Forty-Fourth Annual Allerton Conference
Allerton House, UIUC, Illinois, USA
Sept 27-29, 2006

Finite-Rate Real-Time Navigation

Sridevi V. Sarma and Munther A. Dahleh

Abstract— **In this paper, we consider a set up in which the plant and controller are local to each other, but are together driven by a remote reference signal that is transmitted through a finite-rate noiseless channel. When control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals over the channel and achieving performance in finite-time. Most work in the area of control under communication constraints have addressed infinite-horizon control objectives (eg. stability, disturbance rejection). In this paper, we study a finite-horizon navigation problem. Our task is to navigate the state of the remote system from a nonzero initial condition, which lies in a bounded set, to as close to the origin as possible in finite-time. We compute lower and upper bounds on the worst-case performance as a function of the channel rate, time horizon, and system parameters. We achieve the lower bound with a noncausal coding scheme and show that imposing causality on the coding scheme degrades performance. We illustrate how the bounds behave under various scenarios and show tradeoffs between time and performance accuracy.**

I. Introduction

The classical control paradigm addresses problems where communication between one plant and one controller is essentially perfect. Today new problems in control over networked systems, whose components are connected via communication links that can be very noisy, induce delays, and have finite rate constraints, are emerging. Applications include remote navigation systems (deep-space and sea exploration) and multi-robot control systems (eg. aircraft and spacecraft formation flying control, coordinated control of land robots, control of multiple surface and underwater vehicles), where robots exchange data through communication channels that impose constraints on the design of coordination strategies.

In communication systems, problems entail designing channel encoders and decoders to reconstruct signals sent through noisy channels. Questions about *asymptotic* reconstruction are typically addressed. In control systems, problems often entail designing controllers to generate *real-time* desirable responses from a system. Therefore, when control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals and achieving performance in finite-time.

Control under communication constraints is a research area of growing interest. Much work has focused on stability under finite-rate (or countable) feedback control, where the only excitation to the system is an unknown (but bounded) initial state condition [2], [7], [10], [14], [15], [16], [18], [19], [4], [5], [3], [20], [22], [23]. The questions posed involve conditions on the channel rate that will guarantee that the state of the system (or some function of the state) approach the origin/remain bounded as time goes to infinity. More recently, disturbance rejection limitations were derived for the same setting, assuming stochastic exogenous signals entering the system [17]. Although these studies have contributed greatly to our understanding of the interplay between communication and control, few studies have addressed finite-horizon performance limitations under communication constraints.

A handful of recent studies explore the tradeoffs between finite-horizon performance and control complexity for linear systems and finite automata systems [6], [8], [11], [12]. In [12], Fagnani etal. consider the closed-loop system shown in Figure 1.

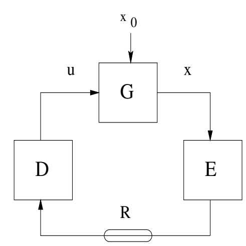

Fig. 1. Equivalent Closed-Loop System

G is a single-input multi-output discrete-time causal LTI system with unknown initial condition $x_0 \in \mathbb{R}^n$, which is a random vector with uniform probability density over a given bounded set $W \subset \mathbb{R}^n$. The feedback control law, $u \in \mathbb{R}$, must be generated over a finite-rate link that transmits exactly R bits per time step. Fagnani etal. ask the following question:

Given a subset V of W, find the minimum expected time, $E\{T_{(W,V)}\}$ that "traps" the state x_t in V for all $t \geq T$.

Fagnani etal. show that for any given $\beta > 0$,

$$\frac{E\{T_{(W,V)}\}}{ln(C)} \leq \beta \Rightarrow \frac{LN}{ln(C)} \geq \delta(\beta),$$

where $C = \frac{\mu[W]}{\mu[V]}$ (μ is the Lebesgue measure in $\mathbb{R}^n$) is a contraction rate that describes how small the target set is with respect to the starting set. L is a measure of the complexity of the coding scheme (E, D) and $\delta(\beta) = H_1 \beta w^{\frac{1}{\beta}}$, for some $w > 1$ and constant H_1, which depends on the plant dynamics. See [12] for details. This result shows that demanding smaller values of the expected minimum time to reach set V, results in requiring more complicated coding schemes.

