The Game Chromatic Number of Some Families of Cartesian Product Graphs

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Abstract

We find exact values for the game chromatic number of the Cartesian product graphs $S_m \square P_n$, $S_m \square C_n$, $P_2 \square W_n$, and $P_2 \square K_{m,n}$. This extends previous results of Bartnicki et al. on the game chromatic number of Cartesian product graphs.

1 Introduction

We consider the following well-known graph coloring game introduced independently by Brams [6] and Bodlaender [4]. Two players, Alice and Bob, alternately color the vertices of a graph G using a fixed set of colors such that no two adjacent vertices have the same color, with Alice going first. Bob wins if, at some time before the graph is completely colored, one of the players has no legal move; otherwise Alice wins, and a proper coloring of G is achieved. The *game chromatic number* of G, denoted $\chi_g(G)$, is the least number of colors for which Alice has a winning strategy.

Clearly, the game chromatic number of G is bounded from below by the usual chromatic number $\chi(G)$, and from above by $\Delta(G)+1$, where $\Delta(G)$ is the maximum vertex degree in G. One of the main problems related to this graph invariant is to find upper bounds on the game chromatic number over families of graphs $\mathcal{G}$. Define $\chi_g(\mathcal{G}) = \sup\{\chi_g(G) : G \in \mathcal{G}\}$. For the class of planar graphs $\mathcal{P}$, it is known that $8 \leq \chi_g(\mathcal{P}) \leq 17$ (see Kierstead and Trotter [8] and Zhu [11]); for the class of outerplanar graphs $\mathcal{OP}$, it is known that $6 \leq \chi_g(\mathcal{OP}) \leq 7$ (see Guan and Zhu [7] and [8]); and for k-trees $\mathcal{KT}$ we have $\chi_g(\mathcal{KT}) = 3k+2$ for $k \geq 2$ (see Wu and Zhu [9] and Zhu [10]). We refer the reader to the survey article by Bartnicki, Grytczuk, Kierstead, and Zhu [3] for a discussion of the techniques used to analyze this problem.

In 2007, Bartnicki, Bresăr, Grytczuk, Kovše, Miechowicz, and Peterin [1] studied the game chromatic number for the Cartesian product $G \square H$ of two graphs G and H. They showed that the game chromatic number is not bounded over the family of Cartesian products of two complete bipartite graphs. Their result implies that the game chromatic number

$\chi_g(G\square H)$ is in general not bounded from above by a function of $\chi_g(G)$ and $\chi_g(H)$. Bartnicki et al. also determined the exact values of $\chi_g(P_2\square P_n)$, $\chi_g(P_2\square C_n)$, and $\chi_g(P_2\square K_n)$, where P_n, C_n, and K_n denote the path graph, the cycle graph, and the complete graph on n vertices respectively. More recently, in 2008, Zhu [12] found a bound for the game chromatic number of the Cartesian product graph $G\square H$ in terms of the game coloring number and acyclic chromatic number of G and of H. Defining $\chi_g(\mathcal{G}\square\mathcal{H}) = \sup\{\chi_g(G\square H) : G \in \mathcal{G}, H \in \mathcal{H}\}$, Zhu's result implies that $\chi_g(\mathcal{F}\square\mathcal{F}) \leq 10$ and $\chi_g(\mathcal{P}\square\mathcal{P}) \leq 105$.

Let S_n and W_n denote the star graph and the wheel graph on $n+1$ vertices respectively, and let $K_{m,n}$ denote the complete bipartite graph with parts of size m and n. In this note, we obtain exact values for the game chromatic number of additional Cartesian product graphs, namely the graphs $S_m\square P_n$, $S_m\square C_n$, $P_2\square W_n$, and $P_2\square K_{m,n}$.

2 Exact Values for $\chi_g(S_m\square P_n)$, $\chi_g(S_m\square C_n)$, $\chi_g(P_2\square W_n)$, and $\chi_g(P_2\square K_{m,n})$

We begin by reviewing the definition of the Cartesian product of graphs. Given two graphs G and H, their *Cartesian product* $G\square H$ is the graph with vertex set $V(G)\times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. The graphs G and H are called *factor graphs* of $G\square H$. Note that the Cartesian product operation is both commutative and associative up to isomorphism. Given a vertex $v \in V(H)$, the subgraph G_v of $G\square H$ induced by $\{(u, v) : u \in V(G)\}$ is called a *G-fiber*; *H-fibers* are defined similarly.

