_i^*)}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \leq r_i \leq \left(\frac{r'(w_{n-i}^*)}{r'(w_{i+1}^*)}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}}$$

Since F is symmetric, $\beta_i = -\beta_{n-i+1}$ and $\beta_{i+1} = -\beta_{n-i}$. Hence, $\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} = 1$. So,

$$1 \leq \left(\frac{r'(w_{n-i+1}^*)}{r'(w_i^*)}\right) \leq r_i \leq \left(\frac{r'(w_{n-i}^*)}{r'(w_{i+1}^*)}\right)$$

Similarly,

$$1 \leq \left(\frac{r'(w_{n-i}^*)}{r'(w_{i+1}^*)}\right) \leq r_{i+1} \leq \left(\frac{r'(w_{n-i-1}^*)}{r'(w_{i+2}^*)}\right)$$

Hence,

$$1 \leq \left(\frac{r'(w_{n-i+1}^*)}{r'(w_i^*)}\right) \leq r_i \leq \left(\frac{r'(w_{n-i}^*)}{r'(w_{i+1}^*)}\right) \leq r_{i+1} \leq \left(\frac{r'(w_{n-i-1}^*)}{r'(w_{i+2}^*)}\right)$$

Therefore, $r_i \geq 1$ and $r_{i+1} \geq r_i$ for all $i \geq \frac{n}{2}$. Now consider $i \leq \frac{n}{2}$. $r_i = \frac{1}{r_{n-i}}$ and $r_{i-1} = \frac{1}{r_{n-i+1}}$. Since $n-i \geq \frac{n}{2}$, $1 \leq r_{n-i} \leq r_{n-i+1}$. Hence, $\frac{1}{r_{n-i+1}} \leq \frac{1}{r_{n-i}}$, or $r_{i-1} \leq r_i$. Therefore, $r_i \leq r_{i+1}$ for all i.

It follows from Proposition 4 that if $R \geq \frac{P}{3}$,

$$\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \leq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \leq q_i \leq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}}$$

Since F is symmetric, $\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} = 1$. So,

$$1 \leq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \leq q_i \leq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right)$$

Similarly,

$$1 \leq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \leq q_{i+1} \leq \left(\frac{z'(u^*_{n-i-1})}{z'(u^*_{i+2})}\right)$$

Hence,

$$1 \leq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \leq q_i \leq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \leq q_{i+1} \leq \left(\frac{z'(u^*_{n-i-1})}{z'(u^*_{i+2})}\right)$$

Therefore, $q_i \geq 1$ and $q_{i+1} \geq q_i$ for all $i \geq \frac{n}{2}$. Now consider $i \leq \frac{n}{2}$. $q_i = \frac{1}{q_{n-i}}$ and $q_{i-1} = \frac{1}{q_{n-i+1}}$. Since $n-i \geq \frac{n}{2}$, $1 \leq q_{n-i} \leq q_{n-i+1}$. Hence, $\frac{1}{q_{n-i+1}} \leq \frac{1}{q_{n-i}}$, or $q_{i-1} \leq q_i$. Therefore, $q_i \leq q_{i+1}$ for all i.

It follows from Proposition 4 that if $R \leq \frac{P}{2}$,

$$\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \geq \left(\frac{r'(w^*_{n-i+1})}{r'(w^*_i)}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \geq r_i \geq \left(\frac{r'(w^*_{n-i})}{r'(w^*_{i+1})}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}}$$

Since F is symmetric, $\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} = 1$. So,

$$1 \geq \left(\frac{r'(w^*_{n-i+1})}{r'(w^*_i)}\right) \geq r_i \geq \left(\frac{r'(w^*_{n-i})}{r'(w^*_{i+1})}\right)$$

Similarly,

$$1 \geq \left(\frac{r'(w^*_{n-i})}{r'(w^*_{i+1})}\right) \geq r_{i+1} \geq \left(\frac{r'(w^*_{n-i-1})}{r'(w^*_{i+2})}\right)$$

Hence,

$$1 \geq \left(\frac{r'(w^*_{n-i+1})}{r'(w^*_i)}\right) \geq r_i \geq \left(\frac{r'(w^*_{n-i})}{r'(w^*_{i+1})}\right) \geq r_{i+1} \geq \left(\frac{r'(w^*_{n-i-1})}{r'(w^*_{i+2})}\right)$$

Therefore, $r_i \leq 1$ and $r_{i+1} \leq r_i$ for all $i \geq \frac{n}{2}$. Now consider $i \leq \frac{n}{2}$. $r_i = \frac{1}{r_{n-i}}$ and $r_{i-1} = \frac{1}{r_{n-i+1}}$. Since $n-i \geq \frac{n}{2}$, $1 \geq r_{n-i} \geq r_{n-i+1}$. Hence, $\frac{1}{r_{n-i+1}} \geq \frac{1}{r_{n-i}}$, or $r_{i-1} \geq r_i$. Therefore, $r_i \geq r_{i+1}$ for all i.

It follows from Proposition 4 that if $R \leq \frac{P}{3}$,

$$\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \geq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} \geq q_i \geq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}}$$

Since F is symmetric, $\frac{\beta_i - \beta_{i+1}}{\beta_{n-i} - \beta_{n-i+1}} = 1$. So,

$$1 \geq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \geq q_i \geq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right)$$

Similarly,

$$1 \geq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \geq q_{i+1} \geq \left(\frac{z'(u^*_{n-i-1})}{z'(u^*_{i+2})}\right)$$

Hence,

$$1 \geq \left(\frac{z'(u^*_{n-i+1})}{z'(u^*_i)}\right) \geq q_i \geq \left(\frac{z'(u^*_{n-i})}{z'(u^*_{i+1})}\right) \geq q_{i+1} \geq \left(\frac{z'(u^*_{n-i-1})}{z'(u^*_{i+2})}\right)$$

Therefore, $q_i \leq 1$ and $q_{i+1} \leq q_i$ for all $i \geq \frac{n}{2}$. Now consider $i \leq \frac{n}{2}$. $q_i = \frac{1}{q_{n-i}}$ and $q_{i-1} = \frac{1}{q_{n-i+1}}$. Since $n - i \geq \frac{n}{2}$, $1 \geq q_{n-i} \geq q_{n-i+1}$. Hence, $\frac{1}{q_{n-i+1}} \geq \frac{1}{q_{n-i}}$, or $q_{i-1} \geq q_i$. Therefore, $q_i \geq q_{i+1}$ for all i. ■

Proof of Corollary 3. If $u(w) = \log(w)$, $R = \frac{P}{2}$. Hence,

$$\frac{w^*_i - w^*_{i+k}}{w^*_j - w^*_{j+l}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$$\frac{w^*_i - w^*_n}{w^*_1 - w^*_n} = \frac{\beta_i - \beta_n}{\beta_1 - \beta_n}$$

$$w^*_i = \frac{\beta_i - \beta_n}{\beta_1 - \beta_n}(w^*_1 - w^*_n) + w^*_n$$

$$w^*_i = \left(\frac{w^*_1 - w^*_n}{\beta_1 - \beta_n}\right)\beta_i + \frac{-\beta_n w^*_1 + \beta_1 w^*_n}{\beta_1 - \beta_n}$$

If $u(w) = \sqrt{w}$, $R = \frac{P}{3}$. Hence,

$$\frac{u^*_i - u^*_{i+k}}{u^*_j - u^*_{j+l}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

$$u^*_i = \left(\frac{u^*_1 - u^*_n}{\beta_1 - \beta_n}\right)\beta_i + \frac{-\beta_n u^*_1 + u_1 w^*_n}{\beta_1 - \beta_n}$$

■

Proof of Lemma 2. In a j tournament, the effort exerted by the players is given by

$$\begin{aligned} c'(e^j) &= \sum_i \beta_i u(w_i) \\ &= \left(\sum_{i=1}^{j} \beta_i\right) u_1 + \left(\sum_{i=j+1}^{n} \beta_i\right) u_2 \end{aligned}$$

Since $\sum_{i=1}^{n} \beta_i = 0$, $\sum_{i=j+1}^{n} \beta_i = -\sum_{i=1}^{j} \beta_i$. Thus,

$$c'(e^j) = \left(\sum_{i=1}^{j} \beta_i \right) (u_1 - u_2)$$

Now, let us consider the effort exerted in an $n - j$ tournament.

$$\begin{aligned} c'(e^{n-j}) &= \left(\sum_{i=1}^{n-j} \beta_i \right) u_1 + \left(\sum_{i=n-j+1}^{n} \beta_i \right) u_2 \\ &= -\left(\sum_{i=n-j+1}^{n} \beta_i \right) u_1 + \left(\sum_{i=n-j+1}^{n} \beta_i \right) u_2 \end{aligned}$$

When F is a symmetric distribution, we know that $\beta_i = -\beta_{n+1-i}$. Thus, for F symmetric, $\sum_{i=n-j+1}^{n} \beta_i = -\sum_{i=1}^{j} \beta_i$ and

$$c'(e^{n-j}) = \left(\sum_{i=1}^{j} \beta_i \right) (u_1 - u_2)$$

This proves the lemma. ■

Proof of Proposition 6. We will begin by considering the case where $R \geq \frac{P}{3}$. We will compare a j tournament ($j \leq \frac{n}{2}$) and an $n - j$ tournament that both meet the IR constraint and lead to the same exertion of effort, e, by players in the IC constraint. We will show that the sum of prizes paid by the principal in the j tournament exceeds the sum of prizes paid by the principal in the $n - j$ tournament. Given this result, we know that we can obtain the same effort with an $n - j$ tournament as a j tournament while meeting the IR constraint and paying out less in prizes. This shows that the optimal j tournament is dominated by the optimal $n - j$ tournament.

