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FINITE-RATE CONTROL: FINITE-HORIZON PERFORMANCE LIMITATIONS

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ABSTRACT

In this paper, we consider a set up in which the plant and controller are local to each other, but are together driven by a remote reference signal that is transmitted through a finite-rate noiseless channel. When control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals over the channel and achieving performance in finite-time. Most work in the area of control under communication constraints have addressed infinite-horizon control objectives (eg. stability, disturbance rejection). In this paper, we compute lower and upper bounds on worst-case performance for a *finite-horizon tracking* objective. We achieve the lower bound with a noncausal coding scheme and show that imposing causality on the coding scheme severely limits achievable performance. We illustrate how the bounds behave under various scenarios and show tradeoffs between time and performance accuracy.

1. INTRODUCTION

The classical control paradigm addresses problems where communication between one plant and one controller is essentially perfect. Today new problems in control over networked systems, whose components are connected via communication links that can be very noisy, induce delays, and have finite rate constraints, are emerging. Applications include remote navigation systems (deep-space and sea exploration) and multi-robot control systems (eg. aircraft and spacecraft formation flying control, coordinated control of land robots, control of multiple surface and underwater vehicles), where robots exchange data through communication channels that impose constraints on the design of coordination strategies.

In communication systems, problems entail designing channel encoders and decoders to reconstruct signals sent through noisy channels. Questions about *asymptotic* reconstruction are typically addressed. In control systems, problems often entail designing controllers to generate *real-time* desirable responses from a system. Therefore, when control must be done over a communication channel, there is a fundamental tradeoff between allowing enough time for reconstruction of signals and achieving stability and performance in finite-time.

Control under communication constraints is a research area of growing interest. Much work has focused on stability under finite-rate (or countable) feedback control, where the only excitation to the system is an unknown (but bounded) initial state condition [1, 6, 8, 12, 13, 14, 16, 17, 3, 4, 2, 19, 21, 22]. The questions posed involve conditions on the channel rate that will guarantee that the state of the system (or some function of the state) approach the origin/remain bounded as time goes to infinity. More recently, disturbance rejection limitations were derived for the same setting, assuming stochastic exogenous signals entering the system [15]. Although these studies have contributed greatly to our understanding of the interplay between communication and control, few studies (see [5, 7, 9, 10]) address finite-horizon performance limitations under communication constraints.

In [10], Fagnani etal. consider the closed-loop system shown in Figure 1.

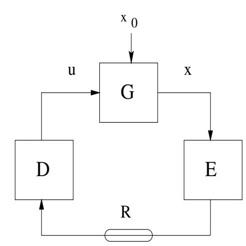

Fig. 1. Equivalent Closed-Loop System

G is a single-input multi-output discrete-time causal LTI system with unknown initial condition $x_0 \in \mathbb{R}^n$, which is a random vector with uniform probability density over a given bounded set $W \subset \mathbb{R}^n$. The feedback control law, $u \in \mathbb{R}$, must be generated over a finite-rate link that transmits exactly R bits per time step. Fagnani etal. ask the following question:

Given a subset V of W, find the minimum expected time, $E\{T_{(W,V)}\}$ that "traps" the state x_t in V for all $t \geq T$.

Fagnani etal. show that for any given $\beta > 0$,

$$\frac{E\{T_{(W,V)}\}}{ln(C)} \leq \beta \Rightarrow \frac{LN}{ln(C)} \geq \delta(\beta),$$

where $C = \frac{\mu[W]}{\mu[V]}$ (μ is the Lebesgue measure in $\mathbb{R}^n$) is a contraction rate that describes how small the target set is

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with respect to the starting set. L is a measure of the complexity of the coding scheme (E, D) and $\delta(\beta) = H_1 \beta w^{\frac{1}{\beta}}$, for some $w > 1$ and constant H_1, which depends on the plant dynamics. See [10] for details. This result shows that demanding smaller values of the expected minimum time to reach set V, results in requiring more complicated coding schemes.