The values of the game chromatic numbers $\chi_g(P_2\square P_n)$, $\chi_g(P_2\square C_n)$, and $\chi_g(P_2\square K_n)$ determined by Bartnicki et al. [1] are equal to the trivial upper bounds obtained by considering the maximum vertex degree of the Cartesian product graph. For the graphs $S_m\square P_n$, $S_m\square C_n$, and $P_2\square W_n$, however, we require a stronger upper bound, which is provided by the *game coloring number* of those graphs. This graph invariant is defined as follows. Suppose that Alice is completely color-blind: she cannot distinguish between any two colors. To accommodate Alice's disability, Alice and Bob modify the rules of the coloring game as follows. The players fix a positive integer k and, instead of coloring vertices, simply mark an unmarked vertex each turn. Bob wins if at some time some unmarked vertex has k marked neighbors, while Alice wins if this never occurs. The *game coloring number* of G, denoted $\mathrm{col}_g(G)$, is defined as the least number k for which Alice has a winning strategy on the graph G. Clearly, if Alice can win the *marking* game for some integer k, then she can also win the coloring game with k colors, that is, $\chi_g(G) \leq \mathrm{col}_g(G)$.

Proposition 1 below, which was first stated in [7], allows us to make use of the structure of Cartesian product graphs to obtain upper bounds on $\mathrm{col}_g(G\square H)$ as a function of $\Delta(G)$, $\Delta(H)$, $\mathrm{col}_g(\bigsqcup_{V(H)} G)$, and $\mathrm{col}_g(\bigsqcup_{V(G)} H)$. Here $\bigsqcup_{V(H)} G$ indicates the disjoint union of $|V(H)|$ copies of G.

Proposition 1 (Guan and Zhu [7, Theorem 2])**.** *Suppose that $G = (V, E)$ is a graph with $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $\chi_g(G) \leq \mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$.*

Proof. Alice plays according to the optimal strategy for G_1, so that at any point in the game, any unmarked vertex has at most $\mathrm{col}_g(G_1) - 1 + \Delta(G_2)$ marked neighbors. □

Corollary 2. *For any Cartesian product graph $G\Box H$, we have*

$$\chi_g(G\Box H) \le \mathrm{col}_g(G\Box H) \le \mathrm{col}_g\Big(\bigsqcup_{V(H)} G\Big) + \Delta(H).$$

Proof. In Proposition 1, set G_1 to be the union of all the G-fibers and G_2 to be the union of all the H-fibers. □

For ease of future reference, we note that for arbitrary positive integers l and n, we have $\mathrm{col}_g(\bigsqcup_{i=1}^{l} S_n) = 2$ and $\mathrm{col}_g(\bigsqcup_{i=1}^{l} W_n) \le 4$. These bounds are attained by having Alice always mark the center vertex of the S_n- or W_n-fiber that Bob last played in, if it is unmarked.

Before stating our results, we introduce some final definitions and notational conventions. Suppose that Alice and Bob play the coloring game with k colors. We say that there is a *threat* to an uncolored vertex v if there are $k-1$ colors in the neighborhood of v, and it is possible to color a vertex adjacent to v with the last color, so that all k colors would then appear in the neighborhood of v. The threat to the vertex v is said to be *blocked* if v is subsequently assigned a color, or it is no longer possible for v to have all k colors in its neighborhood. We shall also use the convention that color numbers are only used to differentiate distinct colors, and should not be regarded as ascribed to particular colors. For example, if only colors 1 and 2 have been used so far and we introduce a new color, color 3, then color 3 can refer to any color that is not the same as color 1 or color 2. Finally, we label figures in the following manner: vertices are labeled in the form "color (player, turn)," with the information in parentheses being omitted if the same configuration can be attained in multiple ways. A pair of asterisks indicates that Alice cannot block the threats to both vertices marked with asterisks in her next turn, so that Bob wins.

We are now ready to state our results.