In this paper, we first compute the smallest allowable ball around the origin that the state of the system can reach in

T time steps, given that the initial condition lies in an n-dimensional ellipsoid. We then construct a non-causal and-causal quantization/coding scheme to derive corresponding upper bounds, and achieve the lower bound with the non-causal scheme. Finally, we illustrate how imposing causality limits performance and show tradeoffs between time and performance accuracy. Our framework is deterministic and our lower bounds are independent of the complexity of the coding scheme.

II. SET UP & PROBLEM FORMULATION

In our set up, we assume that the remote system has some unknown initial condition x_0 which lies in a known bounded ellipsoid in $\mathbb{R}^n$. We want to steer the state of the remote system as close to the origin as possible under the constraint that the control input can take on at most 2^{RT} values after T time steps, *i.e.,* the command is transmitted through a finite-rate noiseless channel as shown in Figure 2.

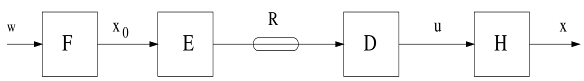

Fig. 2. Finite Horizon Navigation Set Up

Specifically,

- $w \in \mathbb{R}^n$ $s.t.$ $||w||_2 \leq 1$,
- $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear operator,
- $E : \mathbb{R}^n \rightarrow \{0,1\}^{RT}$ is an arbitrary operator (encoder) that maps a real vector to a sequence of 2^{RT} binary symbols,
- R is the channel rate for the finite-rate noiseless channel that maps $\{0,1\}^{RT} \rightarrow \{0,1\}^{RT}$,
- $D : \{0,1\}^{RT} \rightarrow \mathbb{R}^T$ is an arbitrary operator (decoder) that maps a sequence of 2^{RT} binary symbols to a real vector, and
- $H\mathbb{R}^T \rightarrow \mathbb{R}^n$ is a causal SISO LTI system with state-space representation $H = ss(A, B, I, 0)$ with (A, B) reachable and A is full rank.

We investigate two performance metrics for our navigation problem. The first metric fixes the horizon T and looks for the smallest ball around the origin that the state can reach. In particular, we compute lower an upper bounds such that

$$\gamma_{LB} \leq \min_{(E,D)} \ \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2 \leq \gamma_{UB},$$

where $\mathcal{C}_{x_0} \triangleq \{x_0 \in \mathbb{R}^n | x_0 = Fw, w'w \leq 1\} = \{x_0 \in \mathbb{R}^n \mid (F^{-1}x_0)'(F^{-1}x_0) \leq 1\}$. Knowledge of γ_{LB} tells us that regardless of the encoder and decoder that we select, we can do no better than this lower bound. Therefore, we expect it to be independent of E and D. The upper bound tells us that there exists a coding scheme (and encoder an decoder) such that the worst case performance is always less than or equal to γ_{UB}. Therefore, to compute γ_{UB}, we need to construct an encoder and decoder and compute the corresponding worst-case performance.

The second metric fixes γ and looks for the minimum time it takes for the state vector to reach a ball of size γ. Therefore, we fix γ and then look for T_{min} such that $\min_{(E,D)} \ \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2 \leq \gamma$ for all $T \geq T_{min}$.

Before computing γ_{LB} and γ_{UB}, it is worth commenting that in the ideal case of perfect communication ($R = \infty$), it is possible to construct an encoder and decoder ($E = F^{-1}$ and $D = H^{-1}$) such that $||x_T||_2^2 = 0 \ \ \forall x_0 \in \mathcal{C}_{x_0}$ and for all $T \geq n$. However, with a finite-rate R, the control, u, can only take 2^{RT} values over a horizon of T time steps. Therefore, it is not clear what level of performance is achievable. We set out to quantify achievability.

