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REAL-TIME FINITE-RATE TRACKING: PERFORMANCE LIMITATIONS

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ABSTRACT

In this paper, we consider a set up in which the plant and controller are local to each other, but are together driven by a remote reference signal that is transmitted through a finite-rate noiseless channel. When control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals over the channel and achieving performance in finite-time. Most work in the area of control under communication constraints have addressed infinite-horizon control objectives (eg. stability, disturbance rejection). In this paper, we compute lower and upper bounds on worst-case performance for a *finite-horizon tracking* objective. We achieve the lower bound with a noncausal coding scheme and show how imposing causality on the coding scheme severely limits achievable performance. We illustrate how the bounds behave under various scenarios and show tradeoffs between time and performance accuracy.

I. INTRODUCTION

The classical control paradigm addresses problems where communication between one plant and one controller is essentially perfect. Today new problems in control over networked systems, whose components are connected via communication links that can be very noisy, induce delays, and have finite rate constraints, are emerging. Applications include remote navigation systems (deep-space and sea exploration) and multi-robot control systems (eg. aircraft and spacecraft formation flying control, coordinated control of land robots, control of multiple surface and underwater vehicles), where robots exchange data through communication channels that impose constraints on the design of coordination strategies.

In communication systems, problems entail designing channel encoders and decoders to reconstruct signals sent through noisy channels. Questions about *asymptotic* reconstruction are typically addressed. In control systems, problems often entail designing controllers to generate *real-time* desirable responses from a system. Therefore, when control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals and achieving stability and performance in finite-time.

Control under communication constraints is a research area of growing interest. Much work has focused on stability under finite-rate (or countable) feedback control, where the only excitation to the system is an unknown (but bounded) initial state condition [1], [5], [7], [8], [10], [2], [12], [15], [16] (to list a few). The questions posed involve conditions on the channel rate that will guarantee that the state of the system (or some function of the state) approach the origin/remain bounded as time goes to infinity. More recently, disturbance rejection limitations were derived for the same setting, assuming stochastic exogenous signals entering the system [9]. Although these studies have contributed greatly to our understanding of the interplay between communication and cont rol, few studies (see [3], [4], [6]) address finite-horizon performance limitations under communication constraints. In [6], Fagnani etal. explore finite-horizon navigation under finite-rate feedback using a stochastic framework. They ultimately show how demanding faster times to reach a given set in the state space requires more complicated coding schemes.

In this paper we compute lower and upper bounds for finite-horizon tracking objective under finite-rate feedforward control. Specifically, we compute the smallest allowable worst-case performance over a class of reference signals. We also construct quantization/coding schemes to derive upper bounds on performance, and illustrate how imposing causality on the quantizer limits achievable performance. Our framework is deterministic and our lower bound is independent of the complexity of the coding scheme.

II. PROBLEM FORMULATION

In this section, we are interested in tracking a class of reference commands, r, over a finite-horizon and under finite-rate constraints. We consider the cascade of SISO discrete-time systems shown in Figure 1.

Specifically,

- $z \in \mathbb{R}^T \; s.t. \; ||z||_2 \leq 1$,
- $L : \mathbb{R}^T \rightarrow \mathbb{R}^T$ is an invertible linear operator,

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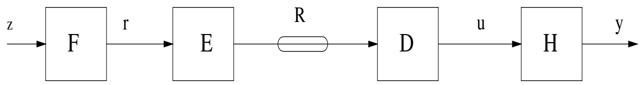

Fig. 1. Finite Horizon Tracking Set Up

- $E : \mathbb{R}^T \to \{0,1\}^{RT}$ is an arbitrary operator (encoder) that maps a real vector to a sequence of 2^{RT} binary symbols,
- R is the channel rate for the finite-rate noiseless channel that maps $\{0,1\}^{RT} \to \{0,1\}^{RT}$,
- $D : \{0,1\}^{RT} \to \mathbb{R}^T$ is an arbitrary operator (decoder) that maps a sequence of 2^{RT} binary symbols to a real vector, and
- $H : \mathbb{R}^T \to \mathbb{R}^T$ is an invertible linear operator (model of remote plant and controller system at rest).