In this paper we compute lower and upper bounds for finite-horizon tracking objective under finite-rate feedforward control. Specifically, we compute the smallest allowable worst-case performance over a class of reference signals. We also construct quantization/coding schemes to derive upper bounds on performance, and illustrate how imposing causality on the quantizer limits achievable performance. Our framework is deterministic and our lower bounds are independent of the complexity of the coding scheme.

2. PROBLEM FORMULATION

In this section, we are interested in tracking a class of reference commands, r, over a finite-horizon and under finite-rate constraints. We consider the cascade of SISO discrete-time systems shown in Figure 2.

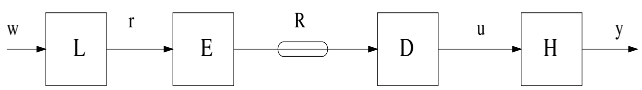

Fig. 2. Finite Horizon Tracking Set Up

Specifically,

- $w \in \mathbb{R}^T \; s.t. \; ||w||_2 \leq 1$,
- $L : \mathbb{R}^T \to \mathbb{R}^T$ is an invertible linear operator,
- $E : \mathbb{R}^T \to \{0,1\}^{RT}$ is an arbitrary operator (encoder) that maps a real vector to a sequence of 2^{RT} binary symbols,
- R is the channel rate for the finite-rate noiseless channel that maps $\{0,1\}^{RT} \to \{0,1\}^{RT}$,
- $D : \{0,1\}^{RT} \to \mathbb{R}^T$ is an arbitrary operator (decoder) that maps a sequence of 2^{RT} binary symbols to a real vector, and
- $H : \mathbb{R}^T \to \mathbb{R}^T$ is an invertible linear operator (model of remote plant and controller system at rest).

Note that L defines a class of signals, $\mathcal{C}_r$, that is generated from a unit ball in $\mathbb{R}^T$. Since L is linear, it maps the unit ball to a bounded ellipsoid (see Theorem 3.1 for details). We set out to minimize a tracking error over all signals, r, in this class (worst-case analysis). Since the input and output signals have finite length, the following performance metric is computed over a finite-horizon: $||W(y - r)||_2^2$, where $W \in \mathbb{R}^T \times \mathbb{R}^T$ is a given full-rank weight matrix.

It is worth commenting that in the ideal case of perfect communication ($R = \infty$), it is possible to construct an encoder and decoder ($E = L^{-1}$ and $D = H^{-1}$) such that $||W(y - r)||_2^2 = 0 \;\; \forall r \in \mathcal{C}_r$. However, with a finite-rate constraint, the control, u, can only take 2^{RT} values over a horizon of T time steps. Furthermore, with H being a one-to-one mapping, the output, y, can only take 2^{RT} values over a horizon of T time steps. Therefore, it is not clear what level of performance is achievable over $\mathcal{C}_r$.

To understand tracking limitations under finite-rate feedforward control, we compute γ_{LB} and γ_{UB}, such that

$$\gamma_{LB} \leq \min_{(E,D)} \; \sup_{r \in \mathcal{C}_r} ||W(y - r)||_2^2 \leq \gamma_{UB}.$$

Knowledge of γ_{LB} tells us that regardless of the encoder and decoder that we select, we can do no better than this lower bound. Therefore, we expect it to be independent of E and D. The upper bound tells us that there exists a coding scheme such that the worst case performance is always less than or equal to γ_{UB}. Therefore, to compute γ_{UB}, we need to construct an encoder and decoder and compute the corresponding worst-case performance. We compute γ_{LB} and γ_{UB} in the following sections.

3. A LOWER BOUND

In this section we derive the lower bound on worst-case performance.

Theorem 3.1. *Given the tracking set up defined above,*

$$\gamma_{LB} = 2^{-2R} \{|det(L)| \; |det(W)|\}^{\frac{2}{T}}.$$

Proof.