III. A LOWER BOUND

In this section we derive the lower bound on worst-case navigation performance.

Theorem 3.1: Consider a reachable SISO LTI causal DT system, $H = ss(A, B, I, 0)$ where A is full-rank and with initial condition $x_0 \in \{x \in \mathbb{R}^n | x = Fw, w \in \mathbb{R}^n, w'w \leq 1\}$, for a given full-rank F. If the control input is constrained to take on at most 2^{RT} values after $T \geq n$ time steps, then $||x_T||_2^2 \leq \gamma$ only if

$$\gamma \geq 2^{-2RT/n}\{|det(F)| \ |det(A^T)|\}^{\frac{2}{n}}.$$

Proof.

We first observe that the set of all possible initial conditions, $\mathcal{C}_{x_0} \triangleq \{x_0 \in \mathbb{R}^n | x_0 = Fw, w'w \leq 1\} = \{x_0 \in \mathbb{R}^n \mid (F^{-1}x_0)'(F^{-1}x_0) \leq 1\}$. $\mathcal{C}_{x_0}$ is a bounded ellipsoid in $\mathbb{R}^n$ centered at the origin with volume $\eta \ det\{((F^{-1})'(F^{-1}))^{-0.5}\} = \eta |det(F)|$, where η is the volume of a unit ball in $\mathbb{R}^n$. Since A is full rank, the set created by multiplying x_0 on the left by A^T is also a bounded ellipsoid in $\mathbb{R}^n$. Denote this new set as $\mathcal{C}_{A^T x_0} \triangleq \{y \in \mathbb{R}^n | y = A^T x_0, x_0 \in \mathcal{C}_{x_0}\}$.

Next, note that $||x_T||_2^2 = ||A^T x_0 + Mu||_2^2$, where A^T is the $(T)^{th}$ power of the matrix A, and $M = \begin{bmatrix} A^{T-2}B & A^{T-3}B & \ldots & AB & B \end{bmatrix}$ is the reachability matrix of system H. Since M is full row rank, one can pick any value $y \in \mathbb{R}^n$ and find a $u \in \mathbb{R}^T$ such that $y = Mu$. Over a horizon T, the channel sends a total of RT bits which limits the control signal to take on no more than 2^{RT} values. Therefore, $y = Mu$ can only take on 2^{RT} values.

Consider a selection of signals $y_1, y_2, ..., y_{2^{RT}}$, which correspond to inputs $u_1, u_2, ..., u_{2^{RT}}$, respectively. We must then map each $A^T x_0 \in \mathcal{C}_{A^T x_0}$ to exactly one y_i $i = 1, 2, ..., 2^{RT}$. Such a mapping induces a partition on $\mathcal{C}_{A^T x_0}$. In particular, define $P_i = \{v \in \mathcal{C}_{A^T x_0} | \ v \rightarrow y_i\}$ for $i = 1, 2, ..., 2^{RT}$.

Now, suppose that the selection $y_1, y_2, ..., y_{2^{RT}}$ were chosen such that $||A^T x_0 + y_i||_2^2 \leq \gamma$ for all $A^T x_0 \in P_i$, and for all i. Then necessarily $P_i \subseteq \mathcal{B}_\gamma$, where $\mathcal{B}_\gamma$ is a ball in $\mathbb{R}^n$ of size γ with volume $\sqrt{\gamma}^n$.

Since $P_i \subseteq \mathcal{B}_\gamma$ for each $i = 1, 2, ..., 2^{RT}$, it is necessary that 2^{RT} γ-balls cover the set $\mathcal{C}_{A^T x_0}$. This implies that $2^{RT} \times volume(\mathcal{B}_\gamma) \geq volume(\mathcal{C}_{A^T x_0})$. Equivalently,

$$2^{RT} \geq \frac{volume(\mathcal{C}_{x_0})}{volume(\mathcal{B}_\gamma)} = \frac{|det(F)| \ |det(A^T)|}{(\sqrt{\gamma})^n}.$$

After rearranging terms, we get that $\gamma \geq 2^{-2RT/n}\{|det(F)| \ |det(A^T)|\}^{\frac{2}{n}}$.