Note that L defines a class of signals, $\mathcal{C}_r$, that is generated from a unit ball in $\mathbb{R}^T$. Since L is linear, it maps the unit ball to a bounded ellipsoid. We set out to minimize a tracking error over all signals, r, in this class (worst-case analysis). Since the input and output signals have finite length, the following performance metric is computed over a finite-horizon: $||W(y-r)||_2^2$, where $W \in \mathbb{R}^T \times \mathbb{R}^T$ is a given full-rank weight matrix.

It is worth commenting that in the ideal case of perfect communication ($R = \infty$), it is possible to construct an encoder and decoder ($E = L^{-1}$ and $D = H^{-1}$) such that $||W(y-r)||_2^2 = 0 \ \ \forall r \in \mathcal{C}_r$. However, with a finite-rate constraint, the control, u, can only take 2^{RT} values over a horizon of T time steps. Furthermore, with H being a one-to-one mapping, the output, y, can only take 2^{RT} values over a horizon of T time steps. Therefore, it is not clear what level of performance is achievable over $\mathcal{C}_r$.

To understand tracking limitations under finite-rate feed-forward control, we compute γ_{LB} and γ_{UB}, such that

$$\gamma_{LB} \leq \min_{(E,D)} \ \sup_{r\in\mathcal{C}_r} ||W(y-r)||_2^2 \leq \gamma_{UB}.$$

Knowledge of γ_{LB} tells us that regardless of the encoder and decoder that we select, we can do no better than this lower bound. Therefore, we expect it to be independent of E and D. The upper bound tells us that there exists a coding scheme such that the worst case performance is always less than or equal to γ_{UB}. Therefore, to compute γ_{UB}, we need to construct an encoder and decoder and compute the corresponding worst-case performance. We compute γ_{LB} and γ_{UB} in the following sections.

III. A LOWER BOUND

In this section we derive the lower bound on worst-case performance.

Theorem III.1. *Given the tracking set up defined above, assume that* $det(W) \neq 0$, $det(L) \neq 0$. *Then,*

$$\gamma_{LB} = 2^{-2R}\{|det(L)|\ |det(W)|\}^{\frac{2}{T}}.$$

Proof.

The set of all possible commands, $\mathcal{C}_r \triangleq \{r \in \mathbb{R}^T | r = Lz, z'z \leq 1\} = \{r \in \mathbb{R}^T \mid (L^{-1}r)'(L^{-1}r) \leq 1\}$. $\mathcal{C}_r$ *is a bounded ellipsoid in* $\mathbb{R}^T$ *centered at the origin with volume* $\eta\ det\{((L^{-1})'(L^{-1}))^{-0.5}\} = \eta|det(L)|$, *where* η *is the volume of a unit ball in* $\mathbb{R}^T$.

Over a horizon T, *the channel sends a total of* RT *bits which limits the control signal to take on no more than* 2^{RT} *values; and, since* H *is a one-to-one mapping, the channel limits the output to take on no more than* 2^{RT} *values. Consider a selection of outputs* $y_1, y_2, ..., y_{2^{RT}}$, *which correspond to inputs* $u_1, u_2, ..., u_{2^{RT}}$, *respectively. We must then map each* $r \in \mathcal{C}_r$ *to exactly one* y_i $i = 1, 2, ..., 2^{RT}$. *Such a mapping induces a partition on* $\mathcal{C}_r$. *In particular, define* $P_i = \{r \in \mathcal{C}_r \mid r \to y_i\}$ *for* $i = 1, 2, ..., 2^{RT}$. *Now, suppose that the selection* $y_1, y_2, ..., y_{2^{RT}}$ *were chosen such that* $||W(y_i - r)||_2^2 \leq \gamma$ *for all* $r \in P_i$, *and for all* i. *Then necessarily* $P_i \subseteq S^{\gamma}_{y_i} \triangleq \{r \in \mathbb{R}^T \mid (r-y_i)'W'W(r-y_i) \leq \gamma\}$. *Note that* $S^{\gamma}_{y_i}$ *is a bounded ellipsoid in* $\mathbb{R}^T$ *centered at point* y_i *with volume* $\eta(\sqrt{\gamma})^T det\{(W'W)^{-0.5}\} = \frac{\eta\sqrt{\gamma}^T}{|det(W)|}$. *See Figure 2 for an illustration.*

