What we did in recitation:

Finished change-of-variable: computed how to change unit of area

Namely, computed \( dA_{u,v} \) from \( dA_{x,y} \) given change-of-variable

\[
\begin{align*}
\{ x \} \to \{ y \} \\
x = x(u,v) \\
y = y(u,v)
\end{align*}
\]

\[
\begin{align*}
\{ u \} \to \{ x \} \\
u = u(x,y) \\
v = v(x,y)
\end{align*}
\]

To go from \( x,y \)-coords to \( u,v \)-coords, use the Jacobian - the determinant of the matrix of partials for \( x,y \):

\[
\begin{bmatrix}
dx \ dy
\end{bmatrix} = |J| \begin{bmatrix}
du \ dv
\end{bmatrix} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \ dv
\]

where

\[
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}
x_u & x_v \\
y_u & y_v
\end{vmatrix} = \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{array} \right| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
\]

where \( x = x(u,v) \); \( y = y(u,v) \)

We call this \( J \), the Jacobian.

Note: Here, Jacobian is the determinant of the matrix of partials. In some other contexts, the Jacobian refers to the matrix of partials itself.

Briefly, we discussed how to determine which variables to switch to.

Idea: choose new variables geometrically: look at region bounding curves, think of them as level curves of some function of one dim higher.

Alternatively: Think of bounding curves of region as defining a new coordinate system on \( \mathbb{R}^2 \) plane. Choose new variables appropriately.

Example 1: Integrate over \( R \) the function

\[
\left( \frac{x-y}{x+y+z} \right)^2
\]
What are the bounding curves of \( R \)? They are 4 lines, which are pairwise parallel:

\[ y - x = \text{const} \]

Thus the bounding curves seem to belong to two families:

\[ \{ y - x = \text{const}\} \quad \text{and} \quad \{ 2y + x = \text{const}\} \]

Combining these lines we get a grid on the \( xy \) plane.

This defines a new coordinate system given by points \((c, d)\) at the intersection of \( x + y = c \) and \( x - y = d \).

We can let the two axes be called "\( u \)" and "\( v \)," set \( u = x + y \) and \( v = x - y \), then the coordinates of point \( P \) will be \( u = c \), \( v = d \).
In this case, the new variables worked well because the integrand was

\[
\left( \frac{x-y}{x+y+2} \right)^2 = \left( \frac{-u}{u+2} \right)^2 = \left( \frac{u}{u+2} \right)^2.
\]

Thus, since \( \mathbf{R} \) translates into \(-1 \leq u = y-x \leq 1, \), we can rewrite the integral

\[
\int \int_{\mathbf{R}} \left( \frac{x-y}{x+y+2} \right)^2 \, dy \, dx = \int \int_{-1}^{1} \left( \frac{u}{u+2} \right)^2 \left| J \right| \, du \, dv = (\ast)
\]

where \( J \) computed from \( x(u,v), y(u,v) \):

\[
\begin{align*}
  u &= y-x \quad \text{solves} \quad 2y &= u+v \\
  v &= y+x \quad \text{or} \quad x &= \frac{v-u}{2}
\end{align*}
\]

\[
J = \left| \begin{array}{cc}
  xu & xv \\
  yu & yv \\
\end{array} \right| = \left| \begin{array}{cc}
  -\frac{1}{2} & \frac{1}{2} \\
  \frac{1}{2} & \frac{1}{2} \\
\end{array} \right| = \frac{1}{4} - \frac{1}{4} = -\frac{1}{2}
\]

Thus \( (\ast) = \int \int_{-1}^{1} \left( \frac{u}{u+2} \right)^2 \left| -\frac{1}{2} \right| \, du \, dv = \frac{1}{2} \int \int_{-1}^{1} \left( \frac{u}{u+2} \right)^2 \, du \, dv \).

Then evaluate normally (Check: answer should be as in Notes, CV.

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Example 2: (3D-7)

Trick: \( x=0 \) is \( z \)-axis, is same as \( xy=0 \).

Thus the bounding curves are level curves of \( xy=\text{const} \) and \( x=\text{const} \):

\[
\begin{align*}
  0 &\leq xy \leq 1 \\
  1 &\leq x \leq 2
\end{align*}
\]

Set \( u=x \), \( v=xy \) to transfer into

\[
\begin{align*}
  u &= x, \quad v=xy \\
  0 &\leq u \leq 2 \\
  0 &\leq v \leq 1
\end{align*}
\]

These are the most convenient vars for the Region. But are they the most convenient for the \( u,v \) integrand?
We are trying to find the polar moment of inertia of $R$.

\[ \iiint_R x^2 + y^2 \, dx \, dy \]  

and the problem suggests a different transfer of vars:

\[ u = xy, \quad v = \frac{y}{x} \]

\[ \|This will allow us to parametrize $x^2 + y^2$ nicely\|

- $x^2 + y^2 = xy \cdot \frac{x}{y} + xy \cdot \frac{y}{x} = u \cdot \frac{1}{v} + u \cdot v = \frac{u}{v} + uv$.