■

Since we often consider classes of inputs generated from LTI systems, *i.e.,* F is LTI, we compute the lower bound for this case in the following corollary.

Corollary 3.1: Given the navigation set up defined above, if F is generated by a causal SISO LTI system with state-space description $ss(A_f, B_f, C_f, D_f)$, then

$$||x_T||_2^2 \geq 2^{-2RT/n}(D_f)^2\{|det(A^T)|\}^{\frac{2}{n}}.$$

Proof. If F is generated by a SISO causal LTI system with state-space description $F = ss(A_f, B_f, C_f, D_f)$, then it can be represented as a $n \times n$ lower triangular Toeplitz matrix operator, with all n eigenvalues equal to D_f. This implies that the $\{det(F)\}^{\frac{2}{n}} = (D_f)^2$. ■

We now comment on γ_{LB}.

- γ_{LB} depends on F (class of initial conditions), n (the dimension of the system state), T (performance horizon), and R (channel rate).
- If A is stable, then necessarily the lower bound approaches 0 and T grows large since A^T is approaching 0. In this case, for large T one does not need to apply a control input u and the state will approach the origin.
- If $det(A)$ and/or if $det(F) = 0$, then the counting argument shown in Theorem 3.1 has to be done in $\mathbb{R}^s$, where $s = \min\{rank(F), rank(A)\}$. Consider a case where $A = diag(\lambda_0, \lambda_1, ..., \lambda_{n-1})$, and $det(A) = 0$ because $\lambda_{k_0} = 0$, for some $0 \leq k_0 \leq n-1$. Then, 0 bits can to be applied in the k-th dimension of x_0 as it will not impact the state output along that dimension. Therefore, the problem reduces to allocating bits to $x_0(k)$ for all $k \neq k_0$. On the other hand, if $det(F) = 0$, then one or more of the components of x_0 are linear combinations of each other, and bits only need to be allocated to some components of x_0, and the decoder can reconstruct the others knowing F.
- It is helpful (as we will see when we compute upper bounds) to rewrite the lower bound in terms of the singular and eigenvalues of the matrix $A^T F$ as follows:

$$\gamma_{LB} = 2^{-2RT/n}\{\prod_{i=0}^{n-1} \sigma_i(A^T F)\}^{\frac{2}{n}} = 2^{-2RT/n}\{\prod_{i=0}^{n-1} |\lambda_i(A^T F)|\}^{\frac{2}{n}}.$$

A. Causality

When computing the lower bound, we made no assumptions on whether the encoder and decoder are causal or noncausal. If E and D are noncausal, then the navigation problem essentially reduces to a vector quantization problem where time need not enter the picture [1]. At time $t = 0$, the decoder "knows" the future, that is it knows u_k for $k = 0, 1, ..., T-1$, which are represented by TR bits over a horison of T steps. On the other hand, if E and D are causal, then the decoder only knows u_k for $k = 0, 1, ..., T-1$, only at time $t \geq k$, and u_k is represented by at most $(k+1)R$ bits.

In the following sections, we compute two upper bounds. One bound is computed by constructing a noncausal encoder and decoder, and the second upper bound is computed by constructing a causal coding scheme, which is more practical.

IV. A Noncausal Upper Bound

In this section, we derive an upper bound on worst-case performance assuming that the encoder and decoder are noncausal. The upper bound is derived using a coding scheme that transmits information about the vector x_0 in terms of a basis derived from the singular value decomposition (SVD) of the matrix $A^T F$.

Consider Figure 3 below. The encoder first uses the SVD of $A^T F = U\Sigma V^*$ to write $A^T x_0 = A^T F w = \sum_{i=0}^{n-1} \sigma_i \alpha_i u_i$, where σ_i is the ith singular value of $A^T F$, $\alpha_i = v_i^* w$ where v_i^* is the ith row vector of V^*, and u_i is the ith column vector of U. The α_i's are then each converted into their binary representations and truncated according to the bit-allocation strategy denoted in $\mathcal{R} = (R_1, R_2, ..., R_n)$. In particular, a total of R_k bits are allocated to α_k, for $k = 1, ..., n$, and the only restriction is that $\sum_{k=1}^n R_k = TR$.