Proposition 3. *Let m and n be positive integers with $m \ge 2$. Then*

$$\chi_g(S_m\Box P_1) = 2;$$
$$\chi_g(S_m\Box P_2) = 3;$$
$$\chi_g(S_m\Box C_n) = \chi_g(S_m\Box P_n) = 4 \text{ for } n \ge 3.$$

Proof. The result is clear for the graphs $S_m\Box P_1$ and $S_m\Box P_2$. For $n \ge 3$, we obtain the upper bounds $\chi_g(S_m\Box P_n) \le 4$ and $\chi_g(S_m\Box C_n) \le 4$ by taking $G = S_m$ and $H = P_n$ or C_n in Corollary 2. It remains to show that Bob can win with three or fewer colors whenever $n \ge 3$. Observe that if Bob can force a subgraph of the form shown in Fig. 1 after his first turn, then he wins after his second turn, since Alice cannot block the threats to both the vertices marked with asterisks. It is easy to see that this can always be done for graphs of the form $S_m\Box C_n$, $n \ge 3$. For graphs of the form $S_m\Box P_n$, Bob cannot create this configuration after his first turn only if Alice makes her first move (say color 1) in the center vertex of one of the two "side" S_m-fibers (see Fig. 2), or if $n = 3$ and Alice plays in a noncentral vertex of the middle S_m-fiber. Suppose that $n \ge 4$, so that we are in the former case. Bob should respond by playing color 2 in the center vertex of the S_m-fiber at a distance 2 away from Alice. This forces Alice to play color 3 in the unique vertex adjacent to both colored vertices, since this is the only way to block the threat to that vertex. Bob then plays color 3 as shown in Fig. 2. Alice cannot block the threats to both the vertices marked with asterisks, so Bob wins.

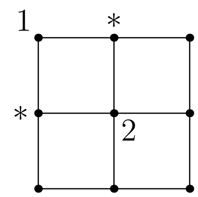

Figure 1: The configuration that Bob attempts to achieve after his first turn.

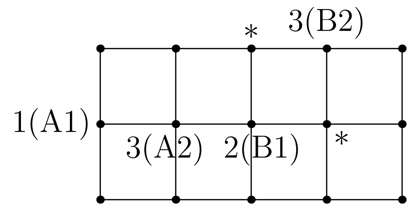

Figure 2: Bob's winning strategy for $S_m \square P_n$, $n \geq 4$. (The graph above is $S_2 \square P_5$.)

Finally, we are left with the graph $S_m \square P_3$. The case analysis in Fig. 3 shows that Bob wins with 3 colors. □

Proposition 4. *For any integer* $n \geq 9$, $\chi_g(P_2 \square W_n) = 5$.

Proof. First, we show that Bob has a winning strategy with four or fewer colors. We give Bob's winning strategy when there are exactly four colors; it will be easy to see that Bob can win using the same starting moves (and replacing color 4 by color 3, if necessary) when there are fewer than four colors. Denote the vertices of one W_n-fiber by $v_0, v_1, v_2, \ldots, v_n$, with v_0 being the center vertex and v_1, v_2, …, v_n the n-cycle, and denote the corresponding vertices of the other fiber by $v'_0, v'_1, v'_2, \ldots, v'_n$. Bob should respond to Alice's first move in such a way that we may assume without loss of generality that v_0 has color 1 and v'_1 has color 2. If Alice responds with color 3 in v'_0, then Bob plays color 4 in v_2. Alice cannot block the threats to both v_1 and v'_2, so Bob wins.

Now suppose that Alice responds to Bob's first move by playing in some vertex that is not v'_0. We may suppose without loss of generality that Alice plays in one of v_1, v_n, v_{n-1}, …, $v_{\lceil n/2 \rceil+1}$, $v'_{\lceil n/2 \rceil+1}$, $v'_{\lceil n/2 \rceil+2}$, …, v'_n. If Alice plays in v_1 or v_n, then Bob responds with color 3 in v'_4, as shown in Fig. 4. This forces Alice to play color 4 in v'_0; otherwise, Bob would win on his next turn by playing color 4 in one of the v'_i. Bob then replies with color 2 in v_5, creating threats to v_4 and v'_5. Alice cannot block both these threats, so Bob wins. On the other hand, if Alice does not play in v_1 or v_n, then Bob responds with color 3 in v'_2, as shown in Fig. 5. As before, this forces Alice to play color 4 in v'_0. Bob then replies with color 2 in v_3, creating threats to v_2 and v'_3. Once again, Alice cannot block both these threats, so Bob wins. Note that we have used the fact that $n \geq 9$ here, so that Alice's move on her second turn does not affect Bob's ability to threaten Alice after her third turn.