Following this argument, we will now consider a j tournament and an $n - j$ tournament that both meet the IR constraint and lead to the same effort exertion. Let w_1 and w_2 denote the prizes paid in the j tournament and let $u_i = u(w_i)$. Similarly, let $\tilde{w}_1$ and $\tilde{w}_2$ denote the prizes paid in the $n - j$ tournament and let $\tilde{u}_i = u(\tilde{w}_i)$. Further, let $\alpha = \frac{j}{n}$. The IR constraints for the j and $n - j$ tournaments imply that

$$\begin{aligned} \alpha u_1 + (1 - \alpha) u_2 &= \bar{u} \\ (1 - \alpha)\tilde{u}_1 + \alpha \tilde{u}_2 &= \bar{u} \end{aligned}$$

where

$$\bar{u} = \bar{U} + c(e)$$

Lemma 1 tells us that effort is the same in the j and $n - j$ tournaments when $u_1 - u_2 = \tilde{u}_1 - \tilde{u}_2$. These three equations tell us that

$$\begin{aligned} \tilde{u}_1 &= \frac{\alpha}{1 - \alpha} u_1 + \frac{1 - 2\alpha}{1 - \alpha} \bar{u} \\ \tilde{u}_2 &= 2\bar{u} - u_1 \\ u_2 &= \frac{-\alpha}{1 - \alpha} u_1 + \frac{1}{1 - \alpha} \bar{u} \end{aligned}$$

Let W denote the sum of prizes in the j tournament and $\tilde{W}$ denote the sum of prizes in the $n - j$ tournament. Also, let $h = u^{-1}$. Then

$$\begin{aligned}W &= \alpha w_1 + (1-\alpha)w_2 = \alpha h(u_1) + (1-\alpha)h(\frac{-\alpha}{1-\alpha}u_1 + \frac{1}{1-\alpha}\bar{u}) \\ \tilde{W} &= \alpha \tilde{w}_2 + (1-\alpha)\tilde{w}_1 = \alpha h(2\bar{u} - u_1) + (1-\alpha)h(\frac{\alpha}{1-\alpha}u_1 + \frac{1-2\alpha}{1-\alpha}\bar{u})\end{aligned}$$

Let $g(x) = \alpha h(x) + (1-\alpha)h(\frac{\bar{u}-\alpha x}{1-\alpha})$ and $\Delta = u_1 - \bar{u} \geq 0$. We need to show that, for $\alpha \leq \frac{1}{2}$, $W \geq \tilde{W}$, or

$$g(\bar{u}+\Delta) - g(\bar{u}-\Delta) \geq 0 \qquad (*)$$

We see that

$$g'(x) = \alpha(h'(x) - h'(\frac{\bar{u}-\alpha x}{1-\alpha}))$$

$h''(y) = \frac{-u''(h(y))}{[u'(h(y))]^3}$. Since u is concave, $h'' \geq 0$. Observe that $g'(x) \geq 0$ for $x \geq \bar{u}$ and $g'(x) \leq 0$ for $x \leq \bar{u}$ since $h'' \geq 0$. Let $\varphi(\Delta) \equiv g(\bar{u}+\Delta) - g(\bar{u}-\Delta)$. A sufficient condition for (*) is that: $\varphi'(\Delta) \geq 0 \ \forall \Delta \geq 0$ since $\varphi(0) = 0$. We see that

$$\begin{aligned}\varphi'(\Delta) &= g'(\bar{u}+\Delta) + g'(\bar{u}-\Delta) \\ &= \alpha(h'(\bar{u}+\Delta) - h'(\bar{u} - \frac{\alpha}{1-\alpha}\Delta)) + \alpha(h'(\bar{u}-\Delta) - h'(\bar{u} + \frac{\alpha}{1-\alpha}\Delta)) \\ &= \alpha(h'(\bar{u}+\Delta) - h'(\bar{u} + \frac{\alpha}{1-\alpha}\Delta)) - \alpha(h'(\bar{u} - \frac{\alpha}{1-\alpha}\Delta) - h'(\bar{u}-\Delta))\end{aligned}$$

Let $\omega(\theta, x, y) = \alpha\left[(h'(x+\theta) - h'(x)) - (h'(y+\theta) - h'(y))\right]$. Then,

$$\varphi'(\Delta) = \omega(\frac{1-2\alpha}{1-\alpha}, \bar{u} + \frac{\alpha}{1-\alpha}\Delta, \bar{u}-\Delta)$$

Observe that $\frac{1-2\alpha}{1-\alpha} \geq 0$ since $\alpha \leq \frac{1}{2}$ and $\bar{u} + \frac{\alpha}{1-\alpha}\Delta \geq \bar{u} - \Delta$. Since, $\omega(0, x, y) = 0$, it is sufficient to show that $\frac{\partial \omega}{\partial \theta}(\theta, x, y) \geq 0$ when $x \geq y$.

$$\frac{\partial \omega}{\partial \theta}(\theta, x, y) = \alpha(h''(x+\theta) - h''(y+\theta))$$

A sufficient condition therefore for $\frac{\partial \omega}{\partial \theta}(\theta, x, y) \geq 0$ is $h''' \geq 0$.

$$\begin{aligned}h'''(y) &= \frac{3u'\left(u''\right)^2 - \left(u'\right)^2 u'''}{(u')^6} \\ &= \frac{-3u''}{(u')^4}\left(R - \frac{P}{3}\right)\end{aligned}$$

Thus, $h''' \geq 0$ when $R \geq \frac{P}{3}$. This proves that $W \geq \tilde{W}$.

Let us now turn to the case where $R \leq \frac{P}{3}$. Again, we will consider a j tournament ($j \leq \frac{n}{2}$) and an $n-j$ tournament that both meet the IR constraint and lead to the same effort exertion. As before,

$$\begin{aligned}W &= \alpha w_1 + (1-\alpha)w_2 = \alpha h(u_1) + (1-\alpha)h(\frac{-\alpha}{1-\alpha}u_1 + \frac{1}{1-\alpha}\bar{u}) \\ \tilde{W} &= \alpha \tilde{w}_2 + (1-\alpha)\tilde{w}_1 = \alpha h(2\bar{u} - u_1) + (1-\alpha)h(\frac{\alpha}{1-\alpha}u_1 + \frac{1-2\alpha}{1-\alpha}\bar{u})\end{aligned}$$

We need to show that, for $\alpha \leq \frac{1}{2}$, $W \leq \tilde{W}$, or

$$g(\bar{u}+\Delta) - g(\bar{u}-\Delta) \leq 0 \tag{**}$$

Recall that

$$g'(x) = \alpha(h'(x) - h'(\frac{\bar{u}-\alpha x}{1-\alpha}))$$

$h''(y) = \frac{-u''(h(y))}{[u'(h(y))]^3}$. Since u is concave, $h'' \geq 0$. As before, $g'(x) \geq 0$ for $x \geq \bar{u}$ and $g'(x) \leq 0$ for $x \leq \bar{u}$ since $h'' \geq 0$. Let $\varphi(\Delta) \equiv g(\bar{u}+\Delta) - g(\bar{u}-\Delta)$. A sufficient condition for (**) is that: $\varphi'(\Delta) \leq 0\ \forall \Delta \geq 0$ since $\varphi(0) = 0$. We see that

$$\begin{aligned}
\varphi'(\Delta) &= g'(\bar{u}+\Delta) + g'(\bar{u}-\Delta) \\
&= \alpha(h'(\bar{u}+\Delta) - h'(\bar{u} - \frac{\alpha}{1-\alpha}\Delta)) + \alpha(h'(\bar{u}-\Delta) - h'(\bar{u} + \frac{\alpha}{1-\alpha}\Delta)) \\
&= \alpha(h'(\bar{u}+\Delta) - h'(\bar{u} + \frac{\alpha}{1-\alpha}\Delta)) - \alpha(h'(\bar{u} - \frac{\alpha}{1-\alpha}\Delta) - h'(\bar{u}-\Delta))
\end{aligned}$$

Again

$$\varphi'(\Delta) = \omega(\frac{1-2\alpha}{1-\alpha}, \bar{u} + \frac{\alpha}{1-\alpha}\Delta, \bar{u}-\Delta)$$

$\frac{1-2\alpha}{1-\alpha} \geq 0$ since $\alpha \leq \frac{1}{2}$ and $\bar{u} + \frac{\alpha}{1-\alpha}\Delta \geq \bar{u} - \Delta$. Since $\omega(0,x,y) = 0$, it is sufficient to show that $\frac{\partial\omega}{\partial\theta}(\theta,x,y) \leq 0$ when $x \geq y$.

$$\frac{\partial\omega}{\partial\theta}(\theta,x,y) = \alpha(h''(x+\theta) - h''(y+\theta))$$

Therefore, a sufficient condition for $\frac{\partial\omega}{\partial\theta}(\theta,x,y) \leq 0$ is $h''' \leq 0$. $R \leq \frac{P}{3}$ implies $h''' \leq 0$, which proves that $W \leq \tilde{W}$. ■

Proof of Proposition 7. We will begin with the case where $R \geq \frac{P}{3}$. Let us consider a j tournament and a j' tournament with $j' > j \geq n/2$. Let $\alpha = \frac{j}{n}$ and $\alpha' = \frac{j'}{n}$. We will compare j and j' tournaments that lead to the same level of effort exertion, e, and consider the amounts paid out in prizes by the principal. Let w_1 and w_2 denote the prizes paid in the j tournament and $\tilde{w}_1$ and $\tilde{w}_2$ denote the prizes paid in the j' tournament. Let $u_i = u(w_i)$, $\tilde{u}_i = u(\tilde{w}_i)$ and let W and $\tilde{W}$ denote the sum of prizes in the j and j' tournaments respectively. Before we proceed, we need to define two functions:

$$\begin{aligned}
\beta(x) &= \beta_{\lceil nx \rceil} \\
\gamma(x) &= n\int_0^x \beta(x)dx
\end{aligned}$$

We see that $\gamma(\frac{j}{n}) = \sum_{i=1}^{j} \beta_i$. Thus, the incentive compatibility constraints for the j and j' tournaments can be written as

$$\begin{aligned}
c'(e) &= \gamma(\alpha)(u(w_1) - u(w_2)) \\
c'(e) &= \gamma(\alpha')(u(\tilde{w}_1) - u(\tilde{w}_2))
\end{aligned}$$