The set of all possible commands, $\mathcal{C}_r \triangleq \{r \in \mathbb{R}^T | r = Lw, w'w \leq 1\} = \{r \in \mathbb{R}^T \mid (L^{-1}r)'(L^{-1}r) \leq 1\}$. $\mathcal{C}_r$ is a bounded ellipsoid in $\mathbb{R}^T$ centered at the origin with volume $\eta \; det\{((L^{-1})'(L^{-1}))^{-0.5}\} = \eta |det(L)|$, where η is the volume of a unit ball in $\mathbb{R}^T$.

Over a horizon T, the channel sends a total of RT bits which limits the control signal to take on no more than 2^{RT} values; and, since H is a one-to-one mapping, the channel limits the output to take on no more than 2^{RT} values. Consider a selection of outputs $y_1, y_2, ..., y_{2^{RT}}$, which correspond to inputs $u_1, u_2, ..., u_{2^{RT}}$, respectively. We must then map each $r \in \mathcal{C}_r$ to exactly one y_i $i = 1, 2, ..., 2^{RT}$. Such a mapping induces a partition on $\mathcal{C}_r$. In particular, define $P_i = \{r \in \mathcal{C}_r | \; r \to y_i\}$ for $i = 1, 2, ..., 2^{RT}$. Now, suppose that the selection $y_1, y_2, ..., y_{2^{RT}}$ were chosen such that $||W(y_i - r)||_2^2 \leq \gamma$ for all $r \in P_i$, and for all i. Then necessarily $P_i \subseteq S_{y_i}^{\gamma} \triangleq \{r \in \mathbb{R}^T \mid (r - y_i)'W'W(r - y_i) \leq \gamma\}$. Note that $S_{y_i}^{\gamma}$ is a bounded ellipsoid in $\mathbb{R}^T$ centered at point y_i with volume $\eta(\sqrt{\gamma})^T det\{(W'W)^{-0.5}\} = \frac{\eta \sqrt{\gamma}^T}{|det(W)|}$. See Figure 3 for an illustration.

Since $P_i \subseteq S_{y_i}^{\gamma}$ for each $i = 1, 2, ..., 2^{RT}$, it is necessary that 2^{RT} bounded ellipsoids (S_y^{γ}) cover the set $\mathcal{C}_r$. This implies that $2^{RT} \times volume(S_y^{\gamma}) \geq volume(\mathcal{C}_r)$. Equivalently,

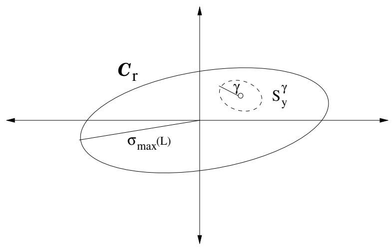

Fig. 3. Bounded Ellipsoids $\mathcal{C}_r$ and S_y^γ

$$2^{RT} \geq \frac{volume(\mathcal{C}_r)}{volume(S_y^\gamma)} = \frac{|det(L)|\ |det(W)|}{(\sqrt{\gamma})^T}.$$

After rearranging terms, we get that $\gamma \geq 2^{-2R}\{|det(L)|\ |det(W)|\}^{\frac{2}{T}}$. ■

Since we often consider classes of inputs generated from LTI systems, *i.e.,* L is LTI, we compute the lower bound for this case in the following corollary.

Corollary 3.1. *Given the tracking set up defined above, if* L *is a causal SISO LTI system with state-space description* $L = ss(A_l, B_l, C_l, D_l)$, *then*

$$\gamma_{LB} = 2^{-2R}(D_l)^2\{|det(W)|\}^{\frac{2}{T}}.$$

Proof. *If* L *is a SISO causal LTI with state-space description* $L = ss(A_l, B_l, C_l, D_l)$, *then for* T *time steps, it can be represented as a* $T \times T$ *lower triangular Toeplitz matrix operator, with all* T *eigenvalues equal to* D_l. *This implies that the* $\{det(L)\}^{\frac{2}{T}} = (D_l)^2$. ■

We now make some comments about γ_{LB}.