Finally, the upper bound $\chi_g(P_2 \square W_n) \leq 5$ follows from taking $G = W_n$ and $H = P_2$ in Corollary 2. □

Bartnicki et al. [1] remarked that, based on their results for $\chi_g(P_2 \square P_n)$, $\chi_g(P_2 \square C_n)$, and $\chi_g(P_2 \square K_n)$, one might falsely conjecture that $\chi_g(P_2 \square G) = \chi_g(G) + 1$ in general. They

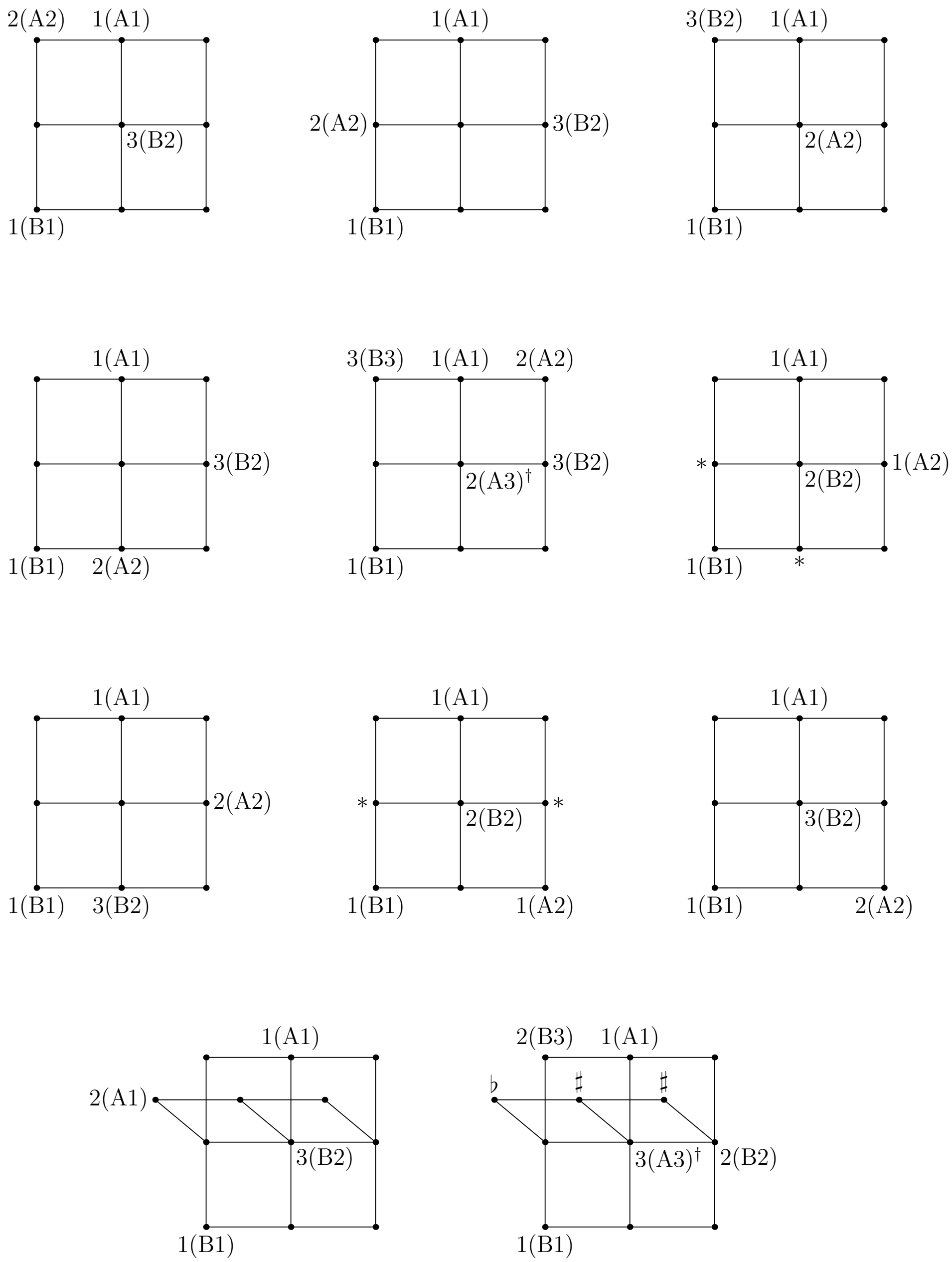

Figure 3: Bob's winning strategy for $S_m \square P_3$. The top nine cases show Bob's strategy when play is confined to a $S_2 \square P_3$ subgraph, while the bottom two cases illustrate Bob's strategy when play is not confined to a $S_2 \square P_3$ subgraph. A dagger (†) indicates that if Alice does not color that vertex on her specified turn, then Bob can play such that that vertex is adjacent to 3 distinct colors after his turn. In the bottom-right diagram, Alice either plays color 1 in the vertex marked with a flat (♭), or any color in a vertex marked with a sharp (♯) on her second turn.