Individual rationality implies that

$$\begin{aligned}
\alpha u_1 + (1-\alpha)u_2 &= \bar{u} \\
\alpha'\tilde{u}_1 + (1-\alpha')\tilde{u}_2 &= \bar{u}
\end{aligned}$$

where

$$\bar{u} = \bar{U} + c(e)$$

Combining these four constraints, we can solve for W and $\tilde{W}$ in terms of u_1. Let us define a few functions:

$$\begin{aligned}
\Phi(\alpha') &= \frac{\gamma(\alpha') - \gamma(\alpha)}{\gamma(\alpha')} + \alpha' \frac{\gamma(\alpha)}{\gamma(\alpha')} \\
h &= u^{-1} \\
g(x, \alpha') &= \beta h(\alpha') + (1 - \alpha') h(\frac{\bar{u} - \alpha' x}{1 - \alpha'}) \\
\psi(\alpha') &= g(\frac{1 - \Phi(\alpha')}{1 - \alpha} u_1 + \frac{\Phi(\alpha') - \alpha}{1 - \alpha} \bar{u}, \beta)
\end{aligned}$$

Then, we find that W and $\tilde{W}$ can be expressed as follows:

$$\begin{aligned}
\tilde{W} &= \psi(\alpha') \\
W &= \psi(\alpha)
\end{aligned}$$

Let us consider $\psi'(x)$. If we find that $\psi'(x) \leq 0$ for $x \in [\alpha, \alpha']$, then it follows that $\tilde{W} \leq W$. This implies that the j' tournament dominates the j tournament.

$$\begin{aligned}
\psi'(x) &= (\bar{u} - u_1)\, \Phi'(x) \left(\frac{1}{1 - x}\right) \left(x h'(u_1) + (\frac{1}{\Phi'(x)} - x) h'(\frac{\bar{u} - x u_1}{1 - x})\right) \\
&\quad + \left(h(u_1) - h(\frac{\bar{u} - x u_1}{1 - x})\right)
\end{aligned}$$

Let us define

$$\begin{aligned}
\Gamma(u) &= (\bar{u} - u)\, \Phi'(x) \left(\frac{1}{1 - x}\right) \left(x h'(u) + (\frac{1}{\Phi'(x)} - x) h'(\frac{\bar{u} - x u}{1 - x})\right) \\
&\quad + \left(h(u) - h(\frac{\bar{u} - x u}{1 - x})\right)
\end{aligned}$$

We see that $\Gamma(\bar{u}) = 0$. Since $u_1 > \bar{u}$, $\psi'(x) = \Gamma(u_1) \leq 0$ if $\Gamma'(u) \leq 0$ for $u > \bar{u}$.

$$\begin{aligned}
\Gamma'(u) &= \left(\frac{\bar{u} - u}{1 - x}\right) x \left(h''(u) - \frac{\frac{1}{\Phi'(x)} - x}{1 - x} h''(\frac{\bar{u} - x u}{1 - x})\right) \\
&\quad + \left(\frac{1 - x(1 + \Phi'(x))}{1 - x}\right) \left(h'(u) - h'(\frac{\bar{u} - x u}{1 - x})\right)
\end{aligned}$$

Suppose it were the case that $\Phi'(x) = 1$. Then,

$$\begin{aligned}
\Gamma'(u) &= \left(\frac{\bar{u} - u}{1 - x}\right) x \left(h''(u) - h''(\frac{\bar{u} - x u}{1 - x})\right) \\
&\quad + \left(\frac{1 - 2x}{1 - x}\right) \left(h'(u) - h'(\frac{\bar{u} - x u}{1 - x})\right)
\end{aligned}$$

Recall that we are assuming $1 > x \geq \frac{1}{2}$ and $u > \bar{u}$. Since $R \geq \frac{P}{3}$, it follows that $h'', h''' \geq 0$ (to see the argument, see the proof of Proposition 6). It therefore follows that the above expression is less than zero. Thus, if $\Phi'(x) = 1$, $\psi'(x) < 0$. From the definition of Φ, it follows that

$$\Phi'(x) = 1 + \frac{\gamma'(x)}{\gamma(x)}(1-x)$$

Since, $\gamma(x) = n\int_0^x \beta(x)dx$, $\gamma'(x) = n\beta(x) = n\beta_{\lceil nx \rceil}$. Thus,

$$\Phi'(x) = 1 + \frac{n\beta_{\lceil nx \rceil}}{\gamma(x)}(1-x)$$

Suppose that F is a symmetric and uniform distribution. It follows from Proposition 1 that $\beta_i = 0$ for $1 < i < n$. This implies that $\Phi'(x) = 1$ for $x \in [\frac{1}{n}, \frac{n-1}{n})$. Hence, when F is a symmetric uniform distribution, the j' tournament dominates the j tournament where $j' > j \geq n/2$. If follows from this and Corollary 3 that, for F a symmetric uniform distribution, the optimal j tournament is the strict loser prize tournament.

Now, we will consider the case where $R \geq \frac{P}{3}$. Let us consider a j tournament and a j' tournament with $j' < j \leq n/2$. Let $\alpha = \frac{j}{n}$ and $\alpha' = \frac{j'}{n}$. As before, we will compare j and j' tournaments that lead to the same level of effort exertion, e, and consider the amounts paid out in prizes by the principal. Let w_1 and w_2 denote the prizes paid in the j tournament and $\tilde{w}_1$ and $\tilde{w}_2$ denote the prizes paid in the j' tournament. Let $u_i = u(w_i)$, $\tilde{u}_i = u(\tilde{w}_i)$ and let W and $\tilde{W}$ denote the sum of prizes in the j and j' tournaments respectively. We will show that, when F is uniform, $\tilde{W} \leq W$.

Again, we find that W and $\tilde{W}$ can be expressed as:

$$\begin{aligned} \tilde{W} &= \psi(\alpha') \\ W &= \psi(\alpha) \end{aligned}$$

If we find that $\psi'(x) \geq 0$ for $x \in [\alpha', \alpha]$, then it follows that $\tilde{W} \leq W$.

$$\begin{aligned} \psi'(x) &= (\bar{u} - u_1)\,\Phi'(x)\left(\frac{1}{1-x}\right)\left(xh'(u_1) + (\frac{1}{\Phi'(x)} - x)h'(\frac{\bar{u} - xu_1}{1-x})\right) \\ &\quad + \left(h(u_1) - h(\frac{\bar{u} - xu_1}{1-x})\right) \end{aligned}$$

Again, let

$$\begin{aligned} \Gamma(u) &= (\bar{u} - u)\,\Phi'(x)\left(\frac{1}{1-x}\right)\left(xh'(u) + (\frac{1}{\Phi'(x)} - x)h'(\frac{\bar{u} - xu}{1-x})\right) \\ &\quad + \left(h(u) - h(\frac{\bar{u} - xu}{1-x})\right) \end{aligned}$$

We see that $\Gamma(\bar{u}) = 0$. Since $u_1 > \bar{u}$, $\psi'(x) = \Gamma(u_1) \geq 0$ if $\Gamma'(u) \geq 0$ for $u > \bar{u}$.

$$\begin{aligned} \Gamma'(u) &= \left(\frac{\bar{u} - u}{1-x}\right) x \left(h''(u) - \frac{\frac{1}{\Phi'(x)} - x}{1-x} h''(\frac{\bar{u} - xu}{1-x})\right) \\ &\quad + \left(\frac{1 - x(1 + \Phi'(x))}{1-x}\right)\left(h'(u) - h'(\frac{\bar{u} - xu}{1-x})\right) \end{aligned}$$

As shown above, when F is uniform, $\Phi'(x) = 1$. Thus,

$$\begin{aligned}\Gamma'(u) &= \left(\frac{\bar{u}-u}{1-x}\right) x \left(h''(u) - h''(\frac{\bar{u}-xu}{1-x})\right) \\ &\quad + \left(\frac{1-2x}{1-x}\right)\left(h'(u) - h'(\frac{\bar{u}-xu}{1-x})\right)\end{aligned}$$

Recall that we are assuming $0 < x \leq \frac{1}{2}$ and $u > \bar{u}$. Since $R \leq \frac{P}{3}$, $h'' \geq 0$ and $h''' \leq 0$ (to see the argument, see the proof of Proposition 6). It therefore follows that the above expression is greater than zero. Hence, $\tilde{W} \leq W$. ■

Proof of Lemma 3. The principal's Lagrangian can be modified slightly to take into account the limited liability constraint:

$$\mathcal{L} = \left((c')^{-1}\left(\sum_i \beta_i u_i\right) - \frac{1}{n}\sum_i u^{-1}(u_i)\right) - \lambda\left(\bar{U} - \frac{1}{n}\sum_i u_i + c\left((c')^{-1}\left(\sum_i \beta_i u_i\right)\right)\right) - \sum_i \gamma_i(\bar{u} - u_i)$$

where $\bar{u} = u^{-1}(\bar{w})$.

Let $h(x) = (c')^{-1}(x)$, $v(x) = u'(x)$, and $v_i = u'(w_i) = u'(u^{-1}(u_i))$.

The first order condition for u_i is as follows:

$$\begin{aligned}\left(\beta_i h'\left(\sum_i \beta_i u_i\right) - \frac{1}{nu'(u^{-1}(u_i))}\right) + \lambda\left(\frac{1}{n} - c'\left(h\left(\sum_i \beta_i u_i\right)\right) h'\left(\sum_i \beta_i u_i\right)\right) + \gamma_i u_i &= 0 \\ \beta_i n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right) + \lambda &= \frac{1}{v_i}\end{aligned}$$

By assumption, $w_i^* > \bar{w}$. This implies that $u_i^* > \bar{u}$, which means that $\gamma_i = 0$.