The decoder uses the bit-allocation strategy $\mathcal{R}$ to reconstruct α and then uses the SVD of $A^T F$ to compute $\hat{x}_0$ from $\hat{\alpha}$. Finally, the decoder computes the minimum 2-norm control signal u such that $A^T \hat{x}_0 + Mu = 0$, which is the least squares solution to an under-constrained set of linear equations. We call this $E-D$ construction the "SVD Coding Scheme."

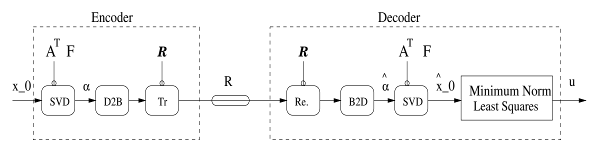

Fig. 3. SVD Coding Scheme

Note that with the above SVD coding scheme,

$$\begin{aligned}\sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2 &= \sup_{x_0 \in \mathcal{C}_{x_0}} ||A^T(\hat{x}_0 - x_0)||_2^2 \\ &= \sup_{\{w| \ ||w||_2 \leq 1\}} ||A^T F(\hat{w} - w)||_2^2 \\ &= \sup_{\{\alpha| \ ||\alpha||_2 \leq 1\}} \\ &\sum_{i=1}^n \sum_{j=1}^n (\hat{\alpha}_i - \alpha_i)(\hat{\alpha}_j - \alpha_j)\sigma_i\sigma_j(u_i' u_j) \\ &\leq \sup_{\{\alpha| \ ||\alpha|_2 \leq 1\}} \sum_{i=1}^n |\alpha_i|^2 2^{-2R_i}\sigma_i^2 \\ &= \max_i 2^{-2R_i}\sigma_i^2.\end{aligned}$$

To derive the upper bound using the above SVD coding scheme, we construct $\mathcal{R} = (R_1, R_1, ..., R_n)$ to solve the following optimization problem:

$$\begin{aligned}&\min_{\mathcal{R}} \max_i 2^{-2R_i}\sigma_i^2 \\ &\text{s.t. } \sum_{i=1}^n R_i = TR \\ &R_i \geq 0 \quad \forall i.\end{aligned}$$

We allow the rates to take on non-integer values to solve for an optimal bit-allocation strategy. The resulting non-integer valued rates can be interpreted as average rates over time. The above problem is computable and it is easy to verify (using Lagrange multipliers) that the optimal solution is $R_i^* = (\frac{RT}{n} - \frac{1}{n}\sum_{i=1}^n log_2(\sigma_i)) + log_2(\sigma_i) \quad \forall i$. Therefore, the larger the singular value σ_i, the more bits are allocated to α_i. It is straightforward to plug in R_i^* back into the cost $\max_i 2^{-2R_i}\sigma_i^2$ to show that the resulting upper bound achieves the lower bound!

V. A Causal Upper Bound

In this section, we derive an upper bound by constructing a modified SVD Coding Scheme, where the encoder has access to the entire vector $x_0 \in \mathcal{C}_{x_0}$ at time $t = 0$, but the decoder is restricted to only process R bits of information per time step. This is a more practical implementation.