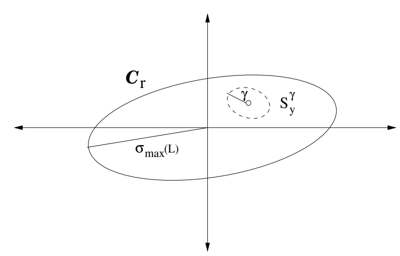

Fig. 2. Bounded Ellipsoids $\mathcal{C}_r$ and S^{γ}_y

Since $P_i \subseteq S^{\gamma}_{y_i}$ *for each* $i = 1, 2, ..., 2^{RT}$, *it is necessary that* 2^{RT} *bounded ellipsoids* (S^{γ}_y) *cover the set* $\mathcal{C}_r$. *This implies that* $2^{RT} \times volume(S^{\gamma}_y) \geq volume(\mathcal{C}_r)$. *Equivalently,*

$$2^{RT} \geq \frac{volume(\mathcal{C}_r)}{volume(S^{\gamma}_y)} = \frac{|det(L)|\ |det(W)|}{(\sqrt{\gamma})^T}.$$

After rearranging terms, we get that $\gamma \geq 2^{-2R}\{|det(L)|\ |det(W)|\}^{\frac{2}{T}}$.

■

Since we often consider classes of inputs generated from LTI systems, *i.e.,* L is LTI, we compute the lower bound for this case in the following corollary.

Corollary III.1. *Assume that* $det(W) \neq 0$, $det(L) \neq 0$, *and* H *is a one-to-one mapping. If* L *is a causal SISO LTI system with state-space description* $L = ss(A_l, B_l, C_l, D_l)$, *then*

$$||W(y-r)||_2^2 \geq 2^{-2R}(D_l)^2\{|det(W)|\}^{\frac{2}{T}}.$$

Proof. *If L is a SISO causal LTI with state-space description $L = ss(A_l, B_l, C_l, D_l)$, then for T time steps, it can be represented as a $T \times T$ lower triangular Toeplitz matrix operator, with all T eigenvalues equal to D_l. This implies that the $\{det(L)\}^{\frac{2}{T}} = (D_l)^2$.* ■

As expected, γ_{LB} depends on L (class of reference commands), W (performance weights), T (performance horizon), and R (channel rate). It is helpful (as we will see when we compute upper bounds) to rewrite the lower bound in terms of the singular and eigenvalues of the matrix WL as follows: $\gamma_{LB} = 2^{-2R}\{\prod_{i=0}^{T-1} \sigma_i(WL)\}^{\frac{2}{T}} = 2^{-2R}\{\prod_{i=0}^{T-1} |\lambda_i(WL)|\}^{\frac{2}{T}}$. When computing the lower bound, we made no assumptions on whether the encoder and decoder are causal or noncausal. If E and D are both noncausal, then our tracking problem reduces vector quantization problem [11], where time need not enter the picture. At time $t = 0$, the decoder "knows" the future, that is, it knows u_k for $k = 0, 1, ..., T-1$, which are represented by TR bits over a horizon of T steps. This is what causes the lower bound to be independent of the coding scheme. On the other hand, if E and D are causal, then the decoder only knows u_k for $k = 0, 1, ..., T-1$, only at time $t \geq k$, and u_k is represented by at most $(k+1)R$ bits.

In the following section, we compute an upper bound by constructing a causal coding scheme, which is more practical. In [14], we also compute an upper bound allowing the coding scheme to be noncausal, and achieve the lower bound.

IV. A CAUSAL UPPER BOUND

In this section, we derive an upper bound assuming that the encoder and decoder are causal and implement a practical coding scheme illustrated in Figure 3. In this scheme, the encoder is a quantizer parameterized by a rate matrix which dictates how bits are allocated to each component in the encoder's memory at each time step. Specifically, the rate matrix has the following form.

$$\mathcal{R} = \begin{bmatrix} R_{00} & 0 & 0 & ... & ... \\ R_{01} & R_{11} & 0 & 0 & ... \\ R_{02} & R_{12} & R_{22} & 0 & ... \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ R_{0,T-1} & R_{1,T-1} & R_{2,T-1} & ... & R_{T-1,T-1} \end{bmatrix},$$

such that $\sum_j R_{ij} = R$ for $i = 0, 1, ..., T-1$.