Therefore

$$\begin{aligned}\left(\beta_i h'\left(\sum_i \beta_i u_i\right) - \frac{1}{nu'(u^{-1}(u_i))}\right) + \lambda\left(\frac{1}{n} - c'\left(h\left(\sum_i \beta_i u_i\right)\right) h'\left(\sum_i \beta_i u_i\right)\right) &= 0 \\ \beta_i n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right) + \lambda &= \frac{1}{v_i}\end{aligned}$$

Since $w_k^* > \bar{w}$,

$$\frac{1}{v_i} - \frac{1}{v_{i+k}} = (\beta_i - \beta_{i+k}) n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right)$$

Similarly, since $w_j^*, w_k^* > \bar{w}$,

$$\frac{1}{v_j} - \frac{1}{v_{j+l}} = \frac{1}{v_i} - \frac{1}{v_{i+k}} = (\beta_j - \beta_{j+l}) n h'\left(\sum_i \beta_i u_i\right)\left(1 - \lambda c'\left(h\left(\sum_i \beta_i u_i\right)\right)\right)$$

Therefore,

$$\frac{\frac{1}{v_i} - \frac{1}{v_{i+k}}}{\frac{1}{v_j} - \frac{1}{v_{j+l}}} = \frac{\beta_i - \beta_{i+k}}{\beta_j - \beta_{j+l}}$$

■

Proof of Proposition 8. Suppose that $(\theta - E(\theta))e^* + \varepsilon$ is distributed according to Q (where ε is distributed according to F and $\theta e^* - E(\theta)$ is distributed according to H, and θ and ε are independent). Then,

$$\begin{aligned} Q(x) &= \Pr((\theta - E(\theta))e^* + \varepsilon \leq x) \\ &= E(F(x - (\theta - E(\theta))e^*)) \\ &= \int_{\mathbb{R}} F(x - ye^*)g(y)dy \end{aligned}$$

Now suppose that F and H are symmetric. We will show that Q will also be symmetric:

$$\begin{aligned} Q(-x) &= \int_{\mathbb{R}} F(-x - ye^*)g(y)dy \\ &= -\int_{\mathbb{R}} F(-x + ye^*)g(-y)d(-y) \\ &= \int_{\mathbb{R}} (1 - F(x - ye^*))g(y)dy \\ &= 1 - \int_{\mathbb{R}} F(x - ye^*)g(y)dy \\ &= 1 - Q(x) \end{aligned}$$

Hence Q is symmetric when F and H are symmetric since $Q(-x) = 1 - Q(x)$. Now let us turn to the agents' problem. Since we are restricting attention to symmetric equilibria, an agent maximizes

$$\sum_{i=1}^{n} \varphi_i(e, e^*)u(w_i) - c(e)$$

where $\varphi_i(e, e^*) = \Pr(i\text{th place}|e, e^*)$. Since agents do not know their types at the point when effort is chosen, φ_i does not depend upon θ. The first-order condition for the agents is given by

$$c'(e) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial e} \varphi_i(e, e^*) \right) u(w_i)$$

Since we are considering symmetric equilibria, the solution for e is $e = e^*$. Hence,

$$\begin{aligned} c'(e^*) &= \sum_{i=1}^{n} \beta_i u(w_i) \\ \text{where } \beta_i &= \left. \frac{\partial}{\partial e} \varphi_i(e, e^*) \right|_{e=e^*} \end{aligned}$$

Let us now derive a formula for β_i.

$$\begin{aligned} \varphi_i(e, e^*) &= \binom{n-1}{i-1} \Pr(\theta e - E(\theta)e^* + \varepsilon \geq (\theta' - E(\theta))e^* + \varepsilon')^{n-i} \Pr(\theta e - E(\theta)e^* + \varepsilon \\ &\leq (\theta' - E(\theta))e^* + \varepsilon')^{i-1} \\ &= \binom{n-1}{i-1} \int_{\mathbb{R}} \int_{\mathbb{R}} (Q(ye - E(\theta)e^* + x)^{n-i} (1 - Q(ye - E(\theta)e^* + x))^{i-1} f(x)h(y)dxdy \end{aligned}$$

$$\begin{aligned}\beta_i &= \left.\frac{\partial}{\partial e}\varphi_i(e,e^*)\right|_{e=e^*} \\ &= \binom{n-1}{i-1}\int_{\mathbb{R}}\int_{\mathbb{R}}[(n-i)\,(Q((y-E(\theta))e^*+x)^{n-i-1}\,(1-Q((y-E(\theta))e^*+x))^{i-1} \\ &\quad -(i-1)\,(Q((y-E(\theta))e^*+x))^{n-i}\,(1-Q((y-E(\theta))e^*+x))^{i-2}]f(x)g(y)q((y-E(\theta))e^*+x)dxdy\end{aligned}$$

Now suppose that F and H are symmetric. From above, we know that this also implies that Q is symmetric.

$$\begin{aligned}\beta_{n-i+1} &= \binom{n-1}{n-i}\int_{\mathbb{R}}\int_{\mathbb{R}}[(i-1)\,(Q((y-E(\theta))e^*+x)^{i-2}\,(1-Q((y-E(\theta))e^*+x))^{n-i} \\ &\quad -(n-i)\,(Q((y-E(\theta))e^*+x))^{i-1}\,(1-Q((y-E(\theta))e^*+x))^{n-i-1}] \\ &\quad \cdot f(x)g(y)q((y-E(\theta))e^*+x)dxdy \\ &= -\binom{n-1}{i-1}\int_{\mathbb{R}}\int_{\mathbb{R}}[(n-i)\left(Q(-(y'-E(\theta))e^*-x'\right)^{i-1}\left(1-Q(-(y'-E(\theta))e^*-x')\right)^{n-i-1} \\ &\quad -(i-1)\left(Q(-(y'-E(\theta))e^*-x')\right)^{i-2}\left(1-Q(-(y'-E(\theta))e^*-x')\right)^{n-i}] \\ &\quad \cdot f(x)g(y)q((y'-E(\theta))e^*+x')d(-x')d(2E(\theta)-y') \\ &= -\binom{n-1}{i-1}\int_{\mathbb{R}}\int_{\mathbb{R}}[(n-i)\left(Q((y'-E(\theta))e^*+x'\right)^{n-i-1}\left(1-Q((y'-E(\theta))e^*+x')\right)^{i-1} \\ &\quad -(i-1)\left(Q((y'-E(\theta))e^*+x')\right)^{n-i}\left(1-Q((y'-E(\theta))e^*+x')\right)^{i-2}] \\ &\quad \cdot f(x')g(y')q((y'-E(\theta))e^*+x')dxdy \\ &= -\beta_i\end{aligned}$$

Therefore, when F and H are symmetric, $\beta_i = -\beta_{n-i+1}$. ■

Proof of Proposition 9. Consider a θ-Additive strict loser-prize tournament and a θ-Additive strict winner-prize tournament, both of which pay out prizes w_1 and w_2. Let $u_i = u(w_i)$. We will assume that F is symmetric and that $c(e) = \frac{e^2}{2}$. Let us denote the effort of the high and low types in the winner-prize case by e_θ^W and e_0^W respectively and let us denote the efforts in the loser-prize case by e_θ^L and e_0^L. Let us denote the sum of efforts by e^W and e^L in the winner- and loser-prize cases respectively We will now proceed to show that $e^W \le e^L$. First consider effort in the winner-prize tournament:

$$\Pr(win|\theta,e) = E_{\varepsilon_i}\left[F(e-e_\theta^W+\varepsilon_i)^{n/2-1}F(e-e_0^W+\theta+\varepsilon_i)^{n/2}\right]$$

$$e_\theta^W = \arg\max_e \Pr(win|\theta,e)(u_1-u_2) - \frac{e^2}{2}$$

$$e_\theta^W = E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(\Delta^W+\theta+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(\Delta^W+\theta+\varepsilon_i)^{n/2-1}f(\Delta^W+\theta+\varepsilon_i)\end{array}\right](u_1-u_2)$$

$$\text{where } \Delta^W = e_\theta^W - e_0^W$$

$$e_\theta^W = \Psi(\Delta^W+\theta)(u_1-u_2)$$

$$\text{where } \Psi(x) = E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(x+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(x+\varepsilon_i)^{n/2-1}f(x+\varepsilon_i)\end{array}\right]$$

$$\Pr(win|0,e) = E_{\varepsilon_i}\left[F(e - e_\theta^W - \theta + \varepsilon_i)^{n/2} F(e - e_0^W + \varepsilon_i)^{n/2-1}\right]$$

$$e_0^W = \arg\max_e \Pr(win|0,e)(u_1 - u_2) - \frac{e^2}{2}$$

$$e_0^W = E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(\varepsilon_i)^{n/2-2} F(-\Delta^W - \theta + \varepsilon_i)^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} F(\varepsilon_i)^{n/2-1} F(-\Delta^W - \theta + \varepsilon_i)^{n/2-1} f(-\Delta^W - \theta + \varepsilon_i) \end{array}\right](u_1 - u_2)$$

$$= \Psi(-\Delta^W - \theta)(u_1 - u_2)$$

Now let us consider effort exertion in the loser-prize tournament:

$$\Pr(notlose|\theta,e) = 1 - \Pr(lose|\theta,e)$$

$$= 1 - E_{\varepsilon_i}\left[(1 - F(e - e_\theta^L + \varepsilon_i))^{n/2-1}(1 - F(e - e_0^L + \theta + \varepsilon_i))^{n/2}\right]$$

$$= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(\varepsilon_i)^{n/2-2} F(-\Delta^L - \theta + \varepsilon_i)^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} F(\varepsilon_i)^{n/2-1} F(-\Delta^L - \theta + \varepsilon_i)^{n/2-1} f(-\Delta^L - \theta + \varepsilon_i) \end{array}\right](u_1 - u_2)$$

$$= \Psi(-\Delta^L - \theta)(u_1 - u_2)$$

$$e_\theta^L = \arg\max_e \Pr(notlose|\theta,e)(u_1 - u_2) - \frac{e^2}{2}$$

$$e_\theta^L = E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) (1 - F(\varepsilon_i))^{n/2-2} (1 - F(\Delta^L + \theta + \varepsilon_i))^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} (1 - F(\varepsilon_i))^{n/2-1} (1 - F(\Delta^L + \theta + \varepsilon_i))^{n/2-1} f(\Delta^L + \theta + \varepsilon_i) \end{array}\right](u_1 - u_2)$$

$$= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(-\varepsilon_i)^{n/2-2} F(-\Delta^L - \theta - \varepsilon_i)^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} F(-\varepsilon_i)^{n/2-1} F(-\Delta^L - \theta - \varepsilon_i)^{n/2-1} f(\Delta^L + \theta + \varepsilon_i) \end{array}\right](u_1 - u_2)$$

$$= (u_1 - u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(-\varepsilon_i)^{n/2-2} F(-\Delta^L - \theta - \varepsilon_i)^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} F(-\varepsilon_i)^{n/2-1} F(-\Delta^L - \theta - \varepsilon_i)^{n/2-1} f(\Delta^L + \theta + \varepsilon_i) \end{array}\right] f(\varepsilon_i) d\varepsilon_i$$

$$= (u_1 - u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(-\varepsilon_i)^{n/2-2} F(-\Delta^L - \theta - \varepsilon_i)^{n/2} f(-\varepsilon_i) \\ +\frac{n}{2} F(-\varepsilon_i)^{n/2-1} F(-\Delta^L - \theta - \varepsilon_i)^{n/2-1} f(-\Delta^L - \theta - \varepsilon_i) \end{array}\right] f(-\varepsilon_i) d\varepsilon_i$$