- γ_{LB} depends on L (class of reference commands), W (performance weights), T (performance horizon), and R (channel rate).
- If $det(W)$ and/or if $det(L) = 0$, then the counting argument shown in Theorem 3.1 has to be done in $\mathbb{R}^s$, where $s = \min\{rank(L), rank(W)\}$. Consider a case where $W = diag(\lambda_0, \lambda_1, ..., \lambda_{T-1})$, and $det(W) = 0$ because $\lambda_{k_0} = 0$, for some $0 \leq k_0 \leq T-1$. Then, 0 bits can be allocated to r_{k_0} and performance will not be impacted. Therefore, the problem reduces to allocated bits to r_k for all $k \neq k_0$. On the other hand, if $det(L) = 0$ then one or more of the r_k's are linear combinations of each other, and bits only need to be allocated to one of these r_k's, and the decoder can reconstruct the others knowing L.
- If L is LTI and if $W = I$, then γ_{LB}, is independent of T.
- It is helpful (as we will see when we compute upper bounds) to rewrite the lower bound in terms of the singular and eigenvalues of the matrix WL as follows: $\gamma_{LB} = 2^{-2R}\{\prod_{i=0}^{T-1}\sigma_i(WL)\}^{\frac{2}{T}} = 2^{-2R}\{\prod_{i=0}^{T-1}|\lambda_i(WL)|\}^{\frac{2}{T}}$.

3.1. Causality

When computing the lower bound, we made no assumptions on whether the encoder and decoder are causal or noncausal. If E and D are both noncausal, then the tracking and navigation problems essentially reduce vector quantization problems [18], where time need not enter the picture. At time $t = 0$, the decoder "knows" the future, that is, it knows u_k for $k = 0, 1, ..., T-1$, which are represented by TR bits over a horizon of T steps. This is what causes the lower bound to be independent of the coding scheme. On the other hand, if E and D are causal, then the decoder only knows u_k for $k = 0, 1, ..., T-1$, only at time $t \geq k$, and u_k is represented by at most $(k+1)R$ bits.

In the following sections, we compute two upper bounds. One bound is computed by constructing a noncausal encoder and decoder, and the second upper bound is computed by constructing a causal coding scheme, which is more practical.

4. A NONCAUSAL UPPER BOUND

In this section, we derive an upper bound on worst-case performance assuming that the encoder is noncausal. The upper bound is derived using a coding scheme that transmits information about the signal r in terms of a basis derived from the singular value decomposition (SVD) of the matrix WL.

Consider Figure 4 below. The encoder first uses the SVD of $WL = U\Sigma V^*$ to write $Wr = \sum_{i=0}^{T-1}\sigma_i\alpha_i u_i$, where σ_i is the ith singular value of WL, $\alpha_i = v_i^* w$ where v_i^* is the ith row vector of V^*, and u_i is the ith column vector of U. The α_i's are then each converted into their binary representations and truncated according to the bit-allocation strategy denoted in $\mathcal{R} = (R_0, R_1, ..., R_{T-1})$. In particular, a total of R_k bits are allocated to α_k, for $k = 0, 1, ..., T$, and the only restriction is that $\sum_{k=0}^{T-1} R_k = TR$.

The decoder uses the bit-allocation strategy $\mathcal{R}$ to reconstruct α and then uses the SVD of WL to compute $\hat{r}$ from $\hat{\alpha}$. Finally, the decoder applies H^{-1} to $\hat{r}$ to generate u. We call this $E - D$ construction the "SVD Coding Scheme."