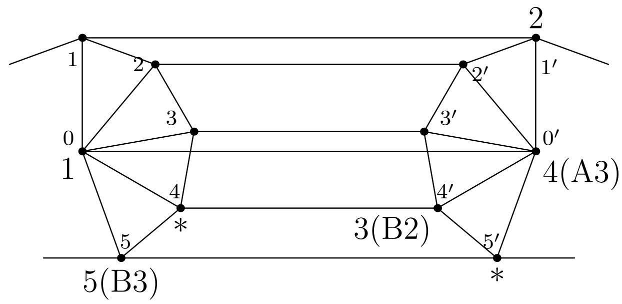

Figure 4: Bob's strategy if Alice plays in v_1 or v_n. Vertices of the form v_i are labeled by a small i, while vertices of the form v_i' are labeled by a small i'.

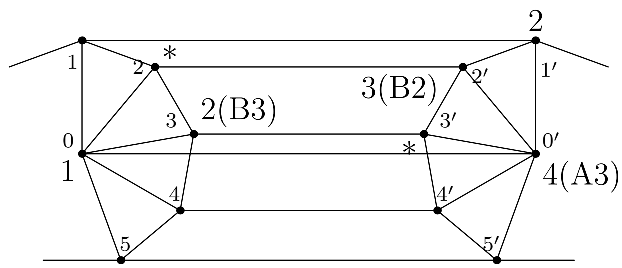

Figure 5: Bob's strategy if Alice does not play in v_1 or v_n.

showed that if G is the graph obtained from the triangle by attaching two leaves to every vertex, then $\chi_g(G) = \chi_g(P_2 \Box G) = 4$. Theorem 5, besides being of independent interest, also illustrates that there exist graphs G such that $\chi_g(P_2 \Box G) > \chi_g(G) + 1$ (since $\chi_g(K_{m,n}) = 3$ for $m, n \geq 2$).

Theorem 5. *For integers $m, n \geq 5$, $\chi_g(P_2 \Box K_{m,n}) = 5$.*

Proof. First, we partition the vertex set of $P_2 \Box K_{m,n}$ into four disjoint sets A, B, A', and B' as follows. Let G and G' be the two $K_{m,n}$-fibers, let A (respectively B) be the independent set of size m (respectively n) in G, and let A' (respectively B') be the independent set of size m (respectively n) in G'. Note that $A \cup B'$ and $A' \cup B$ are independent sets. For convenience, we refer to the sets $A \cup A'$ and $B \cup B'$ as the two *parts* of the graph, and we always mean a $K_{m,n}$-fiber when we refer to the same or opposite fiber. We have to show that Bob wins with four or fewer colors, and that Alice wins with five colors. It is easy to see that Bob wins with three or fewer colors. Indeed, suppose without loss of generality that Alice plays color 1 in A on her first move. Bob responds with color 2 in A, thus forcing a third color in B. If Alice plays color 3 in B, then Bob also plays color 3 in B'. This forces a fourth color in B, so Bob wins. Otherwise, if Alice does not play color 3 in B, then Bob can play color 3 in A, which also forces a fourth color in B.

Now suppose that four colors are available. Again, we suppose without loss of generality that Alice plays color 1 in A for her first move. Bob will respond with color 2 in A. We now consider two cases.

Case 1: Alice plays color 3 in B on her second move. Bob will respond with color 3 in B', forcing a fourth color in B. If Alice does not play color 4 in B immediately after this, then Bob will play color 4 in A, forcing a fifth color in B. Hence Alice must play color 4 in B. Bob will respond with color 4 in B'. At this point, only colors 1 and 2 can appear in A and A'. Alice can prevent Bob from playing at most one of colors 1 and 2 in B' (by playing that color, say 1, in A'). Then Bob plays the other color (say 2) in B'. This forces a fifth color in A', so Bob wins.

Case 2: Alice does not play color 3 in B on her second move. If Alice plays color 3 in A, then we are done, since Bob will respond with color 4 in A, forcing a fifth color in B. Otherwise, Bob plays either color 3 or 4 (say color 3) in A, such that only three colors appear in the graph so far. Alice must play color 4 in B, else Bob would play color 4 in A and win (this is where we require the assumption that $m, n \geq 5$). Bob will then respond by playing color 4 in B'. This forces a fifth color in B, so Bob wins.