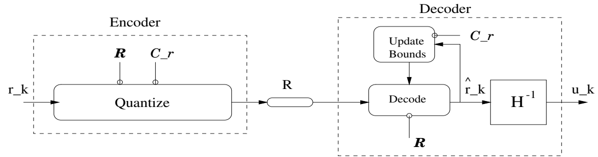

Fig. 3. Causal Coding Scheme

To understand how the above rate matrix dictates a bit-allocation strategy, let $\hat{r}_i(j)$ be the quantized estimate of r_i at time j. Then, $\mathcal{R}$ determines that at time $t = 0$, R_{00} bits are used to quantize r_0 to produce $\hat{r}_0(0)$. At time $t = 1$, an *additional* R_{01} bits are used to quantize r_0 to produce $\hat{r}_0(1)$, and R_{11} bits are used to quantize r_1 to produce $\hat{r}_1(1)$, and so on. The accuracy of $\hat{r}_i(j)$ is within $\pm M_i 2^{-\sum_{k=i}^{j} R_{ik}}$ of r_i for all $i \geq 0$, where $2M_i$ is the length of the interval in which r_i belongs to as computed by the decoder at time step j. The decoder computes M_i from its past inputs and from constraints imposed on $r \in \mathcal{C}_r$. These computations will become clear below.

Consider the following example for horizon length $T = 2$,

$$Q \triangleq (L^{-1})'(L^{-1}) = \begin{bmatrix} q_{00} & q_{01} \\ q_{01} & q_{11} \end{bmatrix}, \quad W'W = \begin{bmatrix} w_{00} & w_{01} \\ w_{01} & w_{11} \end{bmatrix},$$

for some q_{01} and $w_{01} \in \mathbb{R}$.

At time $t = 0$, the encoder receives r_0 which has magnitude less than or equal to M_0,[1] and its quantization region is on the interval $\{-M_0, M_0\}$ which is divided into $2^{R_{00}}$ equal intervals as shown in Figure 4. The union of the representatives for each region comprise the range of E at time 0. Specifically,

$$E(r_0; t = 0) = nM_0 2^{-R_{00}}$$

for $(n-1)M_0 2^{-R_{00}} < r_0 \leq (n+1)M_0 2^{-R_{00}}$, $n = \pm 1, \pm 3, ..., \pm 2^{R_{00}} - 1$. Thus, when the encoder receives r_0 it outputs the centroid value of the interval in which r_0 falls which is represented by R_{00} bits. The decoder then receives $\hat{r}_0(0)$ and updates its bounds for both r_0 and r_1 to prepare for its next input.

The bounds for r_1 are generated from the constraints imposed by the set $\mathcal{C}_r \triangleq \{r \in \mathbb{R}^T | r = Lz, z'z \leq 1\}$. For $T = 2$, $\mathcal{C}_r$ forces $|r_1 + \frac{q_{01}}{q_{11}} r_0| \leq \frac{\sqrt{1+\mu r_0^2}}{\sqrt{q_{11}}}$ given r_0, where $\mu = \frac{q_{01}^2}{q_{11}} - q_{00}$. Therefore, after time step 0, we have that $l_0(0) \leq r_0 \leq u_0(0), l_1(0) \leq r_1 \leq u_1(0)$, [2] where

$$l_0(0) = \hat{r}_0(0) - 2^{-R_{00}} M_0,$$
$$u_0(0) = \hat{r}_0(0) + 2^{-R_{00}} M_0,$$

$$l_1(0) =$$
$$min\{\tfrac{-q_{01}}{q_{11}} l_0(0) - \tfrac{\sqrt{1+\mu(l_0(0))^2}}{\sqrt{q_{11}}}, \tfrac{-q_{01}}{q_{11}} u_0(0) - \tfrac{\sqrt{1+\mu(u_0(0))^2}}{\sqrt{q_{11}}}\},$$

$$u_1(0) =$$
$$max\{\tfrac{-q_{01}}{q_{11}} l_0(0) + \tfrac{\sqrt{1+\mu(l_0(0))^2}}{\sqrt{q_{11}}}, \tfrac{-q_{01}}{q_{11}} u_0(0) + \tfrac{\sqrt{1+\mu(u_0(0))^2}}{\sqrt{q_{11}}}\}.$$

At time $t = 1$, the encoder further quantizes r_0 by dividing the interval $[l_0(0), u_0(0)]$ into $2^{R_{01}}$ equal length intervals, and sends the representative of the new interval