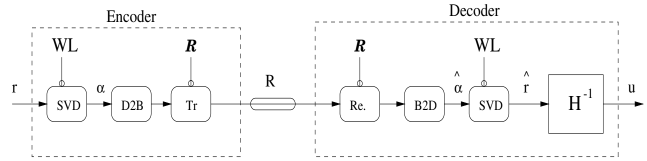

Fig. 4. SVD Coding Scheme

Note that with the above SVD coding scheme,

$$\sup_{r\in\mathcal{C}_r} ||W(y-r)||_2^2 = \sup_{r\in\mathcal{C}_r} ||W(\hat{r}-r)||_2^2$$
$$= \sup_{\{w|\ ||w||_2\le 1\}} ||WL(\hat{w}-w)||_2^2$$
$$= \sup_{\{\alpha|\ ||\alpha|_2\le 1\}} \sum_{i=0}^{T-1}\sum_{j=0}^{T-1}(\hat{\alpha}_i-\alpha_i)(\hat{\alpha}_j-\alpha_j)\sigma_i\sigma_j(u_i'u_j)$$
$$\le \sup_{\{\alpha|\ ||\alpha|_2\le 1\}} \sum_{i=0}^{T-1}|\alpha_i|^2 2^{-2R_i}\sigma_i^2$$
$$\le \sigma_{max}(S)$$

where S is the diagonal matrix $diag(2^{-R_0}\sigma_0, 2^{-R_1}\sigma_1, ..., 2^{-R_{T-1}}\sigma_{T-1})$.

To derive the upper bound using the above SVD coding scheme, we construct $\mathcal{R} = (R_0, R_1, ..., R_{T-1})$ to solve the following optimization problem:

$$\begin{array}{c}\min_{\mathcal{R}} \max_i 2^{-2R_i}\sigma_i^2 \\ \text{s.t. } \sum_{i=0}^{T-1} R_i = TR \\ R_i \ge 0\ \forall i.\end{array}$$

We allow the rates to take on non-integer values to solve for an optimal bit-allocation strategy. The resulting non-integer valued rates can be interpreted as average rates over time. The solution to the above problem forces $2^{-2R_i^*}\sigma_i^2 = 2^{-2R_j^*}\sigma_j^2\ \forall i \ne j$, where $R_i^* = R + log(\sigma_i) - \frac{1}{T}\sum_{j=0}^{T-1} log(\sigma_j)$ for $i = 0, 1, ..., T-1$. This gives us

$$\gamma_{UB} = 2^{-2R}\{log(\textstyle\prod_{i=0}^{T-1}\sigma_j)\}^{\frac{2}{T}} = \gamma_{LB}.$$

Therefore, we achieve the lower bound with a noncausal coding scheme. However, this is not a practical implementation as the encoder does not have access to the entire signal (or its new representation, α) at time 0.

5. A CAUSAL UPPER BOUND

In this section, we derive an upper bound assuming that the encoder is causal and implements a coding scheme illustrated in Figure 5. In the scheme below, the encoder is a quantizer parameterized by a rate matrix which dictates how bits are allocated to each component in the encoder's memory at each time step. Specifically, the rate matrix has the following form.

$$\mathcal{R} = \begin{bmatrix} R_{00} & 0 & 0 & ... & ... \\ R_{01} & R_{11} & 0 & 0 & ... \\ R_{02} & R_{12} & R_{22} & 0 & ... \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ R_{0,T-1} & R_{1,T-1} & R_{2,T-1} & ... & R_{T-1,T-1}\end{bmatrix},$$

such that $\sum_j R_{ij} = R\ \forall i$.