Let $z = -\varepsilon_i$

$$e_\theta^L = (u_1 - u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(z)^{n/2-2} F(-\Delta^L - \theta + z)^{n/2} f(z) \\ +\frac{n}{2} F(z)^{n/2-1} F(-\Delta^L - \theta + z)^{n/2-1} f(-\Delta^L - \theta + z) \end{array}\right] f(z) dz$$

$$= (u_1 - u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(z)^{n/2-2} F(-\Delta^L - \theta + z)^{n/2} f(z) \\ +\frac{n}{2} F(z)^{n/2-1} F(-\Delta^L - \theta + z)^{n/2-1} f(-\Delta^L - \theta + z) \end{array}\right] f(z) dz$$

$$= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right) F(\varepsilon_i)^{n/2-2} F(-\Delta^L - \theta + \varepsilon_i)^{n/2} f(\varepsilon_i) \\ +\frac{n}{2} F(\varepsilon_i)^{n/2-1} F(-\Delta^L - \theta + \varepsilon_i)^{n/2-1} f(-\Delta^L - \theta + \varepsilon_i) \end{array}\right](u_1 - u_2)$$

$$e_\theta^L = \Psi(-\Delta^L - \theta)(u_1 - u_2)$$

$$\begin{aligned} \Pr(notlose|0,e) &= 1 - \Pr(lose|\theta,e) \\ &= 1 - E_{\varepsilon_i}\left[(1 - F(e - e_\theta^L - \theta + \varepsilon_i))^{n/2}(1 - F(e - e_0^L + \varepsilon_i))^{n/2-1}\right] \end{aligned}$$

$$e_0^L = \arg\max_e \Pr(notlose|0,e)(u_1-u_2) - \frac{e^2}{2}$$

$$\begin{aligned} e_0^L &= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)(1-F(\varepsilon_i))^{n/2-2}(1-F(-\Delta^L-\theta+\varepsilon_i))^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}(1-F(\varepsilon_i))^{n/2-1}(1-F(-\Delta^L-\theta+\varepsilon_i))^{n/2-1}f(-\Delta^L-\theta+\varepsilon_i) \end{array}\right] \\ &\cdot(u_1-u_2) \end{aligned}$$

$$= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(\Delta^L+\theta-\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(\Delta^L+\theta-\varepsilon_i)^{n/2-1}f(-\Delta^L-\theta+\varepsilon_i) \end{array}\right](u_1-u_2)$$

$$= (u_1-u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(\Delta^L+\theta-\varepsilon_i)^{n/2}f(-\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(\Delta^L+\theta-\varepsilon_i)^{n/2-1}f(\Delta^L+\theta-\varepsilon_i) \end{array}\right]f(\varepsilon_i)d\varepsilon_i$$

$$\text{Let } z = -\varepsilon_i$$

$$= (u_1-u_2)\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(z)^{n/2-2}F(\Delta^L+\theta+z)^{n/2}f(z) \\ +\frac{n}{2}F(z)^{n/2-1}F(\Delta^L+\theta+z)^{n/2-1}f(\Delta^L+\theta+z) \end{array}\right]f(z)dz$$

$$= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(\Delta^L+\theta+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(\Delta^L+\theta+\varepsilon_i)^{n/2-1}f(\Delta^L+\theta+\varepsilon_i) \end{array}\right](u_1-u_2)$$

$$= \Psi(\Delta^L+\theta)(u_1-u_2)$$

We see then that

$$\begin{aligned} e^W &= e_\theta^W + e_0^W \\ &= \left[\Psi(\Delta^W+\theta)+\Psi(-\Delta^W-\theta)\right](u_1-u_2) \\ &= h(\Delta^W+\theta)(u_1-u_2) \\ \text{where } h(x) &= \Psi(x)+\Psi(-x) \\ e^L &= e_\theta^L + e_0^L \\ &= \left[\Psi(\Delta^L+\theta)+\Psi(-\Delta^L-\theta)\right](u_1-u_2) \\ &= h(\Delta^L+\theta)(u_1-u_2) \end{aligned}$$

There are two things that we need to show to complete the proof. (1) $\Delta^W \geq \Delta^L$, and (2) $h(x)$ is decreasing. It immediately follows from (1) and (2) that $e^W \leq e^L$.

Proof of (1):

$$\begin{aligned} \Delta^W &= \left[\Psi(\Delta^W+\theta)-\Psi(-\Delta^W-\theta)\right](u_1-u_2) \\ \Delta^L &= \left[\Psi(-\Delta^L-\theta)-\Psi(\Delta^L+\theta)\right](u_1-u_2) \end{aligned}$$

Let $g(x) = [\Psi(x)-\Psi(-x)](u_1-u_2)$. Then $\Delta^W = g(\Delta^W+\theta)$ and $\Delta^L = -g(\Delta^L+\theta)$. Let us now find a simpler expression for g:

$$\Psi(x) = E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(x+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(x+\varepsilon_i)^{n/2-1}f(x+\varepsilon_i) \end{array}\right]$$

$$\begin{aligned} &= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(x+\varepsilon_i)^{n/2}f(\varepsilon_i)\right] \\ &\quad +E_{\varepsilon_i}\left[\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(x+\varepsilon_i)^{n/2-1}f(x+\varepsilon_i)\right] \end{aligned}$$

$$\begin{aligned}\Psi(-x) &= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-x+\varepsilon_i)^{n/2}f(\varepsilon_i)\right] \\ &\quad +E_{\varepsilon_i}\left[\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(-x+\varepsilon_i)^{n/2-1}f(-x+\varepsilon_i)\right]\end{aligned}$$

$$\begin{aligned}&= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-x+\varepsilon_i)^{n/2}f(\varepsilon_i)\right] \\ &\quad +\int_{\mathbb{R}}\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(-x+\varepsilon_i)^{n/2-1}f(-x+\varepsilon_i)f(\varepsilon_i)d\varepsilon_i\end{aligned}$$

Let $z = -x+\varepsilon_i$

$$\begin{aligned}&= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-x+\varepsilon_i)^{n/2}f(\varepsilon_i)\right] \\ &\quad +\int_{\mathbb{R}}\frac{n}{2}F(x+z)^{n/2-1}F(z)^{n/2-1}f(z)f(x+z)dz\end{aligned}$$

$$\begin{aligned}&= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-x+\varepsilon_i)^{n/2}f(\varepsilon_i)\right] \\ &\quad +E_{\varepsilon_i}\left[\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(x+\varepsilon_i)^{n/2-1}f(x+\varepsilon_i)\right]\end{aligned}$$

$$g(x) = \Psi(x) - \Psi(-x)$$

$$\begin{aligned}&= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}F(x+\varepsilon_i)^{\frac{n}{2}}f(\varepsilon_i)\right] \\ &\quad -E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(-x+\varepsilon_i)^{\frac{n}{2}}F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\right]\end{aligned}$$

$$= E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\left(F(x+\varepsilon_i)^{\frac{n}{2}} - F(-x+\varepsilon_i)^{\frac{n}{2}}\right)\right]$$

$$g'(x) = E_{\varepsilon_i}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\left(\begin{array}{c}F(x+\varepsilon_i)^{\frac{n}{2}-1}f(x+\varepsilon_i) \\ +F(-x+\varepsilon_i)^{\frac{n}{2}-1}f(-x+\varepsilon_i)\end{array}\right)\right] \geq 0$$

So, $g'(x) \geq 0$. Observe that

$$\lim_{x\longrightarrow\infty}|g(x)| = \lim_{x\longrightarrow\infty}\left|E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\left(\begin{array}{c}F(x+\varepsilon_i)^{\frac{n}{2}} \\ -F(-x+\varepsilon_i)^{\frac{n}{2}}\end{array}\right)\right]\right|$$

$$\begin{aligned}&= \left|E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\left(1-0\right)\right]\right| \\ &= \left|E_{\varepsilon_i}\left[\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{\frac{n}{2}-2}f(\varepsilon_i)\right]\right| < \infty\end{aligned}$$

Since this limit is finite, it follows that for x sufficiently large, $g(x+\theta) < x$. It is obvious that $g(0) = 0$. Since g is increasing, it follows that $g(\theta) \geq 0$. Thus, for $x = 0$, $g(x+\theta) \geq x$. By continuity, there is a value of x, $x \geq 0$, with $g(x+\theta) = x$. Hence, $\Delta^W \geq 0$. Now, suppose $\Delta^W < \Delta^L$. Δ^L solves $\Delta^L = -g(\Delta^L+\theta)$. Since $\Delta^W + \theta \geq 0$ and $g(0) = 0$, we know that $-g(\Delta^W+\theta) \leq 0$. Thus, $\Delta^W \geq -g(\Delta^W+\theta)$. Since g is increasing, $\Delta^W < \Delta^L$ implies that $\Delta^L > -g(\Delta^L+\theta)$, which is a contradiction. It therefore follows that, if a solution for Δ^L exists, $\Delta^W \geq \Delta^L$. For $x = -\theta$, $x < -g(x+\theta)$ and for $x = \Delta^W$, $x \geq -g(x+\theta)$. Thus, by continuity, there exists a solution for Δ^L

with $-\theta \leq \Delta^L \leq \Delta^W$. This proves (1).