The causal coding scheme transmits information about x_0 in terms of a basis derived from the singular value decomposition (SVD) of the matrix F. Consider Figure 4. Here, the encoder first uses the SVD of $F = U\Sigma V^*$ to write $x_0 = \sum_{i=0}^{n-1} \sigma_i \alpha_i u_i$, where σ_i is the ith singular value of F, $\alpha_i = v_i^* w$ where v_i^* is the ith row vector of V^*, and u_i is the ith column vector of U. The α_i's are then each converted into their binary representations and truncated according to the bit-allocation strategy dictated by a $T \times n$ rate matrix

$$\mathcal{R} = \begin{bmatrix} R_{01} & R_{12} & & \dots & R_{0n} \\ R_{11} & R_{12} & & & R_{1n} \\ R_{21} & R_{22} & & & R_{2n} \\ \vdots & \vdots & & & \vdots \\ R_{T-1,1} & R_{T-1,2} & R_{T-1,3} & \dots & R_{T-1,n} \end{bmatrix},$$

such that $\sum_j R_{ij} = R$ for $i = 0, 1, ..., T-1$. Let $R_i(t) = \sum_{j=0}^{t} R_{ji}$ for $i = 1, ..., n$ and $t = 0, 1, ..., T-1$. Then, a total of $R_i(t)$ bits are allocated to α_i, for $i = 1, ..., n$ at time t.

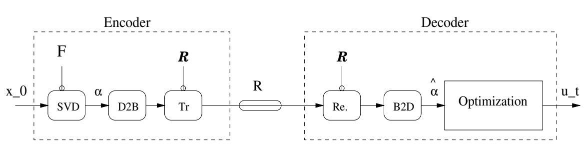

Fig. 4. SVD Coding Scheme for Navigation

At time t, the decoder uses the bit-allocation strategy $\mathcal{R}$ to reconstruct α as $\hat{\alpha}(t)$. The decoder then computes the scalar control input u_t that minimizes $||x_t||_2$, by solving the following optimization problem

$$\min_{u_t} \ \sup_{\alpha \in S_t} ||AU\Sigma\alpha + \textstyle\sum_{i=0}^{t-2} A^{t-1-i}Bu_i + Bu_t||_2$$
$$\text{s.t.}$$
$$S_t = \{\alpha \in \mathbb{R}^n |\ |\alpha_i - \hat{\alpha}_i(t)| \le 2^{-R_i(t)} \quad i = 1, 2, ..., n\}.$$

The solution to the above min-sup problem can be computed exactly. First, we use the property that the supremum of a convex function over a bounded interval occurs at the boundaries, which gives us that $\alpha^*(t)$, the solution to $\sup_{\alpha \in S_t} ||AU\Sigma\alpha + \sum_{i=0}^{t-2} A^{t-1-i}Bu_i + Bu_t||_2$, as one of the following vectors:

$$\alpha^*(t) = \begin{bmatrix} \hat{\alpha}_1(t) \pm 2^{-R_1(t)} \\ \hat{\alpha}_2(t) \pm 2^{-R_2(t)} \\ \ddots \\ \hat{\alpha}_n(t) \pm 2^{-R_n(t)} \end{bmatrix}.$$

Then, we minimize $||AU\Sigma\alpha^*(t) + \sum_{i=0}^{t-2} A^{t-1-i}Bu_i + Bu_t||_2$ by taking its derivative with respect to u_t and then setting it to 0. We get that the optimal control input at time t is $u_t^* = \frac{(AU\Sigma\alpha^*(t) + \sum_{i=0}^{t-2} A^{t-1-i}Bu_i)'B}{(B'B)}$. Finally, we show in [21] that $sup_{\mathcal{C}_{x_0}} ||x_T||_2$ occurs at $x_0 = 0$ and correspondingly $\alpha = 0$, which gives

$$\alpha^*(t) = \begin{bmatrix} 2^{-R_1(t)} \\ 2^{-R_2(t)} \\ \ddots \\ 2^{-R_n(t)} \end{bmatrix}.$$

Therefore, the decoder computes $u_t^* = \frac{(AU\Sigma\alpha^*(t) + \sum_{i=0}^{t-2} A^{t-1-i}Bu_i)'B}{(B'B)}$ using the above $\alpha^*(t)$.

The critical difference between the above coding scheme and that introduced in section IV is the restriction of allocating a total of R bits each time step, which is captured by the rate matrix.