To understand how the above rate matrix dictates a bit-allocation strategy, let $\hat{r}_i(j)$ be the quantized estimate of r_i at time j. Then, $\mathcal{R}$ determines that at time $t = 0$, R_{00} bits

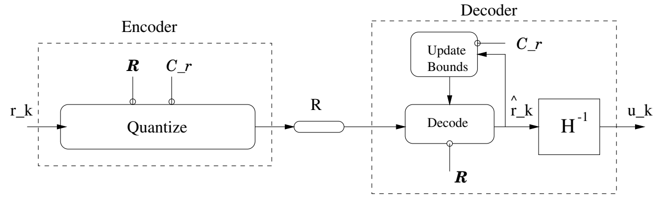

Fig. 5. Causal Coding Scheme

are used to quantize r_0 to produce $\hat{r}_0(0)$. At time $t = 1$, an *additional* R_{01} bits are used to quantize r_0 to produce $\hat{r}_0(1)$, and R_{11} bits are used to quantize r_1 to produce $\hat{r}_1(1)$, and so on. The accuracy of $\hat{r}_i(j)$ is within $\pm M_i(j)2^{-\sum_{k=i}^{j} R_{ik}}$ of r_i for all $i \ge 0$, where $M_i(j)$ is the half-length of the interval in which r_i belongs to as computed by the decoder at time step j. The decoder computes $M_i(j)$ from constraints imposed on $r \in \mathcal{C}_r$.

Consider the following example for horizon length $T = 2$,

$$L = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \qquad W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{bmatrix},$$

for some $\beta \in \mathbb{R}$. Then, the set $\mathcal{C}_r \triangleq \{r \in \mathbb{R}^T | r = Lw, w'w \le 1\}$ imposes the following constraints on r_0 and r_1 : $|r_0| \le 1$ and $|r_1 - \beta r_0| \le \sqrt{1-r_0^2}$.

At time $t = 0$, the encoder's quantization region is on the interval $\{-M_0(0), M_0(0)\} = \{-1, 1\}$, which is divided into $2^{R_{00}}$ equal intervals as shown in Figure 6. The union of the centers of each region comprise the range of E at time 0. The encoder receives r_0 and then outputs the center of the interval in which r_0 falls (see Figure 6) which is represented by R_{00} bits. The decoder then receives $\hat{r}_0(0)$ and then updates its bounds a time 0 for both r_0 and r_1 to prepare for its next input of R bits as follows:

$$\begin{array}{c} lb_0(0) \le r_0 \le ub_0(0), \\ lb_1(0) \le r_1 \le ub_1(0),\end{array}$$

where

$$\begin{array}{c} lb_0(0) = \hat{r}_0(0) - 2^{-R_{00}}M_0(0), \\ ub_0(0) = \hat{r}_0(0) + 2^{-R_{00}}M_0(0), \\ lb_1(0) = \\ min\{\beta lb_0(0) - \sqrt{1-(lb_0(0))^2}, \beta ub_0(0) - \sqrt{1-(ub_0(0))^2}\}, \\ ub_1(0) = \\ max\{\beta lb_0(0) + \sqrt{1-(lb_0(0))^2}, \beta ub_0(0) + \sqrt{1-(ub_0(0))^2}\}.\end{array}$$

At time $t = 1$, the encoder further quantizes r_0 by dividing the interval $\{lb_0(0), ub_0(0)\}$ into $2^{R_{01}}$ equal length intervals, and sends the center of the new interval in which r_0 lies, denoted $\hat{r}_0(1)$. The encoder then uses the remainder R_{11} bits to quantize r_1 by dividing the updated interval $\{lb_1(1), ub_1(1)\}$ into $2^{R_{11}}$ intervals, and then sends the center of the interval in which r_1 falls. It is straightforward to compute

$$lb_0(1) = \hat{r}_0(0) - 2^{-(R_{00}+R_{01})} M_0(0),$$
$$ub_0(1) = \hat{r}_0(0) + 2^{-(R_{00}+R_{01})} M_0(0),$$
$$lb_1(1) =$$
$$min\{\beta lb_0(1) - \sqrt{1-(lb_0(1))^2}, \beta ub_0(1) - \sqrt{1-(ub_0(1))^2}\},$$
$$ub_1(1) =$$
$$max\{\beta lb_0(1) + \sqrt{1-(lb_0(1))^2}, \beta ub_0(1) + \sqrt{1-(ub_0(1))^2}\}.$$