Next, we show that Alice has a winning strategy using 5 colors. Observe that it suffices to show that Alice can achieve one of the following two configurations.

Configuration 1: There are two colors x and y such that x appears in A and B', while y appears in A' and B.

Configuration 2: There are four colors w, x, y, and z such that w and x both appear in A and A', while y and z both appear in B and B'.

Indeed, if either of the two configurations is achieved, then Alice can color any vertex in the graph using one of the colors of that configuration, regardless of how Bob plays.

Alice begins by playing color 1 in A. We then consider several cases. To aid the reader in following the proofs, we have included figures of a possible sequence of moves whenever a new idea is introduced. To avoid cluttering up the figures, edges between A and B, and between A' and B', have been omitted from the figures. Vertex labels not enclosed in square brackets indicate a sequence of moves that could have led to the case in consideration, while vertex labels enclosed in square brackets indicate a subsequent possible sequence of moves using the given strategy.

Case 1: Bob plays color 1 in B'. Alice should then play the other four colors in her next four turns, twice in A' and twice in B, unless she has a move that enables her to create Configuration 1 immediately, as shown in Fig. 6. Bob's only way to prevent Alice from creating Configuration 1 is to play the same color in the same part of the opposite fiber after each of Alice's turns (e.g., if Alice plays color 2 in A', then Bob must also play color 2 in A). However, this causes Bob to create Configuration 2 after his fifth turn, so Alice wins.

Case 2: Bob plays color 1 in A'. Alice responds with color 2 in B.

Case 2.1: Bob plays color 2 in A'. Alice continues in a manner similar to Case 1.

Case 2.2: Bob plays color 2 in B'. Alice then responds with color 3 in A.

Case 2.2.1: Bob plays color 3 in A'. Alice then plays the remaining two colors in her next two turns, once in B and once in B', unless she has a move that enables her to create Configuration 2 immediately, as shown in Fig. 7. A

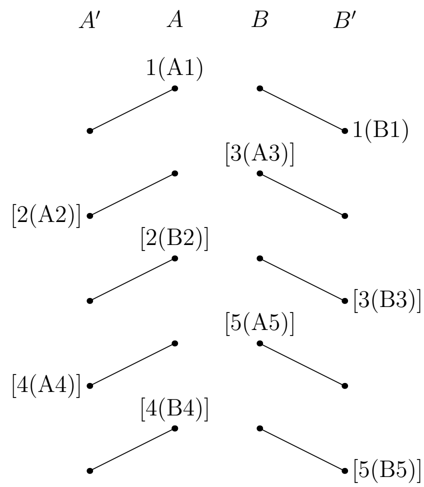

Figure 6: A possible sequence of moves in Case 1. (Edges between A and B, and between A' and B', have been omitted to avoid cluttering up the figure.)

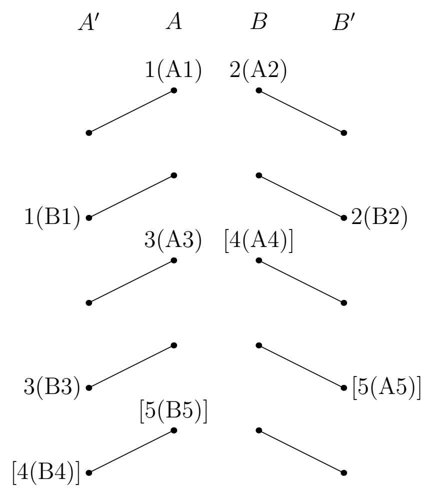

Figure 7: A possible sequence of moves in Case 2.2.1.

similar argument to Case 1 shows that either Configuration 1 or 2 will be created, so Alice wins.

Case 2.2.2: Bob plays color 3 in B'. Alice then plays the remaining two colors in her next two turns, once in B and once in A', unless she has a move that enables her to create Configuration 1 immediately. A similar argument to Case 1 shows that either Configuration 1 or 2 will be created, so Alice wins.

Case 2.2.3: Bob plays color 4. Alice then plays color 5 in B. If Bob then plays one of colors 3, 4, or 5 in a fiber different from the one that it first appeared in, then Alice can create either Configuration 1 or 2, so we are done. Otherwise, Bob either plays one of colors 3, 4, or 5 in the same fiber, or color 1 or 2. From the way Alice played color 5, exactly one of colors 3, 4, or 5 is the only color in its part. Alice should then play that color in the same part of the opposite fiber, as shown in Fig. 8. No matter how Bob plays afterwards, he cannot block Alice from creating Configuration 2, so Alice wins.