Proof of (2): Since $\Delta^L = -g(\Delta^L + \theta)$, it follows that

$$\frac{d(\Delta^L + \theta)}{d\theta} = \frac{1}{1 + g'(\Delta^L + \theta)} \geq 0$$

Since, $\Delta^L(\theta) + \theta \leq \Delta^W(\theta) + \theta$, it follows that there exists $\theta' \geq \theta$ such that

$$\Delta^L(\theta') + \theta' = \Delta^W(\theta) + \theta$$

Since $e^W(\theta) = h(\Delta^W(\theta)+\theta)(u_1-u_2)$ and $e^L(\theta) = h(\Delta^L(\theta)+\theta)(u_1-u_2)$, we see that $e^W(\theta) = e^L(\theta')$ where $\theta' \geq \theta$. We will now argue that $e^L(\theta') \leq e^L(\theta)$ for $\theta' \geq \theta$. From this, it immediately follows that $e^W(\theta) \leq e^L(\theta)$, which proves the claim.

Consider $\theta' \geq \theta$. High ability types are more certain to perform well relative to low ability types than they were before. When all high types exert the same level of effort, as they will in equilibrium, their prospects do not change at all relative to one another. Therefore, for a given level of effort by the low types, e_0^L, high types choose to reduce their level of effort exertion. Low ability types, on the other hand, are more certain to perform poorly relative to high ability types than they were before. When all low ability types exert the same level of effort, as they will in equilibrium, their prospects do not change at all relative to one another. Therefore, low ability types also have less incentive to exert effort for a given e_θ^L than they did before. As a result, both types reduce their effort levels in equilibrium. Hence, $e^L(\theta') \leq e^L(\theta)$, which proves the proposition. ■

Proof of Proposition 10. Let us denote the effort of the high and low types in the winner-prize case by e_θ^W and e_0^W respectively and let us denote the efforts in the loser-prize case by e_θ^L and e_0^L. Let us denote the sum of efforts by e^W and e^L in the winner- and loser-prize cases respectively In the homogeneous case, holding $u_1 - u_2$ constant produced the same effort in the winner-prize and loser-prize tournaments, and this allowed us to conclude that the loser-prize tournament dominated. We will show that, holding $u_1 - u_2$ constant, as $\theta \to \infty$ $\frac{e^W}{e^L} \to \infty$. This suggests that, for θ sufficiently large, the winner-prize tournament produces higher profits than the loser-prize tournament. Hence, $\frac{e^W}{e^L} \to \infty$ is a sufficient condition for the result. First consider effort in the winner-prize tournament:

$$\begin{aligned}
\Pr(win|\theta, e) &= E_{\varepsilon_i}\left[F(e\theta - e_\theta^W\theta + \varepsilon_i)^{n/2-1}F(e\theta - e_0^W + \varepsilon_i)^{n/2}\right] \\
e_\theta^W &= \arg\max_e \Pr(win|\theta, e)(u_1 - u_2) - c(e) \\
c'(e_\theta^W) &= \theta E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(\Delta^W + \varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i\theta)^{n/2-1}F(\Delta^W + \varepsilon_i)^{n/2-1}f(\Delta^W + \varepsilon_i) \end{array}\right] \\
&\quad \cdot(u_1 - u_2) \\
\text{where } \Delta^W &= e_\theta^W\theta - e_0^W \\
c'(e_\theta^W) &= \theta\Psi(\Delta^W)(u_1 - u_2) \\
\text{where } \Psi(x) &= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(x + \varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(x + \varepsilon_i)^{n/2-1}f(x + \varepsilon_i) \end{array}\right]
\end{aligned}$$

$$
\begin{aligned}
\Pr(win|0,e) &= E_{\varepsilon_i}\left[F(e-e_\theta^W\theta+\varepsilon_i)^{n/2}F(e-e_0^W+\varepsilon_i)^{n/2-1}\right] \\
c'(e_0^W) &= \arg\max_e \Pr(win|0,e)(u_1-u_2)-c(e) \\
c'(e_0^W) &= E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-\Delta^W+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(-\Delta^W+\varepsilon_i)^{n/2-1}f(-\Delta^W+\varepsilon_i)\end{array}\right] \\
&\quad \cdot(u_1-u_2) \\
&= \Psi(-\Delta^W)(u_1-u_2)
\end{aligned}
$$

Now let us consider effort exertion in the loser-prize tournament:

$$
\begin{aligned}
\Pr(notlose|\theta,e) &= 1-\Pr(lose|\theta,e) \\
&= 1-E_{\varepsilon_i}\left[(1-F(e\theta-e_\theta^L\theta+\varepsilon_i))^{n/2-1}(1-F(e\theta-e_0^L+\varepsilon_i))^{n/2}\right] \\
e_\theta^L &= \arg\max_e \Pr(notlose|\theta,e)(u_1-u_2)-c(e)
\end{aligned}
$$

$$
\begin{aligned}
c'(e_\theta^L) &= \theta E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)(1-F(\varepsilon_i))^{n/2-2}(1-F(\Delta^L+\varepsilon_i))^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}(1-F(\varepsilon_i))^{n/2-1}(1-F(\Delta^L+\varepsilon_i))^{n/2-1}f(\Delta^L+\varepsilon_i)\end{array}\right] \\
&\quad \cdot(u_1-u_2)
\end{aligned}
$$

$$
= E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(-\Delta^L-\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(-\Delta^L-\varepsilon_i)^{n/2-1}f(\Delta^L+\varepsilon_i)\end{array}\right](u_1-u_2)
$$

$$
= (u_1-u_2)\int_{\mathbb{R}}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(-\Delta^L-\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(-\Delta^L-\varepsilon_i)^{n/2-1}f(\Delta^L+\varepsilon_i)\end{array}\right] \cdot f(\varepsilon_i)d\varepsilon_i
$$

$$
= (u_1-u_2)\theta\int_{\mathbb{R}}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(-\Delta^L-\varepsilon_i)^{n/2}f(-\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(-\Delta^L-\varepsilon_i)^{n/2-1}f(-\Delta^L-\varepsilon_i)\end{array}\right] \cdot f(-\varepsilon_i)d\varepsilon_i
$$

Let $z=-\varepsilon_i$

$$
= (u_1-u_2)\theta\int_{\mathbb{R}}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(z)^{n/2-2}F(-\Delta^L+z)^{n/2}f(z) \\ +\frac{n}{2}F(z)^{n/2-1}F(-\Delta^L+z)^{n/2-1}f(-\Delta^L+z)\end{array}\right]f(z)dz
$$

$$
= \theta E_{\varepsilon_i}\left[\begin{array}{c}\left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(-\Delta^L+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(-\Delta^L+\varepsilon_i)^{n/2-1}f(-\Delta^L+\varepsilon_i)\end{array}\right](u_1-u_2)
$$

$$
= \theta\Psi(-\Delta^L)(u_1-u_2)
$$

$$
\begin{aligned}
\Pr(notlose|0,e) &= 1-\Pr(lose|\theta,e) \\
&= 1-E_{\varepsilon_i}\left[(1-F(e-e_\theta^L\theta+\varepsilon_i))^{n/2}(1-F(e-e_0^L+\varepsilon_i))^{n/2-1}\right] \\
e_0^L &= \arg\max_e \Pr(notlose|0,e)(u_1-u_2)-c(e) \\
c'(e_0^L) &= E_{\varepsilon_i} \\
&\quad \cdot\left[\begin{array}{c} \left(\frac{n}{2}-1\right)(1-F(\varepsilon_i))^{n/2-2}(1-F(-\Delta^L+\varepsilon_i))^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}(1-F(\varepsilon_i))^{n/2-1}(1-F(-\Delta^L+\varepsilon_i))^{n/2-1}f(-\Delta^L+\varepsilon_i) \end{array}\right] \\
&\quad \cdot(u_1-u_2) \\
&= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(\Delta^L-\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(\Delta^L-\varepsilon_i)^{n/2-1}f(-\Delta^L+\varepsilon_i) \end{array}\right] \\
&\quad \cdot(u_1-u_2) \\
&= (u_1-u_2) \\
&\quad \cdot\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(-\varepsilon_i)^{n/2-2}F(\Delta^L-\varepsilon_i)^{n/2}f(-\varepsilon_i) \\ +\frac{n}{2}F(-\varepsilon_i)^{n/2-1}F(\Delta^L-\varepsilon_i)^{n/2-1}f(\Delta^L-\varepsilon_i) \end{array}\right]f(\varepsilon_i)d\varepsilon_i \\
\text{Let } z &= -\varepsilon_i \\
&= (u_1-u_2) \\
&\quad \cdot\int_{\mathbb{R}}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(z)^{n/2-2}F(\Delta^L+z)^{n/2}f(z) \\ +\frac{n}{2}F(z)^{n/2-1}F(\Delta^L+z)^{n/2-1}f(\Delta^L+z) \end{array}\right]f(z)dz \\
&= E_{\varepsilon_i}\left[\begin{array}{c} \left(\frac{n}{2}-1\right)F(\varepsilon_i)^{n/2-2}F(\Delta^L+\varepsilon_i)^{n/2}f(\varepsilon_i) \\ +\frac{n}{2}F(\varepsilon_i)^{n/2-1}F(\Delta^L+\varepsilon_i)^{n/2-1}f(\Delta^L+\varepsilon_i) \end{array}\right](u_1-u_2) \\
&= \Psi(\Delta^L)(u_1-u_2)
\end{aligned}
$$