[1]The encoder can compute $M_0 = |\{\Sigma_l\}_{11}\{U_l\}_{11}|$ from the SVD composition of $L = U_l \Sigma_l V_l^*$

[2]Our notation of $l_j(k)$ is the lower bound for component r_j right after time step k as computed by the decoder. The upper bound is similarly defined.

in which r_0 lies, denoted $\hat{r}_0(1)$. The encoder then uses the remainder R_{11} bits to quantize r_1 by dividing the updated interval $[l_1(1), u_1(1)]$ into $2^{R_{11}}$ intervals, and then sends the center of the interval in which r_1 falls. It is straightforward to compute

$$\begin{gathered} l_0(1) = \hat{r}_0(0) - 2^{-(R_{00}+R_{01})}M_0, \\ u_0(1) = \hat{r}_0(0) + 2^{-(R_{00}+R_{01})}M_0, \\ l_1(1) = \min\{\tfrac{-q_{01}l_0(1)}{q_{11}} - \tfrac{\sqrt{1+\mu l_0(1)^2}}{\sqrt{q_{11}}}, \tfrac{-q_{01}u_0(1)}{q_{11}} - \tfrac{\sqrt{1+\mu u_0(1)^2}}{\sqrt{q_{11}}}\}, \\ u_1(1) = \\ \max\{\tfrac{-q_{01}l_0(1)}{q_{11}} + \tfrac{\sqrt{1+\mu l_0(1)^2}}{\sqrt{q_{11}}}, \tfrac{-q_{01}u_0(1)}{q_{11}} + \tfrac{\sqrt{1+\mu u_0(1)^2}}{\sqrt{q_{11}}}\}. \end{gathered}$$

It is important to note that past allocation of bits to r_0 at time $t = 1$ allows for a tradeoff of a smaller known interval in which r_1 lies, $[l_1(1), u_1(1)]$, and finer quantization of the interval itself.

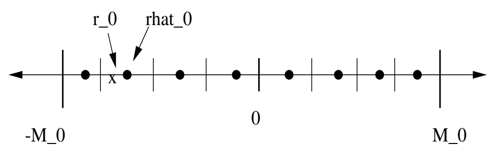

Fig. 4. Quantization region at time $t = 0$

Since $\mathcal{C}_r$ is an ellipsoid in $\mathbb{R}^T$, knowledge of r_j impacts the lower and upper bounds on r_k for $k \neq j$ Therefore, it appears that allocating bits to past signal components may be advantageous. It turns out however, that when $\mathcal{C}_r$ is any ellipsoid, it is always optimal in our worst-case setting to allocate all R bits to the current value r_k at time k, *i.e.*, it is *never* optimal to allocate bits to past values $r_0, r_1, ..., r_{k-1}$ to quantize r_k.

Theorem IV.1. *Consider the tracking problem that implements the causal coding scheme above parameterized by a rate matrix $\mathcal{R}$. Then, the optimal solution to $\min_{\mathcal{R}} \sup_{r \in \mathcal{C}_r} ||W(r - \hat{r})||_2^2$ is a diagonal rate matrix, $\mathcal{R}^* = RI$ and $r^* = 0$, the centroid of $\mathcal{C}_r$.*

Proof.

Here, we prove the theorem assuming L is diagonal for a simpler read. The general case holds and will be published at a later date.

Our argument is outlined below.

1) *We compute an upper bound, U, for $\sup_{r \in \mathcal{C}_r} ||W(r - \hat{r})||_2^2$.*
2) *We show that $r^* = \mathbf{0}$ achieves U and compute $||W(r^* - \hat{r^*})||_2^2$ as a function of the rate matrix.*
3) *We show that minimizing $\sup_{r \in \mathcal{C}_r} ||W(r - \hat{r})||_2^2 = ||W(r^* - \hat{r^*})||_2^2$ over all possible rate matrices gives rise to a diagonal rate matrix.*

Consider the set up for where L (and hence Q) is diagonal.

$$Q \triangleq (L^{-1})'(L^{-1}) = \begin{bmatrix} q_0 & & & \\ & q_1 & & \\ & & \ddots & \\ & & & q_{T-1} \end{bmatrix},$$

$$W'W = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0,T-1} \\ w_{01} & w_{11} & \dots & w_{1,T-1} \\ \vdots & & & \vdots \\ w_{T-1,1} & w_{T-1,1} & \dots & w_{T-1,T-1} \end{bmatrix}.$$