It is important to note that past allocation of bits to r_0 at time $t = 1$ allows for a tradeoff of a smaller known interval in which r_1 lies and finer quantization of the interval itself. I

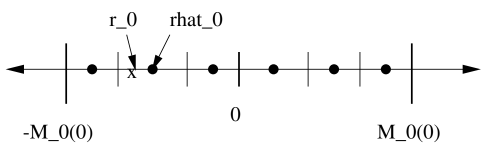

Fig. 6. Quantization region at time $t = 0$

Since $\mathcal{C}_r$ is an ellipse in $\mathbb{R}^T$, knowledge of r_0 impacts the lower and upper bounds on r_k for $k = 1, 2, ..., T-1$. Therefore, it appears that allocating bits to the past signal may be advantageous. It turns out however, that when $\mathcal{C}_r$ is any ellipse, it is always optimal to allocate all R bits to the current value r_k at time k, *i,e,*, it is never optimal to allocate bits to past values $r_0, r_1, ..., r_{k-1}$ to quantize r_k. A proof of the following theorem is shown in [20].

Theorem 5.1. *Consider the tracking problem defined that implements a causal coding scheme parameterized by a rate matrix $\mathcal{R}$. Then, the optimal solution to $\min_{\mathcal{R}} \sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2$ is a diagonal rate matrix, i.e.,*

$$\mathcal{R}^* = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}.$$

6. COMPARISON OF BOUNDS

We now compare the lower bound and causal upper bound to each other for different LTI causal systems $L = ss(A_l, B_l, C_l, D_l)$, and for different time horizons T. We consider diagonal weight matrices $W = diag(\lambda_0, \lambda_1, ..., \lambda_{T-1})$ with $|\lambda_i| \leq 1$, $\forall i$, and fix the rate $R = 10$. Under such conditions, we note that $\gamma_{LB} = 2^{-2R}(D_l)^2\{\prod_{i=0}^{T-1}|\lambda_i|\}^{\frac{2}{T}}$.

Figures 7 and 8 illustrate the bounds for various scenarios, and we make a few observations.

- When the eigenvalues of W are chosen randomly from an i.i.d. process, then we see that the lower bound plateaus for large T. To see why this makes sense, one can show that the expected value of $\gamma_{LB} \to 2^{-2R}$ as T gets large and the variance of $\gamma_{LB} \to 0$ as T gets large. The causal upper bound increases as T increases.