Let $\gamma=(c')^{-1}$. Since $c(x)=dx^\alpha$, $\gamma(x)=\frac{x}{d\alpha}^{\frac{1}{\alpha-1}}$. Since $\alpha>1$, $\lim_{x\to\infty}\gamma(x)=\infty$. We see then that

$$
\begin{aligned}
e^W &= e_\theta^W\theta+e_0^W \\
&= \gamma(\Psi(\Delta^W)\theta^2(u_1-u_2))+\gamma(\Psi(-\Delta^W)(u_1-u_2)) \\
e^L &= e_\theta^L+e_0^L \\
&= \gamma(\Psi(\Delta^L)(u_1-u_2))+\gamma(\Psi(-\Delta^L)\theta^2(u_1-u_2))
\end{aligned}
$$

As $\theta\to\infty$, we see that $\frac{e^W}{e^L}\to\lim\limits_{\theta\to\infty}\frac{\gamma(\Psi(\Delta^W)\theta^2(u_1-u_2))}{\gamma(\Psi(-\Delta^L)\theta^2(u_1-u_2))}=\lim\limits_{\theta\to\infty}\left(\frac{\Psi(\Delta^W)}{\Psi(-\Delta^L)}\right)^{\frac{1}{\alpha-1}}$. We know that

$$
\begin{aligned}
\Delta^W &= \gamma(\Psi(\Delta^W)\theta^2(u_1-u_2))-\gamma(\Psi(-\Delta^W)(u_1-u_2)) \\
\Delta^L &= \gamma(\Psi(-\Delta^L)\theta^2(u_1-u_2))-\gamma(\Psi(\Delta^L)(u_1-u_2))
\end{aligned}
$$

It is easily shown that $\lim\limits_{x\to\infty}\Psi(x)=c>0$ and $\lim\limits_{x\to-\infty}\Psi(x)=0$ and that $\Psi(x)$ is finite for all x. It therefore follows that $\lim\limits_{\theta\to\infty}\Delta^W=\infty$ and $\lim\limits_{\theta\to\infty}\Delta^L=\infty$. From this, we conclude that $\lim\limits_{\theta\to\infty}\frac{\Psi(\Delta^W)}{\Psi(-\Delta^L)}=\infty$, and that $\frac{e^W}{e^L}\to\infty$ as $\theta\to\infty$.

We have shown, therefore, that for the same prizes w_1 and w_2 in the strict winner- and loser-prize tournaments, the ratio of induced efforts, $\frac{e^W}{e^L}$, goes to infinity as $\theta\to\infty$. It immediately follows that, for θ sufficiently large, the principal will prefer the optimal strict winner-prize tournament to the optimal strict loser-prize tournament. ■

Proof of Proposition 11. Let

$$w_n(q) = \begin{cases} \left(\frac{E(\varepsilon^n_{(i)})-(q-e^*_n)}{E(\varepsilon^n_{(i)})-E(\varepsilon^n_{(i+1)})}\right) w^n_{i+1} + \left(\frac{(q-e^*)-E(\varepsilon^n_{(i+1)})}{E(\varepsilon^n_{(i)})-E(\varepsilon^n_{(i+1)})}\right) w^n_i, & q-e^*_n \in [E(\varepsilon^n_{(i+1)}), E(\varepsilon^n_{(i)})] \\ w^n_1 + \left(\frac{w^n_1-w^n_2}{E(\varepsilon^n_{(1)})-E(\varepsilon^n_{(2)})}\right)\Big((q-e^*_n) - E(\varepsilon^n_{(1)})\Big), & q-e^*_n \geq E(\varepsilon^n_{(1)}) \\ w^n_n - \left(\frac{w^n_{n-1}-w^n_n}{E(\varepsilon^n_{(n-1)})-E(\varepsilon^n_{(n)})}\right)\Big(E(\varepsilon^n_{(n)}) - (q-e^*_n)\Big), & q-e^*_n \leq E(\varepsilon^n_{(n)}) \end{cases}$$

where w^n_i is the optimal prize to give for ith place, e^*_n is the induced effort level of the optimal tournament in the case where there are n agents, and $\varepsilon^n_{(i)}$ is the ith order statistic of the idiosyncratic noise.

Green and Stokey (1983) implies that:

$$\begin{aligned} \lim_{n\to\infty} w_n(q) &= w^*(q) \text{ for all } q \in \mathbb{R} \\ \lim_{n\to\infty} e^*_n &= e^* \end{aligned}$$

where $w^*(q)$ is the optimal individual contract and e^* is the effort level induced by $w^*(q)$.

$\{\beta^n_i\}^n_{i=1}$ decreasing in i implies that $\{w^n_i\}^n_{i=1}$ is decreasing in i, which means that $w_n(q)$ is increasing in q. Since $w_n(q)$ is increasing in q for all n, Green and Stokey (1983) implies that $w^*(q)$ is increasing in q.

F symmetric implies that: $E(\varepsilon^n_{(i)}) = E(\varepsilon^n_{(n-i+1)})$. Consider $\theta_1, \theta_2 \geq 0$. It is possible to write θ_1 and $\theta_1+\theta_2$ as: $\theta_1 = \alpha_1 E(\varepsilon_{(i+1)}) + (1-\alpha_1)E(\varepsilon_{(i)})$ and $\theta_1+\theta_2 = \alpha_2 E(\varepsilon_{(j+1)}) + (1-\alpha_2)E(\varepsilon_{(j)})$ where $0 \leq \alpha_1, \alpha_2 \leq 1$ and $j \leq i \leq \lceil n/2 \rceil$. The symmetry of F implies that: $-\theta_1 = \alpha_1 E(\varepsilon_{(n-i)}) + (1-\alpha_1)E(\varepsilon_{(n-i+1)})$ and $-(\theta_1+\theta_2) = \alpha_2 E(\varepsilon_{(n-j)}) + (1-\alpha_2)E(\varepsilon_{(n-j+1)})$. As a result:

$$\begin{aligned} w_n(e^*_n+\theta_1) &= \alpha_1 w^n_{i+1} + (1-\alpha_1)w^n_i \\ w_n(e^*_n+(\theta_1+\theta_2)) &= \alpha_2 w^n_{j+1} + (1-\alpha_2)w^n_j \\ w_n(e^*_n-\theta_1) &= \alpha_1 w^n_{n-i} + (1-\alpha_1)w^n_{n-i+1} \\ w_n(e^*_n-(\theta_1+\theta_2)) &= \alpha_2 w^n_{n-j} + (1-\alpha_2)w^n_{n-j+1} \end{aligned}$$

$$\begin{aligned} w_n(e^*_n+(\theta_1+\theta_2)) - w_n(e^*_n+\theta_1) &= \alpha_2 w^n_{j+1} + (1-\alpha_2)w^n_j - \alpha_1 w^n_{i+1} - (1-\alpha_1)w^n_i \\ &= \alpha_1(\alpha_2 w^n_{j+1} + (1-\alpha_2)w^n_j - w^n_{i+1}) \\ &\quad +(1-\alpha_1)(\alpha_2 w^n_{j+1} + (1-\alpha_2)w^n_j - w^n_i) \\ &= \alpha_1(\alpha_2(w^n_{j+1} - w^n_{i+1}) + (1-\alpha_2)(w^n_j - w^n_{i+1})) \\ &\quad +(1-\alpha_1)(\alpha_2(w^n_{j+1} - w^n_i) + (1-\alpha_2)w^n_j - w^n_i) \end{aligned}$$

Suppose that $R \geq \frac{P}{2}$. Since F is symmetric and $\{\beta_i\}$ is decreasing in i, Proposition 4 implies that, for $j \leq k \leq \lceil n/2 \rceil$

$$w^n_j - w^n_k \leq w^n_{n-k+1} - w^n_{n-j+1}$$

Case 1: $i = j$. Then, since $\theta_2 \geq 0$, it follows that $\alpha_2 \leq \alpha_1$.

$$\begin{aligned}
w_n(e_n^* + (\theta_1 + \theta_2)) - w_n(e_n^* + \theta_1) &= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{i+1}^n - (1-\alpha_1)w_i^n \\
&= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{j+1}^n - (1-\alpha_1)w_j^n \\
&= (\alpha_1 - \alpha_2)w_j^n - (\alpha_1 - \alpha_2)w_{j+1}^n \\
&\leq (\alpha_1 - \alpha_2)w_{n-j}^n - (\alpha_1 - \alpha_2)w_{n-j+1}^n \\
&= \alpha_1 w_{n-i}^n + (1-\alpha_1)w_{n-i+1}^n - \alpha_2 w_{n-j}^n - (1-\alpha_2)w_{n-j+1}^n \\
&= w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1 + \theta_2))
\end{aligned}$$

Case 2: $i \geq j + 1$.

$$\begin{aligned}
w_n(e_n^* + (\theta_1 + \theta_2)) - w_n(e_n^* + \theta_1) &= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{i+1}^n - (1-\alpha_1)w_i^n \\
&= \alpha_1(\alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - w_{i+1}^n) + (1-\alpha_1)(\alpha_2 w_{j+1}^n \\
&\quad +(1-\alpha_2)w_j^n - w_i^n) \\
&= \alpha_1(\alpha_2(w_{j+1}^n - w_{i+1}^n) + (1-\alpha_2)(w_j^n - w_{i+1}^n)) \\
&\quad +(1-\alpha_1)(\alpha_2(w_{j+1}^n - w_i^n) + (1-\alpha_2)(w_j^n - w_i^n)) \\
&\leq \alpha_1(\alpha_2(w_{n-i}^n - w_{n-j}^n) + (1-\alpha_2)(w_{n-i}^n - w_{n-j+1}^n)) \\
&\quad +(1-\alpha_1)(\alpha_2(w_{n-i+1}^n - w_{n-j}^n) + (1-\alpha_2)(w_{n-i+1}^n - w_{n-j+1}^n)) \\
&= \alpha_1 w_{n-i}^n + (1-\alpha_1)w_{n-i+1}^n - \alpha_2 w_{n-j}^n - (1-\alpha_2)w_{n-j+1}^n \\
&= w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1 + \theta_2))
\end{aligned}$$