VI. Comparison of Bounds

We now compare the lower and upper navigation bounds on γ to each other for different LTI causal systems $H = ss(A, B, C, D)$, and for different time horizons T. We consider diagonal 4×4 ($n = 4$) state-transition matrices $A = diag(a_0, a_1, a_2, a_3)$, an F that is generated by LTI system $ss(A_f, B_f, C_f, D_f)$, and we fix the rate $R = 5$. Under such conditions,

$$\gamma_{LB} = 2^{-2RT/n}(D_f)^2\{\textstyle\prod_{i=0}^{n-1} |a_i|^T\}^{\frac{2}{n}}.$$

Figures 5 and 6 illustrate the bounds for the following scenarios.

1) $F = ss(0.99, 0.99, 1, 1)$, $A = diag([0.2\ 0.8\ 0.9\ 0.8])$, and $B = [1\ 1\ 1\ 1]'$.
2) $F = ss(0.01, 0.01, 1, 1)$, and A and B are the same matrices as those generated in 1.
3) $F = ss(0.99, 0.99, 1, 1)$, $A = diag([1.2\ 1.8\ 1.9\ 1.8])$, and $B = [1\ 1\ 1\ 1]'$.
4) $F = ss(0.01, 0.01, 1, 1)$, and A and B are the same matrices as those generated in 3.

We make a few observations.

- *Stability of A*: All bounds decay when A is stable as T grows. When A is unstable, then the causal upper bound only decays when the pole of the system that generates F is closer to the unit disk.
- *Pole of F*: When the pole of the system that generates F is closer to the origin, the singular values of F are all comparable and therefore using the SVD basis to represent x_0 in the causal coding scheme is less helpful. Therefore, we expect causal coding performance to deteriorate, which it does. Put another way, when F has a pole close to the unit disk, then the ellipsoid set $\mathcal{C}_{x_0}$ has more structure, that is knowing some components of x_0 give alot of information about the remaining component of x_0. When the pole of F is closer to 0, then $\mathcal{C}_{x_0}$ looks more and more like an n-dimensional sphere.

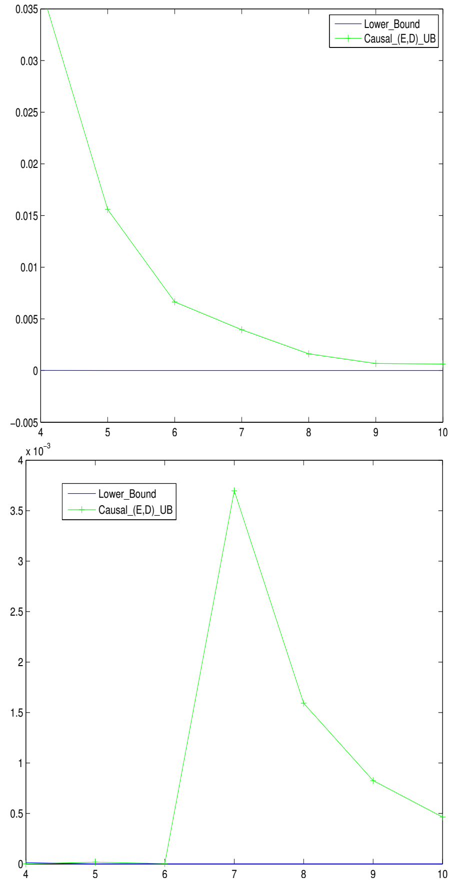

Fig. 5. Left: Bounds for $F = ss(0.99, 0.99, 1, 1)$ and A stable, Right: Bounds for $F = ss(0.01, 0.01, 1, 1)$ and A stable