Then, $\mathcal{C}_r = \{r \in \mathbb{R}^2 | \sum_{i=0}^{T-1} q_i r_i^2 \leq 1\}$. Now, since we transmit the signal r in a causal manner, the class of signals forces $|r_i| \leq M_i$, where

$$M_i = \sqrt{\tfrac{1-\sum_{k=0}^{i-1} q_k r_k^2}{q_i}}, \quad i = 1, 2, , , T-1,$$

and $M_0 = |\{\Sigma_l\}_{11}\{U_l\}_{11}|$ from the SVD composition of $L = U_l \Sigma_l V_l^*$. At the decoder end, at time i, the transmitted signal $(r_0, r_1, ..., r_i)$ is not perfectly known. Therefore, it computes each bound as

$$M_i = \sqrt{\tfrac{1-\sum_{k=0}^{i-1} q_k(\hat{r}_k(i) - M_k 2^{-R_k(i)})^2}{q_i}},$$

where $R_k(i) = \sum_{j=k}^{i} R_{kj}$. Note that M_i is a function of $\hat{r}_0(i), \hat{r}_1(i), ..., \hat{r}_i(i)$ and $\mathcal{R}_i$, which is the first $i \times i$ elements of the full $T \times T$ rate matrix. We suppress these dependencies for an easier read. The above bound assumes, without loss of generality, the following encoder operator at time j:

$$E(r_i; j) = n_i M_i 2^{-R_i(j)}$$

$$n_i \in S_i \triangleq \{\pm 1, \pm 3, ..., \pm 2^{R_i(j)} - 1\},$$

for $(n_i - 1)M_i 2^{-R_i(j)} \leq r_i < (n_i + 1)M_i 2^{-R_i(j)}$. To see where the above expression for M_i comes from, consider a simple example where $T = 2$ and $R = 3$ illustrated in Figure 5. Given $\hat{r}_0(1)$ at $t = 1$, the uncertainty set as computed by the decoder for r shrinks from $\mathcal{C}_r$ to one of $2^{(R_{00}+R_{01})}$ "strips" of the 2D-ellipsoid. For example, for $r_0 \in [4M_0 2^{-(R_{00}+R_{01})}, 6M_0 2^{-(R_{00}+R_{01})})$, the uncertainty computed by the decoder is the shaded strip shown in Figure 5, and the corresponding $\hat{r}_0(1) = 5M_0 2^{-(R_{00}+R_{01})}$. As shown in Figure 5, the bound $M_1 = \sqrt{\frac{1-q_0(\hat{r}_0(1)-M_0 2^{(-(R_{00}+R_{01}))})^2}{q_1}}$.

Now, we can begin computing an upper bound on the cost function. Note that

$$||W(r - \hat{r})||_2^2 \leq \sum_{i=0}^{T-1} w_{ii}(M_i 2^{-R_{ii}})^2 + \sum_{i=0}^{T-1} \sum_{j=i+1}^{T-1} w_{ij}[M_i 2^{-R_{ii}}][M_j 2^{-R_{jj}}].$$

Then, the following is an upper bound on the cost.

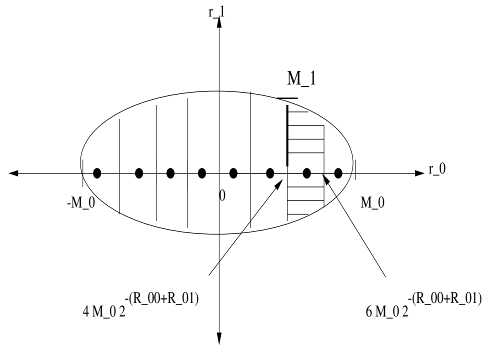

Fig. 5. Intervals for r_1 when $r_0 \in [4M_0 2^{-(R_{00}+R_{01})}, 6M_0 2^{-(R_{00}+R_{01})})$.