Do Jacobian to compute area unit:

\[
J = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} \]

\[
= \frac{1}{\left| \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \right|} = \frac{1}{\left| \begin{vmatrix} y & x \\ -\frac{y^2}{x^2} & \frac{y}{x} \end{vmatrix} \right|} = \frac{1}{2y/x} = \frac{x}{2y}.
\]

Thus, \[ dx \, dy = \frac{x}{2y} \, du \, dv. \]

Furthermore:

- $1 \leq x \leq 2$
- $0 \leq y \leq xy$
- $0 \leq u \leq \sqrt{\frac{1}{y^2}}$
- $0 \leq v \leq 1$

So we have integral \[ (\#5) = \iint_{R} \left( \frac{u \cdot v}{u + v} \right) \frac{1}{2v} \, du \, dv. \]

Once again, use standard techniques to solve these integrals.

\[ \iint_{R} x^2 + y^2 \, dx \, dy = \int_{1/4}^{1} \int_{0}^{1} \frac{u \cdot v}{u + v} \frac{1}{2v} \, du \, dv. \]
Finally... **Probability** and its connection to **double integrals and Jacobians!**

**Probability in Pictures**

Given region $R$ and subregion $S$

the probability that $x$ in $R$ lies in $S$ is

$$P(x \text{ is in } S) = \frac{\text{Part}}{\text{Whole}} = \frac{\text{Area (S)}}{\text{Area (R)}}, \text{ if } x \text{ is uniformly distributed in } R$$

"Uniformly distributed" means exactly what it sounds like: any "value" of $x$—any point in $R$—is just as likely, in a sense, as any other position.

**Note:** usually, for a continuous distribution of $x$ over an interval or an area,

$$P(x = a) = 0$$

one specific value,

i.e., for $0 < x < 1$,

$$P(x = 0.75) = 0$$

**So, this is a caveat, but for uniformly distributed over continuous distribution, we say probability of $x$ in small interval of some area at one point is same as prob of $x$ in other interval of same small area elsewhere in area.**

Kind of hard to understand, leave for later, if cleared.

However, very rarely work w/ uniformly distributed variables, usually variables are given a probability density function that depends on one or more dimensions
For instance, on a line

\[ \text{region of interest} \]

If \( 0 < x < 1 \), dimension is \( x \), and variable \( y \) is distributed on line with density function \( f(x) \) [\( x \)-position on line]

Then, the probability that \( y \in (0,1) \) can be computed:

\[
P(0 < x < 1 \text{ in } y \in (0,1)) = \text{sum of prob. density on area of interest} = \int_0^1 f(x) \, dx
\]

**There is one requirement** for this formula to work:

\[
\text{Total prob. of \( y \) in any area} = \int_{-\infty}^{\infty} f(x) \, dx = 1
\]

(i.e. \( y \) lies somewhere in entire line, max region \( R \)).

What if this is not true? Then we need to rescale. Let \( \delta = \) density function, \( S \)-subregion of \( R \)-total region

\[
P(\text{var } \in S) = \frac{\text{mass}(S)}{\text{mass of } R} = \frac{1}{M} \int_S \delta \, dA = \frac{\int_S \delta \, dA}{\int_R \delta \, dA}
\]

Thus, since \( M \) is independent of subregions \( S \), and \( R \) is a global value, we can precompute constant

\[
M = \int_R \delta \, dA,
\]

and then RENORMALIZE

\[
P(\text{var } \in S) = \frac{1}{M} \int_S \delta \, dA = \int_S \left( \frac{\delta}{M} \right) \, dA
\]

(normalized probability density function)
Calculate probability density of some dimension in 2-d problem w/ given 2-d density:

\[ R = \text{disk} \quad 2-d, \quad S = \text{washer} \quad a < r < b, \quad \delta = \delta(r, \theta) \]

2-d density function.

To calculate prob. density of one of the dimensions, say \( r \), we need to determine what

\[ P(\text{dime}(a, b)) = P(a < r < b) \quad \text{is.} \]

Namely, we need to find a prob. density function \( f = f(r) \)

\[ \int_a^b f(r) \, dr = P(a < r < b) \quad \text{in the disk} R, \quad \text{i.e., if} \quad S = \begin{array}{c} a \end{array} \quad = P(x \in S) \]

Just from definition of probability.

The easiest way to do this here is to

- Set up \( P(x \in S) \) as 2-d integral in \( r, \theta \)
- Integrate out \( \theta \) to be left w/ integral in \( r \)
- Pattern match for \( f \)

Example: For \( R = \begin{array}{c} \end{array} \), \( S = \begin{array}{c} \end{array} \), \( \delta = r^2 \theta \)

\[ P(a < r < b) = \int_S \delta \, dr \, d\theta = \int_a^b r^3 \begin{array}{c} \int_0^{2\pi} (r^2 \theta) \, r \, d\theta \end{array} \, dr \]

Compute to get \( f \)

\[ = \int_a^b r^3 \left( \frac{4\pi^2 \theta}{2} - 0 \right) \, dr = \int_a^b 2\pi^2 r^3 \, dr \]

This is our prob. density function:

\[ f(r) = 2\pi^2 r^3 \]

---
Finally, some notes on **conditional probability**

**Definition: (Intuition)**

Let \( A, B \) be sets, \( A \cup B \) (a union of \( B \)) - be the entire space and \( A \cap B = C \), the intersection.

Suppose \( x \) is distributed uniformly over \( A \cup B \).

Then the conditional probability that \( x \in C \) given that \( x \in B \) is

\[
P(x \in C \mid x \in B) = \frac{\text{Area}(C)}{\text{Area}(B)} = \frac{\text{Part}}{\text{Whole}}
\]

Main idea from notes:

Support \( C \) is interval \( a < x < b \) and \( B \) is constraint or curve \( g(x,y) = v_0 \).

Then doing change of variables \( u = x, v = g(x,y) \), we get

\[
P(a < x < b \mid g(x,y) = v_0) = \int_{a}^{b} |J| \, du
\]

with normalization factor \( = \int_{v_0}^{v_0} |J| \, du \).

In particular, in notes, for \( g = xy \),

\[
\begin{align*}
U &= x, & |J| &= \frac{1}{|\frac{\partial (uv)}{\partial (x,y)}|} = \frac{1}{|y \cdot x|} = \frac{1}{1} = 1 \\
V &= xy
\end{align*}
\]

Thus, the probability density is the renormalized Jacobian

\[
\int |J| = \frac{1}{M(u)}
\]