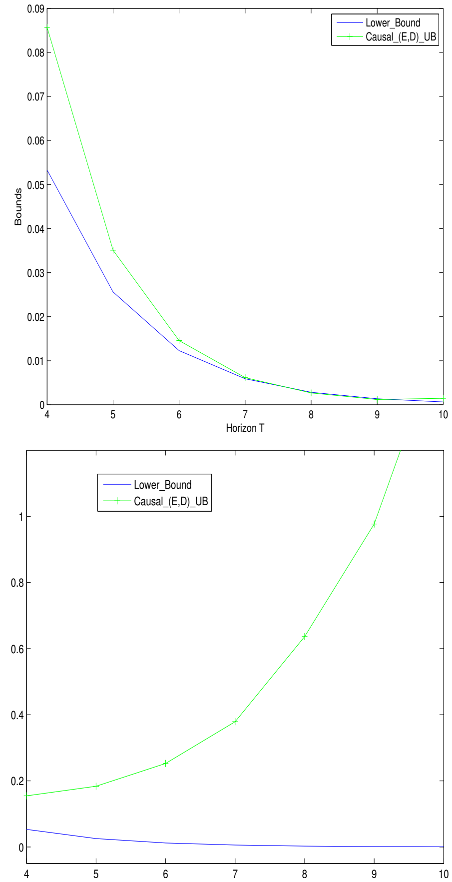

Fig. 6. Left: Bounds for $F = ss(0.99, 0.99, 1, 1)$ and A unstable, Right: Bounds for $F = ss(0.01, 0.01, 1, 1)$ and A unstable,

Finally, we note that when the encoder has access to the entire vector x_0 at time 0, it can allocate R bits to any of the n components of x_0 each time step. If for some reason, as is the case when the input to the encoder is a signal, r, that must be tracked by the remote system, then the encoder only has acces to $r_0, r_1, ..., r_t$ at time step t. Therefore, it must allocate R bits to any of the components in the set $r_0, r_1, ..., r_t$. This problem is studied in [21].

VII. A Lower Bound on Time Horizon

In this section, we seek to minimize the time it takes for the state vector to reach a ball of size γ. Therefore, we fix γ and then look for the smallest horizon T to meet performance.

Theorem 7.1: Consider a reachable SISO LTI causal DT system, $H = ss(A, B, I, 0)$ with A being full rank and with initial condition $x_0 \in \{x \in \mathbb{R}^n | x = Fw, w \in \mathbb{R}^n, w'w \leq 1\}$, for a given F. If the control input is constrained to take on at most 2^{RT} values after $T \geq n$ time steps, then $||x_T||_2^2 \leq \gamma$ (for any given $\gamma > 2^{-2R}|det(A)|^2$) only if

$$T \geq max(\left|\frac{2(log_2(|det(F)|) - log_2(|det(A)|)}{log_2(\gamma) + 2R - 2log(|det(A)|)}\right|, n).$$

Proof.

As shown in the proof for Theorem 3.1, an equivalent expression for navigation performance $||Mu + A^T Fw||_2^2 \leq \gamma$ is $||U_m\alpha + A^T Fw||_2^2 \leq \gamma$. We apply the same counting argument as that given in the proof for Theorem 3.1, but isolate T on one side of the inequality instead of γ. We then get that the lower bound on the number of time steps as $\left|\frac{2(log_2(|det(F)|) - log_2(|det(A)|)}{log_2(\gamma) + 2R - 2log(|det(A)|)}\right|$ as long as $log_2(\gamma) + 2R - 2log(|det(A)|) > 0$, which is equivalent to $\gamma > 2^{-2R}|det(A)|^2$. Finally, since we assumed that $T \geq n$ (M full row rank), this lower bound is only valid if it is greater than n. If $\left|\frac{2(log_2(|det(F)|) - log_2(|det(A)|)}{log_2(\gamma) + 2R - 2log(|det(A)|)}\right| < n$, then it is possible to reach a ball of size γ in n time steps.

■

Note that the lower bound depends on R, F, γ, and the system dynamics A.

VIII. Conclusions & Future Work

In this paper, we compute finite-horizon navigation limitations under finite-rate control. We also show tradeoffs between time and performance accuracy. Future work entails computing finite-horizon performance limitations for finite-capacity channels, *i.e.,* control over noisy finite-rate channels.

IX. Acknowledgements

The authors would like to thank Alexander Megretski for insightful discussions.

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