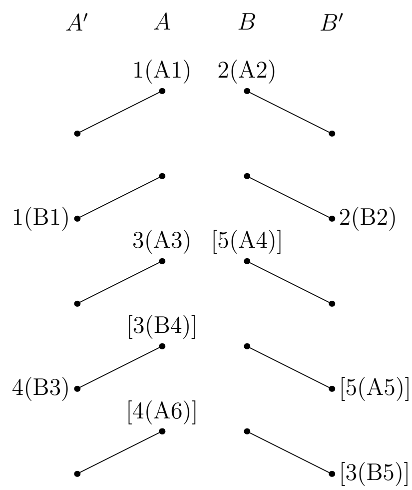

Figure 8: A possible sequence of moves in Case 2.2.3, when in his fourth step, Bob either plays one of colors 3, 4, or 5 in the same fiber, or color 1 or 2.

Case 2.2.4: Bob plays color 3 in A, or one of colors 1 or 2. Alice responds with color 4 in B. If Bob then plays color 3 or 4 in $A' \cup B'$, Alice immediately has a winning move; otherwise, Alice plays color 5 and reduces the scenario to Case 2.2.3, from which she has a winning strategy.

Case 2.3: Bob plays color 3. Alice responds by playing color 4 such that (1) there are exactly two colors in each of $A \cup A'$ and $B \cup B'$, and (2) colors 2, 3, 4 are not all in $A' \cup B$, as shown in Fig. 9.

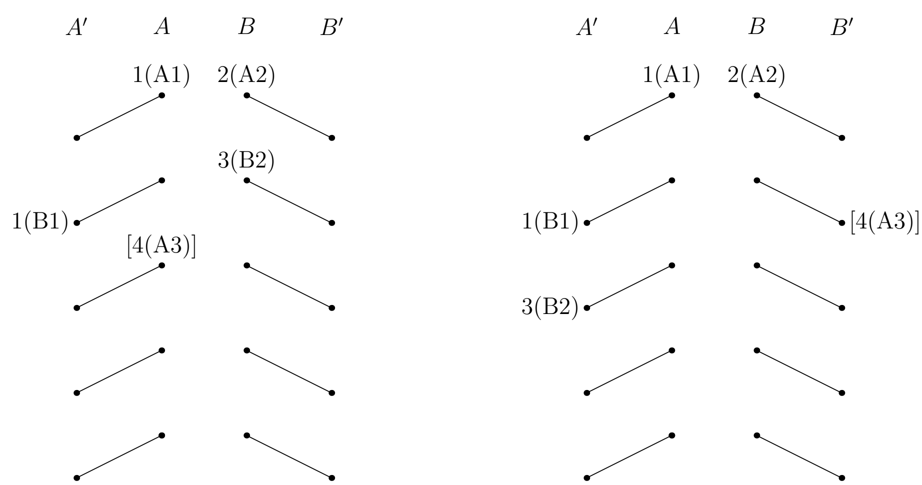

Figure 9: Examples of how Alice should respond to Bob's move in Case 2.3.

Case 2.3.1: Bob plays one of colors 1, 2, 3, or 4 in the same fiber as the one it first appeared in, or he plays color 5. From the way Alice played color 4, exactly one of colors 2, 3, or 4 is the only color in $A \cup B'$ or $A' \cup B$. Alice should then play that color in the opposite part of the opposite fiber. No matter how Bob plays afterwards, he cannot block Alice from creating Configuration 1, so Alice wins.

Case 2.3.2: Bob plays one of colors 2, 3, or 4 in the opposite part of the opposite fiber, with respect to the one that it first appeared in. Alice then creates Configuration 1 and wins.

Case 2.3.3: Bob plays one of colors 2, 3, or 4 in the same part of the opposite fiber. Alice responds by playing a different one of those three colors in the same part of the fiber opposite to that which it first appeared in. This reduces the scenario to Case 2.2.1, from which Alice has a winning strategy.

Case 2.4: Bob plays color 2 in B, or color 1. Alice responds by playing color 2 in A'. If Bob does not play a new color, then Alice can continue as in Case 1. So suppose that Bob plays color 3. If he plays color 3 in A or B', then Alice creates Configuration 1 and wins. Otherwise, Bob plays color 3 in either A' or B. If Bob plays in A', then Alice should play the remaining two colors in her next two turns, both in B'. In order for Bob to prevent Alice from creating Configuration 1, he must play the same color in the same part of the opposite fiber after each of Alice's two turns. Alice then plays color 3 in A to create Configuration 2 and wins. If Bob plays color 3 in B instead, then Alice should play the remaining two colors in her next two turns, once in A and once in B'. A similar argument shows that either Configuration 1 or 2 will be created, so Alice also wins.