$$\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2 \le sup_{r\in\mathcal{C}_r} \sum_{i=0}^{T-1} w_{ii}(M_i 2^{-R_{ii}})^2$$
$$+2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T-1} w_{ij} M_i M_j 2^{-(R_{ii}+R_{jj})} \le$$
$$sup_{n_i\in S_i} \ \sum_{i=0}^{T-1} w_{ii}(\sqrt{\frac{1-\sum_{k=0}^{i-1} q_k((n_k-1)M_k 2^{-R_k(i)})^2}{q_i}}2^{-R_{ii}})^2$$
$$+2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T-1} w_{ij}\sqrt{\frac{1-\sum_{k=0}^{i-1} q_k((n_k-1)M_k 2^{(-R_k(i)})^2}{q_i}}\cdots$$
$$\sqrt{\frac{1-\sum_{k=0}^{j-1} q_k((n_k-1)M_k 2^{-R_k(j)})^2}{q_j}}2^{-(R_{ii}+R_{jj})}.$$

We rewrite the above inequality as

$$\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2 \le sup_{n_i\in S_i} \ \{\sum_{i=0}^{T-1} C_{1,i} i M_i^2(n_i) +$$
$$2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T-1} C_{2,ij} M_i(n_i) M_j(n_j)\}.$$

where $M_0(n_i) = |\{\Sigma_l\}_{11}\{U_l\}_{11}|$, *and* $M_i(n_i) = \sqrt{\frac{1-\sum_{k=0}^{i-1} q_k((n_k-1)M_k 2^{-R_k(i)})}{q_i}}$, *for* $i = 1, ..., T-1$, $C_{1,i} = w_{ii}2^{-2R_{ii}}$, *and* $C_{2,ij} = 2w_{ij}2^{-(R_{ii}+R_{jj})}$. *We can then take partial derivatives of* $f(n_0, ..., n_{T-1}) = \sum_{i=1}^{T-1} C_{1,i}M_i^2(n_i) + 2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T-1} C_{2,ij}M_i(n_i)M_j(n_j)$ *with respect to* n_i *assuming* n_i *is continuous for* $i = 0, 1..., T-1$, *and set each to zero to see if the corresponding* n_i^**'s lie in the corresponding sets* S_i*'s. When we take partial derivatives, we get*

$$\frac{\partial f}{\partial n_i} = 2\sum_{i=1}^{T-1} C_{1,i}M_i(n_i)\frac{\partial M_i(n_i)}{\partial n_i} + 2\sum_{j=i+1}^{T-1} C_{2,ij}\frac{\partial M_i(n_i)}{\partial n_i}M_j(n_j).$$

After some algebra, one can show that $n_i^* = 1$ *results in* $\frac{\partial f}{\partial n_i} = 0$ *and* $\frac{\partial^2 f}{\partial n_i^2}|_{n_i^*} < 0$, *for* $i = 0, 1, ..., T-1$. *Since the* $n_i^* \in S_i$, $\hat{r}_i^*(i) = M_i 2^{-R_i(i)}$, *and*

$$\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2 \le \sum_{i=0}^{T-1} w_{ii}(\frac{2^{-R_{ii}}}{\sqrt{q_i}})^2$$
$$+2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T-1} w_{ij}\sqrt{\frac{1}{q_iq_j}}2^{-(R_{ii}+R_{jj})} \triangleq U.$$

Finally, it is straightforward to show that when $r_i^* = 0$, *for* $i = 0, 1, ..., T-1$, *then* $\hat{r}_i^*(i) = M_i 2^{-R_i(i)}$, *and* $||W(r^* - \hat{r^*})||_2^2 = U$. *Therefore,* $\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2$ *for any given rate matrix occurs at the centroid of the ellipsoid.*

Now, we minimize $\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2$ *over all rate matrices as follows:*

$$\begin{array}{ll} min_{\mathcal{R}} & \sum_{i=1}^{T-1} w_{ii}\frac{2^{-2R_{ii}}}{q_i} + 2\sum_{i=0}^{T-1}\sum_{j=i+1}^{T-1} w_{ij}\frac{2^{-(R_{ii}+R_{jj})}}{q_iq_j} \\ & s.t. \ \sum_{i=0}^{j} R_{ij} = R \quad for\ j = 0, 1, ..., T-1 \\ & \quad R_{ij} > 0 \quad for\ all\ i, j. \end{array}$$

By inspection, the optimal rate matrix is diagonal.

■

One can show that past allocation may be useful for signal sets that are non-symmetric or finite or for different performance metrics. Also, if the encoder has access to the entire signal r at time $t = 0$, but still transmits R bits per time step, past allocation improves performance as shown in our navigation problem [13].