Therefore, for $\theta_1, \theta_2 \geq 0$ and $R \geq \frac{P}{2}$,

$$w_n(e_n^* + (\theta_1 + \theta_2)) - w_n(e_n^* + \theta_1) \leq w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1 + \theta_2))$$

From Green and Stokey (1983), it follows that:

$$w^*(e^* + (\theta_1 + \theta_2)) - w^*(e^* + \theta_1) \leq w^*(e^* - \theta_1) - w^*(e^* - (\theta_1 + \theta_2))$$

Suppose that $\frac{P}{3} \leq R \leq \frac{P}{2}$. Since F is symmetric and $\{\beta_i\}$ is decreasing in i, Proposition 4 implies that, for $j \leq k \leq \lceil n/2 \rceil$

$$w_j^n - w_k^n \geq w_{n-k+1}^n - w_{n-j+1}^n$$

Case 1: $i = j$. Then, since $\theta_2 \geq 0$, it follows that $\alpha_2 \leq \alpha_1$.

$$\begin{aligned}
w_n(e_n^* + (\theta_1 + \theta_2)) - w_n(e_n^* + \theta_1) &= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{i+1}^n - (1-\alpha_1)w_i^n \\
&= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{j+1}^n - (1-\alpha_1)w_j^n \\
&= (\alpha_1 - \alpha_2)w_j^n - (\alpha_1 - \alpha_2)w_{j+1}^n \\
&\geq (\alpha_1 - \alpha_2)w_{n-j}^n - (\alpha_1 - \alpha_2)w_{n-j+1}^n \\
&= \alpha_1 w_{n-i}^n + (1-\alpha_1)w_{n-i+1}^n - \alpha_2 w_{n-j}^n - (1-\alpha_2)w_{n-j+1}^n \\
&= w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1 + \theta_2))
\end{aligned}$$

Case 2: $i \geq j+1$.

$$\begin{aligned} w_n(e_n^* + (\theta_1+\theta_2)) - w_n(e_n^*+\theta_1) &= \alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - \alpha_1 w_{i+1}^n - (1-\alpha_1)w_i^n \\ &= \alpha_1(\alpha_2 w_{j+1}^n + (1-\alpha_2)w_j^n - w_{i+1}^n) + (1-\alpha_1)(\alpha_2 w_{j+1}^n \\ &\quad +(1-\alpha_2)w_j^n - w_i^n) \\ &= \alpha_1(\alpha_2(w_{j+1}^n - w_{i+1}^n) + (1-\alpha_2)(w_j^n - w_{i+1}^n)) \\ &\quad +(1-\alpha_1)(\alpha_2(w_{j+1}^n - w_i^n) + (1-\alpha_2)(w_j^n - w_i^n)) \\ &\geq \alpha_1(\alpha_2(w_{n-i}^n - w_{n-j}^n) + (1-\alpha_2)(w_{n-i}^n - w_{n-j+1}^n)) \\ &\quad +(1-\alpha_1)(\alpha_2(w_{n-i+1}^n - w_{n-j}^n) + (1-\alpha_2)(w_{n-i+1}^n - w_{n-j+1}^n)) \\ &= \alpha_1 w_{n-i}^n + (1-\alpha_1)w_{n-i+1}^n - \alpha_2 w_{n-j}^n - (1-\alpha_2)w_{n-j+1}^n \\ &= w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1+\theta_2)) \end{aligned}$$

Therefore, for $\theta_1, \theta_2 \geq 0$ and $\frac{P}{3} \leq R \leq \frac{P}{2}$,

$$w_n(e_n^* + (\theta_1+\theta_2)) - w_n(e_n^*+\theta_1) \geq w_n(e_n^* - \theta_1) - w_n(e_n^* - (\theta_1+\theta_2))$$

From Green and Stokey (1983), it follows that:

$$w^*(e^* + (\theta_1+\theta_2)) - w^*(e^*+\theta_1) \geq w^*(e^* - \theta_1) - w^*(e^* - (\theta_1+\theta_2))$$

Consider the following function analogous to $w_n(q)$.

$$u_n(q) = \begin{cases} \left(\frac{E(\varepsilon_{(i)}^n)-(q-e_n^*)}{E(\varepsilon_{(i)}^n)-E(\varepsilon_{(i+1)}^n)}\right) u(w_{i+1}^n) + \left(\frac{(q-e^*)-E(\varepsilon_{(i+1)}^n)}{E(\varepsilon_{(i)}^n)-E(\varepsilon_{(i+1)}^n)}\right) u(w_i^n), & q-e_n^* \in [E(\varepsilon_{(i+1)}^n), E(\varepsilon_{(i)}^n)] \\ u(w_1^n) + \left(\frac{u(w_1^n)-u(w_2^n)}{E(\varepsilon_{(1)}^n)-E(\varepsilon_{(2)}^n)}\right)\left((q-e_n^*) - E(\varepsilon_{(1)}^n)\right), & q-e_n^* \geq E(\varepsilon_{(1)}^n) \\ u(w_n^n) - \left(\frac{u(w_{n-1}^n)-u(w_n^n)}{E(\varepsilon_{(n-1)}^n)-E(\varepsilon_{(n)}^n)}\right)\left(E(\varepsilon_{(n)}^n) - (q-e_n^*)\right), & q-e_n^* \leq E(\varepsilon_{(n)}^n) \end{cases}$$

Green and Stokey (1983) implies that $u_n(q)$ converges pointwise to $u(w^*(q))$. Since F is symmetric and $\{\beta_i\}$ is decreasing in i, Proposition 4 implies that, when $R \geq \frac{P}{3}$,

$$u(w_j^n) - u(w_k^n) \leq u(w_{n-k+1}^n) - u(w_{n-j+1}^n)$$

for $j \leq k \leq \lceil n/2 \rceil$. Following an identical logic to that given above, we find that, when $R \geq \frac{P}{3}$,

$$u(w_n(e_n^* + (\theta_1+\theta_2))) - u(w_n(e_n^*+\theta_1)) \leq u(w_n(e_n^* - \theta_1)) - u(w_n(e_n^* - (\theta_1+\theta_2)))$$

for $\theta_1, \theta_2 \geq 0$. From Green and Stokey (1983), it follows that:

$$u(w^*(e^* + (\theta_1+\theta_2))) - u(w^*(e^*+\theta_1)) \geq u(w^*(e^* - \theta_1)) - u(w^*(e^* - (\theta_1+\theta_2))).$$

Now let us suppose that F is a normal distribution. It can be shown that $E(\varepsilon_{(i)}^n) = n\sigma^2\beta_i^n$ in this case, where σ is the standard deviation of F. $w_n(q)$ is a piecewise linear function. We can define the segments over which the function is linear:

$$\begin{aligned} S_{n+1}^n &= (-\infty, E(\varepsilon_{(n)}^n)) \\ S_i^n &= (E(\varepsilon_{(i)}^n), E(\varepsilon_{(i-1)}^n)),\ 1 < i < n \\ S_1^n &= (E(\varepsilon_{(1)}^n), \infty) \end{aligned}$$

Let $slope_i^n$ denote the slope of $w_n(q)$ on segment S_i^n. $w_n(q)$ is constructed so that $slope_{n+1}^n = slope_n^n$

and $slope_2^n = slope_1^n$. When $1 < i < n$:

$$\begin{aligned} slope_i^n &= \frac{w_{i-1}^n - w_i^n}{E(\varepsilon_{(i-1)}^n) - E(\varepsilon_{(i)}^n)} \\ &= \frac{w_{i-1}^n - w_i^n}{n\sigma^2(\beta_i^n - \beta_{i-1}^n)} \end{aligned}$$

$slope_{i+1}^n \geq slope_i^n$ if and only if:

$$\begin{aligned} \frac{w_i^n - w_{i+1}^n}{n\sigma^2(\beta_{i+1}^n - \beta_i^n)} &\geq \frac{w_{i-1}^n - w_i^n}{n\sigma^2(\beta_i^n - \beta_{i-1}^n)} \\ \frac{w_i^n - w_{i+1}^n}{w_{i-1}^n - w_i^n} &\geq \frac{\beta_{i+1}^n - \beta_i^n}{\beta_i^n - \beta_{i-1}^n} \end{aligned}$$

Proposition 4 implies that this is true when $R \geq \frac{P}{2}$. $slope_{i+1}^n \leq slope_i^n$ if and only if:

$$\frac{w_i^n - w_{i+1}^n}{w_{i-1}^n - w_i^n} \leq \frac{\beta_{i+1}^n - \beta_i^n}{\beta_i^n - \beta_{i-1}^n}$$

Proposition 4 implies that this is true when $\frac{P}{3} \leq R \leq \frac{P}{2}$. It follows, therefore, that $slope_i^n$ is increasing in i when $R \geq \frac{P}{2}$ and decreasing in i when $\frac{P}{3} \leq R \leq \frac{P}{2}$. Therefore, $w_n(q)$ is concave for all n when $R \geq \frac{P}{2}$ and convex for all n when $\frac{P}{3} \leq R \leq \frac{P}{2}$. Green and Stokey (1983) implies that $w^*(q)$ is concave when $R \geq \frac{P}{2}$ and convex for all n when $\frac{P}{3} \leq R \leq \frac{P}{2}$.
Using the facts that $u_n(q)$ converges to $u(w^*(q))$ and

$$\frac{u(w_i^n) - u(w_{i+1}^n)}{u(w_{i-1}^n) - u(w_i^n)} \geq \frac{\beta_{i+1}^n - \beta_i^n}{\beta_i^n - \beta_{i-1}^n}$$

for $R \geq \frac{P}{3}$, we can construct an analogous argument to show that $u(w^*(q))$ is concave when $R \geq \frac{P}{3}$.
■