Case 3: Bob plays color 2. Alice responds by playing color 3 in B.

Case 3.1: Bob plays one of colors 1, 2, or 3 in a fiber different from the one that it first appeared in. If Bob played such that some color appears in both A and B', or in both A' and B, then Alice completes Configuration 1 and wins. If Bob plays such that some color appears in both A and A' (respectively in both B and B'), then Alice should play one of colors 1, 2, or 3 such that some color now appears in both B and B' (respectively in both A and A'), hence reducing to Case 2.2.

Case 3.2: Bob plays one of colors 1, 2, or 3 in the same fiber as the one it first appeared in, or he plays color 4. From the way Alice played color 3, exactly one of colors 1, 2, or 3 is the only color in $A \cup B'$ or $A' \cup B$. Alice should then play that color in the opposite part of the opposite fiber. No matter how Bob plays afterwards, he cannot block Alice from creating Configuration 1, so Alice wins.

Case 4: Bob plays color 1 in A. Alice responds by playing color 1 in B'.

Case 4.1: Bob plays color 1 in A or B'. Alice continues as in Case 1.

Case 4.2: Bob plays color 2 in A' or B. Then Alice creates Configuration 1 and wins.

Case 4.3: Bob plays color 2 in A or B'. Without loss of generality, suppose that Bob plays color 2 in A. Alice responds by playing color 3 in B. In order to prevent Alice from creating Configuration 1 immediately, Bob is forced to play color 3 in B'. This creates a scenario similar to that in Case 2.4, so Alice wins.

This completes the proof. □

3 Further Directions

Few exact values are known for the game chromatic number of Cartesian product graphs. In particular, the game chromatic numbers $\chi_g(P_m \square P_n)$ and $\chi_g(C_m \square C_n)$ of the grid graph and toroidal grid graph are unknown for general values of m and n. For $m, n \geq 3$, $(m, n) \neq (3, 3)$, the trivial upper bound shows that Alice wins with five colors, while it is easy to see that Bob has a winning strategy with three colors, which implies that $4 \leq \chi_g(P_m \square P_n) \leq 5$ and $4 \leq \chi_g(C_m \square C_n) \leq 5$. Another problem of interest is to obtain nontrivial bounds on $\chi_g(K_m \square K_n)$; this value is equal to the *game chromatic index*—the corresponding graph invariant for the variant of the game where Alice and Bob color *edges* of the graph instead of vertices—of the complete bipartite graph $K_{m,n}$. We refer the reader to the paper of Bartnicki and Grytczuk [2] for known results on the game chromatic index of graphs.

Dinski and Zhu [5] proved the upper bound $\chi_g(G) \leq \chi_a(G)(\chi_a(G) + 1)$ for an arbitrary graph G, where $\chi_a(G)$ denotes the *acyclic chromatic number* of G, defined as the least number of colors required for a proper vertex coloring of G with no 2-colored cycle. Using this result, Bartnicki et al. [1] obtained the bounds $\chi_g(\mathcal{T} \square \mathcal{T}) \leq 12$ and $\chi_g(\mathcal{P} \square \mathcal{P}) \leq 650$, where $\mathcal{T}$ denotes the class of trees. Zhu [12] later proved the upper bound $\chi_g(G \square H) \leq \chi_a(H)(\chi_a(H) + \mathrm{col}_g(G^*)) - 1)$, where G^* is the graph obtained from G by adding to each vertex a set of $|H|$ degree 1 neighbors. From this result, he obtained the improved bounds $\chi_g(\mathcal{F} \square \mathcal{F}) \leq 10$, $\chi_g(\mathcal{F} \square \mathcal{OP}) \leq 16$, $\chi_g(\mathcal{F} \square \mathcal{P}) \leq 36$, $\chi_g(\mathcal{P} \square \mathcal{OP}) \leq 55$, and $\chi_g(\mathcal{P} \square \mathcal{P}) \leq 105$, where $\mathcal{F}$ is the class of forests. It is believed that some of the larger bounds can be improved significantly, while it would be interesting to determine whether some of the smaller bounds, in particular $\chi_g(\mathcal{F} \square \mathcal{F}) \leq 10$, are tight.

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