V. COMPARISON OF BOUNDS

We now compare the lower bound and causal upper bound to each other for different LTI causal systems $L = ss(A_l, B_l, C_l, D_l)$, and for different time horizons T. We consider diagonal weight matrices $W = diag(\lambda_0, \lambda_1, ..., \lambda_{T-1})$ with $|\lambda_i| \le 1, \ \forall i$, and fix the rate $R = 10$. Under such conditions, we note that $\gamma_{LB} = 2^{-2R}(D_l)^2\{\prod_{i=0}^{T-1}|\lambda_i|\}^{\frac{2}{T}}$.

Figure 6 illustrates the bounds for various scenarios, and we make a few observations.

- When the eigenvalues of W are chosen randomly from an i.i.d. process, then we see that the lower bound plateaus for large T. To see why this makes sense, one can show that the expected value of $\gamma_{LB} \to 2^{-2R}$ as T gets large and the variance of $\gamma_{LB} \to 0$ as T gets large. The causal upper bound increases as T increases.
- When the eigenvalues of W are exponentially decaying, *i.e.,* $\lambda_i = (\beta)^i$ for $i = 0, 1, ..., T-1$, and for some $0 < \beta < 1$, then the lower bound and noncausal upper bound approach 0 as $T \to \infty$. This can be verified by showing that the ratio $\frac{\gamma_{LB}(T+1)}{\gamma_{LB}(T)} = \frac{\{\prod_{i=0}^{T}\beta^i\}^{\frac{2}{T+1}}}{\{\prod_{i=0}^{T-1}\beta^i\}^{\frac{2}{T}}} = \beta < 1$. The causal upper bound increases but plateaus

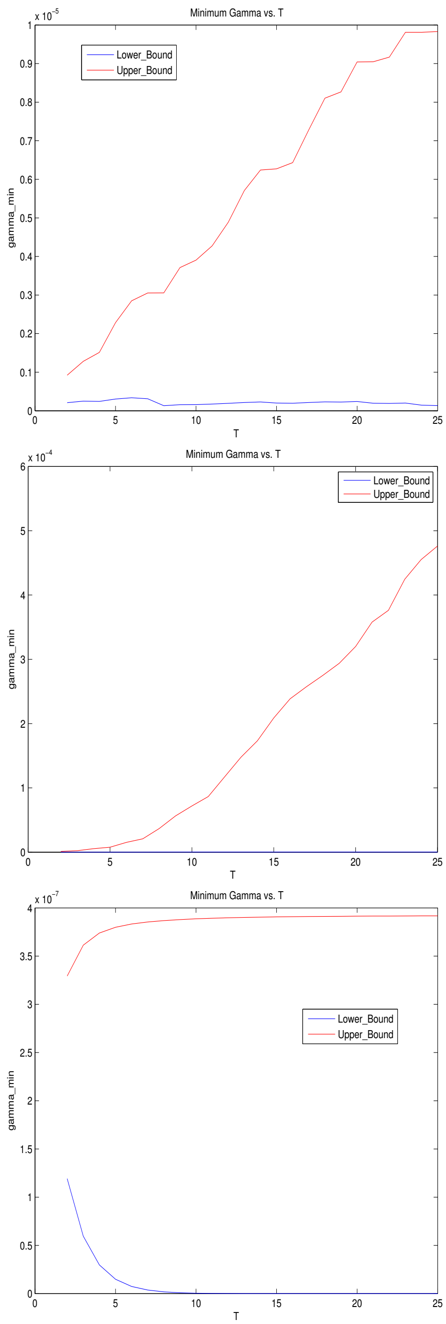

Fig. 6. Top: Bounds for $L = ss(0.01, 0.01, 1, 1)$ and random weights. Middle: Bounds for $L = ss(0.9, 0.9, 1, 1)$ and random weights. Bottom: Bounds for $L = ss(0.1, 0.1, 1, 1)$ and decaying weights.

for large T, but at a much slower rate when the pole of L is close to the unit disk.

- The causal upper bound is closer to the lower bound when the pole of the LTI system of L or that which generates a noncausal L is close to the origin than if the pole is close to the unit disk.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we compute finite-horizon tracking limitations under finite-rate control. We also show tradeoffs between time and performance accuracy and discuss how causality of coding impacts achievable performance. Future work entails computing finite-horizon tracking and navigation performance limitations for finite-capacity channels, *i.e.,* control over noisy finite-rate channels with and without feedback.

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