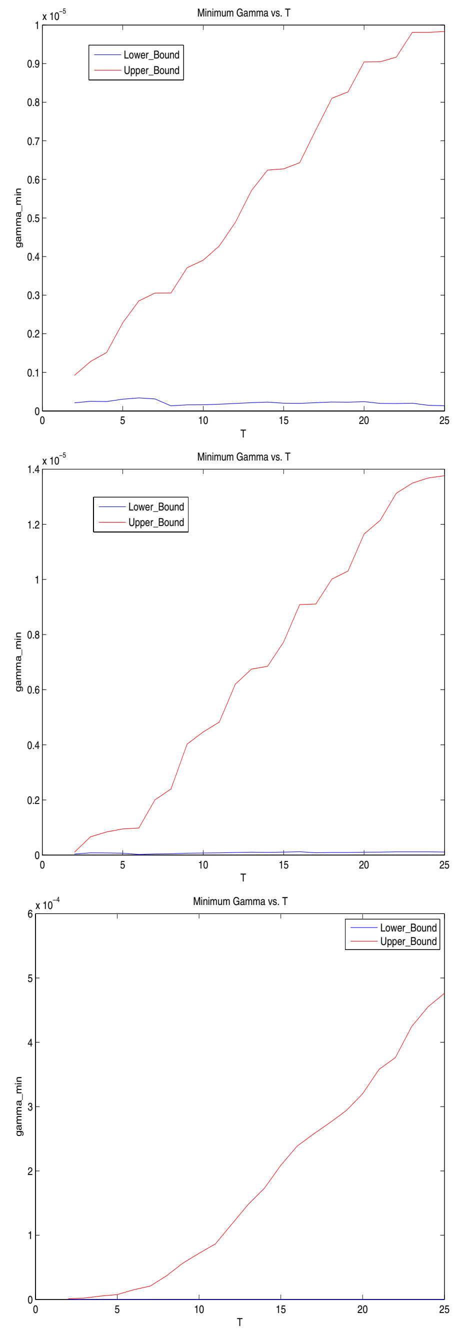

Fig. 7. Top: Bounds for $L = ss(0.01, 0.01, 1, 1)$ and random weights, Middle: Bounds for $L = ss(0.3, 0.3, 1, 1)$ and random weights, Bottom: Bounds for $L = ss(0.9, 0.9, 1, 1)$ and random weights.

- When the eigenvalues of W are exponentially decaying, *i.e.*, $\lambda_i = (\beta)^i$ for $i = 0, 1, ..., T-1$, and for some $0 < \beta < 1$, then the lower bound and non-causal upper bound approach 0 as $T \to \infty$. This can be verified by showing that the ratio $\frac{\gamma_{LB}(T+1)}{\gamma_{LB}(T)} =$

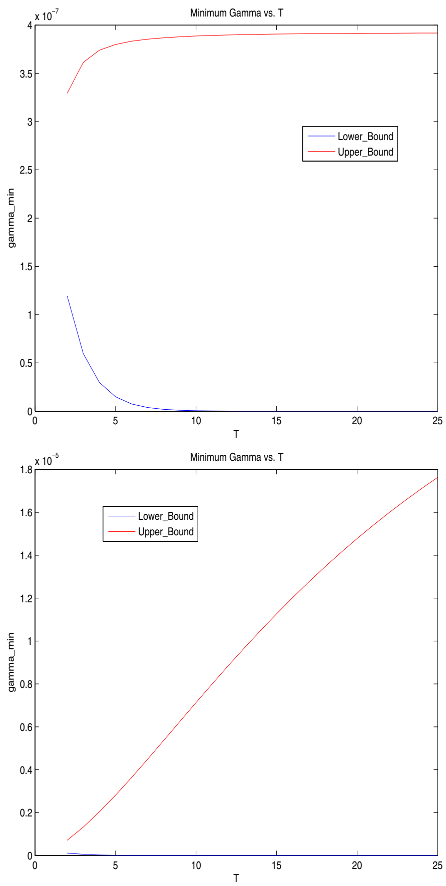

Fig. 8. Top: Bounds for $L = ss(0.1, 0.1, 1, 1)$ and decaying weights, Bottom: Bounds for $L = ss(0.9, 0.9, 1, 1)$ and decaying weights

$\frac{\{\prod_{i=0}^{T}\beta^i\}^{\frac{2}{T+1}}}{\{\prod_{i=0}^{T-1}\beta^i\}^{\frac{2}{T}}} = \beta < 1$. The causal upper bound increases but plateaus for large T, but at a much slower rate when the pole of L is close to the unit disk.

- The causal upper bound is closer to the lower bound when the pole of the LTI system of L or that which generates a noncausal L is close to the origin than if the pole is close to the unit disk.

7. CONCLUSIONS AND FUTURE WORK

In this paper, we compute finite-horizon tracking limitations under finite-rate control. We also show tradeoffs between time and performance accuracy and discuss how causality of coding impacts achievable performance. Future work entails computing finite-horizon tracking and navigation performance limitations for finite-capacity channels, *i.e.,* control over noisy finite-rate channels with and without feedback.

8. ACKNOWLEDGEMENTS

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